A GIAMBELLI FORMULA FOR ISOTROPIC GRASSMANNIANS

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Abstract. Let X be a symplectic or odd orthogonal Grassmannian. We prove a Giambelli formula which expresses an arbitrary Schubert class in $\mathrm{H}^*(X,\mathbb{Z})$ as a polynomial in certain special Schubert classes. This polynomial, which we call a *theta polynomial*, is defined using raising operators, and we study its image in the ring of Billey-Haiman Schubert polynomials.

0. Introduction

Let $\mathrm{G} = \mathrm{G}(m,N)$ denote the Grassmannian of m-dimensional subspaces of $\mathbb{C}^N$. To each integer partition $\lambda = (\lambda_1,\ldots,\lambda_m)$ whose Young diagram is contained in an $m\times(N-m)$ rectangle, we associate a Schubert class σ_λ in the cohomology ring of G. The *special* Schubert classes $\sigma_1,\ldots,\sigma_{N-m}$ are the Chern classes of the universal quotient bundle $\mathcal{Q}$ over $\mathrm{G}(m,N)$; they generate the cohomology ring $\mathrm{H}^*(\mathrm{G},\mathbb{Z})$. The classical *Giambelli formula* [G]

$$\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{i,j} \tag{1}$$

is an explicit expression for σ_λ as a polynomial in the special classes; as is customary, we agree here and in later formulas that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$.

The relation between the Schubert calculus on the Grassmannian $\mathrm{G}(m,N)$ and the algebra of Schur's S-functions s_λ (originally defined by Cauchy [C] and Jacobi [J]) is well known. Given an infinite list $x = (x_1,x_2,\ldots)$ of commuting independent variables, we define the elementary symmetric functions $e_r(x)$ by the formal relation

$$\prod_{i=1}^{\infty}(1+x_it) = \sum_{r=0}^{\infty} e_r(x)t^r$$

and set, for any partition λ, $s_{\lambda'}(x) = \det(e_{\lambda_i+j-i}(x))_{i,j}$. Here λ' is the partition whose Young diagram is the transpose of the diagram of λ. The ring $\Lambda = \mathbb{Z}[e_1,e_2,\ldots]$ of symmetric functions in x has a free $\mathbb{Z}$-basis consisting of the Schur functions s_λ, for all partitions λ. These Schur S-functions enjoy many good combinatorial properties, such as nonnegativity of their coefficients, and multiply exactly like the Schubert classes on $\mathrm{G}(m,N)$, when m and N are sufficiently large.

There is a closely analogous story to the above for the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$ which parametrizes maximal isotropic subspaces of $\mathbb{C}^{2n}$, with respect to a symplectic form. The Schubert classes in $\mathrm{H}^*(\mathrm{LG},\mathbb{Z})$ are indexed by *strict* partitions, i.e., partitions with distinct (non-zero) parts, whose diagrams fit in a square of side n. The special Schubert classes $\sigma_r = c_r(\mathcal{Q})$ again generate the cohomology ring,

Date: June 12, 2013.

2000 *Mathematics Subject Classification.* Primary 14N15; Secondary 05E15, 14M15.

The authors were supported in part by NSF Grants DMS-0603822 and DMS-0906148 (Buch), the Swiss National Science Foundation (Kresch), and NSF Grants DMS-0639033 and DMS-0901341 (Tamvakis).

and there is a Giambelli-type formula due to Pragacz [Pra]. This latter may be described in two steps: For partitions $\lambda = (a,b)$ with only two parts, we have

$$\sigma_{a,b} = \sigma_a\sigma_b - 2\sigma_{a+1}\sigma_{b-1} + 2\sigma_{a+2}\sigma_{b-2} - \cdots \tag{2}$$

while for λ with 3 or more parts,

$$\sigma_\lambda = \text{Pfaffian}(\sigma_{\lambda_i,\lambda_j})_{i<j}. \tag{3}$$

The identities (2) and (3) in fact also go back to the work of Schur [S], who considered a family of symmetric functions $\{Q_\lambda\}$ known as Schur Q-functions. We define $q_r(x)$ by the equation

$$\prod_{i=1}^{\infty} \frac{1+x_i t}{1-x_i t} = \sum_{r=0}^{\infty} q_r(x) t^r$$

and then use the same relations (2) and (3) with $q_r(x)$ in place of σ_r to define $Q_{a,b}(x)$ and then $Q_\lambda(x)$, for each strict partition λ. If we let $\Gamma = \mathbb{Z}[q_1, q_2, \ldots]$ denote the ring of Schur Q-functions, then the $\{Q_\lambda\}$ for λ strict form a $\mathbb{Z}$-basis for Γ, whose algebra agrees with Schubert calculus on $\text{LG}(n,2n)$, as $n \to \infty$. Moreover, there is a well developed combinatorial theory for the Q-functions, analogous to that for the S-functions.

Choose $k \geq 0$ and consider now the Grassmannian $\text{IG}(n-k, 2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, equipped with a symplectic form. We call a partition λ *k-strict* if no part greater than k is repeated, i.e., $\lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1}$. The Schubert classes on IG are indexed by k-strict partitions whose diagrams fit in an $(n-k) \times (n+k)$ rectangle. Given such a λ and a complete flag of subspaces $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$ such that $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \text{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}. \tag{4}$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, using Poincaré duality, a Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $\text{H}^{2|\lambda|}(\text{IG}, \mathbb{Z})$. As above, we consider the special Schubert classes $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$ for $1 \leq r \leq n+k$.

In [BKT1], we proved a Pieri rule for the products $\sigma_r\sigma_\lambda$ in $\text{H}^*(\text{IG})$. Equipped with this rule and the help of a computer, we observed that (i) when $\lambda_j \leq k$ for all j, then σ_λ is given by the determinantal formula (1); (ii) when $\lambda_j > k$ for all non-zero λ_j, then λ is strict and σ_λ is given by the Pfaffian formulas (2), (3). It is tempting to ask for an analogous Giambelli formula for σ_λ when λ is a general k-strict partition. Note that the formula is determined only up to an ideal of relations; whatever the answer, it must naturally interpolate between the Jacobi-Trudi determinant (1) and the Schur Pfaffian (3). A similar question was also raised by Pragacz and Ratajski [PR], who were using a different set of special Schubert classes.

The answer we give depends crucially on our choice of k-strict partitions to index the Schubert classes, and uses Young's *raising operators* [Y, p. 199]. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support and $i < j$, we define $R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$; a raising operator R is any monomial in these R_{ij}'s. Set $m_\alpha = \prod_i \sigma_{\alpha_i}$ and $R\, m_\alpha = m_{R\alpha}$ for any raising operator R.[1] Using these operators, the Giambelli formulas (1) and (2)–(3) can be expressed as

$$\sigma_\lambda = \prod_{i<j}(1-R_{ij})\, m_\lambda \quad \text{and} \quad \sigma_\lambda = \prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}\, m_\lambda, \tag{5}$$

respectively.

Definition 1. For a general k-strict partition λ, we define the operator

$$R^\lambda = \prod_{i<j}(1-R_{ij}) \prod_{\lambda_i+\lambda_j>2k+j-i}(1+R_{ij})^{-1}$$

where the first product is over all pairs $i<j$ and second product is over pairs $i<j$ such that $\lambda_i+\lambda_j>2k+j-i$.

Theorem 1. *For any k-strict partition λ, we have $\sigma_\lambda = R^\lambda m_\lambda$ in the cohomology ring of* $\mathrm{IG}(n-k,2n)$.

For example, in the ring $\mathrm{H}^*(\mathrm{IG}(4,10))$ (where $k=1$) we have

$$\sigma_{321} = \frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\, m_{321} = (1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23})\, m_{321}$$

$$= m_{321}-2m_{411}+m_{42}+2m_{51}-m_{33} = \sigma_3\sigma_2\sigma_1 - 2\sigma_4\sigma_1^2+\sigma_4\sigma_2+2\sigma_5\sigma_1-\sigma_3^2.$$

Furthermore, the theorem implies that if the k-strict partition λ satisfies $\lambda_i+\lambda_j \leq 2k+j-i$ for all $i<j$, then equation (1) is valid, while if $\lambda_i+\lambda_j>2k+j-i$ for $i<j\leq \ell(\lambda)$, then equations (2) and (3) hold.

Our proof of Theorem 1 proceeds by showing directly that the expression $R^\lambda m_\lambda$ satisfies the Pieri rule for isotropic Grassmannians from [BKT1]. This is sufficient because the Pieri rule can be used recursively to show that a general Schubert class may be written as a polynomial in the special Schubert classes. The argument is challenging because the operator R^λ depends on λ, in contrast to the fixed raising operator expressions in (5). We remark that the equations corresponding to (5) for the Schur S- and Q-functions may be deduced from the formal identities

$$\det(x_i^{\ell-j}) = \prod_{i<j}(x_i-x_j) \quad \text{and} \quad \text{Pfaffian}\left(\frac{x_i-x_j}{x_i+x_j}\right) = \prod_{i<j}\frac{x_i-x_j}{x_i+x_j} \tag{6}$$

due to Vandermonde and Schur, respectively (see e.g. [M, I.3 and III.8]).

We next use raising operators to define a family of polynomials $\{\Theta_\lambda\}$ indexed by k-strict partitions whose algebra is the same as the Schubert calculus in the stable cohomology ring of IG. Fix an integer $k\geq 0$ and consider a finite set of variables $y=(y_1,\ldots,y_k)$. For any $r\geq 0$, define ϑ_r by the equation

$$\prod_{i=1}^{\infty}\frac{1+x_it}{1-x_it}\prod_{j=1}^{k}(1+y_jt) = \sum_{r=0}^{\infty}\vartheta_r(x\,;y)t^r,$$

so that $\vartheta_r(x\,;y) = \sum_i q_{r-i}(x)e_i(y)$. We call $\Gamma^{(k)} := \mathbb{Z}[\vartheta_1,\vartheta_2,\ldots]$ the ring of *theta polynomials*. For any finite integer sequence α, let $\vartheta_\alpha = \prod_i \vartheta_{\alpha_i}$, and for any k-strict

[1] As is customary, we slightly abuse the notation and consider that the raising operator R acts on the index α, and not on the monomial m_α itself.

partition λ, define the theta polynomial[2]

$$\Theta_\lambda := R^\lambda \vartheta_\lambda.$$

When $k = 0$, we have that $\Theta_\lambda(x\,;y) = Q_\lambda(x)$ is a Schur Q-function. As a first application of Theorem 1, we obtain the next two results. The first implies that the algebra of theta polynomials agrees with the Schubert calculus on isotropic Grassmannians $\mathrm{IG}(n-k, 2n)$ when n is sufficiently large.

Theorem 2. *The Θ_λ, for λ k-strict, form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$. There is a surjective ring homomorphism $\Gamma^{(k)} \to \mathrm{H}^*(\mathrm{IG}(n-k,2n))$ such that Θ_λ is mapped to σ_λ, if λ fits inside an $(n-k)\times(n+k)$ rectangle, and to zero, otherwise.*

Theorem 3. *Let λ be a k-strict partition.*

(a) *If $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, then*

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda} S_\mu(x) s_{\lambda'/\mu'}(y), \quad \textit{where} \quad S_\mu(x) = \det(q_{\mu_i+j-i}(x)).$$

(b) *If $\lambda_i + \lambda_j > 2k + j - i$ for all $i < j \leq \ell(\lambda)$, then*

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda} Q_\mu(x) s_{\mathcal{S}(\lambda/\mu)'}(y),$$

where the sum is over all strict partitions $\mu \subset \lambda$ such that $\ell(\mu) \geq \ell(\lambda) - 1$, and $\mathcal{S}(\lambda/\mu)$ denotes a shifted skew diagram.

Billey and Haiman [BH] have introduced a theory of type C Schubert polynomials $\mathfrak{C}_w(x,z)$ indexed by elements w of the hyperoctahedral group. To any k-strict partition λ we associate a k-Grassmannian element w_λ, and prove (Proposition 6.2) that $\Theta_\lambda(x\,;z_1,\ldots,z_k) = \mathfrak{C}_{w_\lambda}(x,z)$. Note that this is an equality in the full ring $\Gamma[z_1,z_2,\ldots]$ of Billey-Haiman polynomials, where there are relations among the generators $q_r(x)$ of Γ. Our Giambelli formula may therefore be used to further understand these and related polynomials. For instance, it follows that the type C Stanley symmetric function $F_{w_\lambda}(x)$ of [BH, FK, L] is equal to $R^\lambda q_\lambda(x)$ (Corollary 6.4). Moreover, this connection implies that the coefficients of $\Theta_\lambda(x\,;y)$ are nonnegative integers. These integers have several combinatorial interpretations; the one we provide stems from the work of Kraśkiewicz [Kr] and Lam [L].

Theorem 4. *For any k-strict partition λ, the polynomial Θ_λ is a linear combination of products of Schur Q-functions and S-polynomials:*

$$\Theta_\lambda(x\,;y) = \sum_{\mu,\nu} e^\lambda_{\mu\nu} Q_\mu(x) s_{\nu'}(y)$$

where the sum is over partitions μ and ν such that μ is strict and $\nu \subset \lambda$ with $\nu_1 \leq k$. Moreover, the coefficients $e^\lambda_{\mu\nu}$ are nonnegative integers, equal to the number of Kraśkiewicz tableaux for $w_\lambda w_\nu^{-1}$ of shape μ.

The definition of Kraśkiewicz tableaux is recalled in §6. In [T2], an approach to tableau formulas via raising operators is applied to obtain a different expression for $\Theta_\lambda(x\,;y)$, which writes it as a sum of monomials $2^{n(U)}(xy)^U$ over all 'k-bitableaux' U of shape λ.

[2]We use the term 'theta polynomial' to denote both the Giambelli polynomial in Theorem 1 and its image in $\Gamma^{(k)}$; see Definition 5.3.

The Billey-Haiman polynomials $\mathfrak{C}_{w_\lambda}(x,z)$ represent the pullbacks of the Schubert classes σ_λ in the stable cohomology ring of the complete flag variety Sp_{2n}/B, as explained in [BH]. We emphasize that this theory does not imply our raising operator Giambelli formula, which is an expression for σ_λ in terms of *special Schubert classes* σ_r, analogous to the classical one in type A. Even in the case of Lagrangian Grassmannian classes, when $\mathfrak{C}_{w_\lambda}(x,z) = Q_\lambda(x)$, one deduces Pragacz' formulas (2)–(3) from op. cit. only by appealing to the corresponding known identities for Schur's Q-functions. The present paper provides a new proof of the classical Giambelli formulas for cominuscule Grassmannians which extends to cover all symplectic and orthogonal Grassmannians, where the Schubert calculus is less well understood.

We have described the theory here in the symplectic case, but there are entirely analogous results for the odd orthogonal groups. In fact, for technical reasons, our proof of Theorem 1 is obtained in the setting of orthogonal type B. We also have analogues of these Giambelli formulas for the quantum cohomology rings of symplectic and odd orthogonal Grassmannians; this application appears in [BKT2]. In a sequel to this paper [BKT3], we discuss the Giambelli formula for even orthogonal Grassmannians, which is more involved.

This article is organized as follows. The proof of Theorem 1 occupies §1–§4. Section 5 develops the theory of theta polynomials in a manner parallel to the theory of Schur Q-functions, and contains our proofs of Theorems 2 and 3. Finally, in §6 we show that theta polynomials are equal to certain Billey-Haiman Schubert polynomials for the hyperoctahedral group, and prove Theorem 4.

1. Preliminary Results

1.1. The Schubert varieties in $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ are indexed by k-strict partitions λ which are contained in an $(n-k)\times(n+k)$ rectangle; we denote the set of all such partitions by $\mathcal{P}(k,n)$. Consider the exact sequence of vector bundles over IG

$$0 \to \mathcal{S} \to E \to \mathcal{Q} \to 0,$$

where E denotes the trivial bundle of rank $2n$ and $\mathcal{S}$ is the tautological subbundle of rank $n-k$. The special Schubert class σ_p is equal to the Chern class $c_p(\mathcal{Q})$.

The symplectic form on E gives a pairing $\mathcal{S}\otimes\mathcal{Q}\to\mathcal{O}_{\mathrm{IG}}$, which in turn produces an injection $\mathcal{S}\hookrightarrow\mathcal{Q}^*$. For $r>k$ we therefore have

$$c_{2r}(\mathcal{Q}\oplus\mathcal{Q}^*) = c_{2r}(E/\mathcal{S}\oplus\mathcal{Q}^*) = c_{2r}(\mathcal{Q}^*/\mathcal{S}) = 0,$$

which implies that the relations

$$\sigma_r^2 + 2\sum_{i=1}^{n+k-r}(-1)^i\sigma_{r+i}\sigma_{r-i} = 0 \quad \text{for } r>k \tag{7}$$

hold in $\mathrm{H}^*(\mathrm{IG},\mathbb{Z})$.

1.2. A *composition* $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_r)$ is a vector of integers from the set $\mathbb{N} = \{0,1,2,\ldots\}$; we let $|\alpha| = \sum\alpha_i$. For λ any sequence of (possibly negative) integers, we say that λ has length ℓ if $\lambda_i = 0$ for all $i>\ell$ and $\ell\geq 0$ is the smallest number with this property. All integer sequences in this paper have finite length, and we will identify any integer sequence of length ℓ with the vector consisting of its first ℓ entries. In analogy with Young diagrams of partitions, we will say that a pair $[i,j]$ is a *box* of the integer sequence λ if $i\geq 1$ and $1\leq j\leq\lambda_i$.

Let $\Delta^\circ = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \leq i < j\}$ and define a partial order on Δ° by agreeing that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. We call a finite subset D of Δ° a *valid set of pairs* if it is an order ideal, i.e., $(i,j) \in D$ implies $(i',j') \in D$ for all $(i',j') \in \Delta^\circ$ with $(i',j') \leq (i,j)$.

Any valid set of pairs D defines the raising operator

$$R^D = \prod_{i<j}(1-R_{ij}) \prod_{i<j\,:\,(i,j)\in D}(1+R_{ij})^{-1}.$$

Given a composition α and an integer $\ell > 0$, we denote by $m(D,\alpha,\ell)$ the number of non-zero coordinates α_i such that $(i,\ell) \in D$. We say that α is (D,ℓ)-*compatible* if $\alpha_i \in \{0,1\}$ whenever $(i,\ell) \notin D$.

Definition 1.1. For any valid set of pairs D and any integer sequence λ of length ℓ we define a cohomology class $T_\lambda = T(D,\lambda)$ recursively as follows. Set $T_p = \sigma_p$, and for an arbitrary integer sequence $\mu = (\mu_1,\ldots,\mu_{\ell-1})$ and $r \in \mathbb{Z}$, set

$$T_{\mu,r} = \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\mu+\alpha} T_{r-|\alpha|}\,, \tag{8}$$

where the sum is over all (D,ℓ)-compatible vectors $\alpha \in \mathbb{N}^{\ell-1}$.

The sum (8) is well defined because only finitely many of its summands are non-zero; we also have $T_{\mu,r} = 0$ if $r < 0$. Notice that definition (8) of $T(D,\lambda)$ is equivalent to expanding the raising operator formula

$$R^D m_\lambda = \prod_{i<j<\ell}(1-R_{ij}) \prod_{i<j<\ell\,:\,(i,j)\in D}(1+R_{ij})^{-1} \prod_{i=1}^{\ell-1}(1-R_{i\ell}) \prod_{i\,:\,(i,\ell)\in D}(1+R_{i\ell})^{-1} m_{\mu,r}$$

after the last (i.e., the ℓ-th) entry of $\lambda = (\mu, r)$. Therefore $T_\lambda = R^D m_\lambda$.

1.3. If $D = \emptyset$ then for any integers r and s we have

$$T_{r,s} = T_r T_s - T_{r+1} T_{s-1}$$

and so $T_{r,r+1} = 0$, while more generally $T_{r,s} = -T_{s-1,r+1}$.

We claim that if $D \neq \emptyset$ and $r,s \in \mathbb{Z}$ are such that $r+s > 2k$, then $T_{s,r} = -T_{r,s}$; in particular $T_{r,r} = 0$ whenever $r > k$. Indeed, from the definition we obtain

$$T_{r,s} = \sigma_r\sigma_s - 2\,\sigma_{r+1}\sigma_{s-1} + 2\,\sigma_{r+2}\sigma_{s-2} - \cdots$$

and hence $T_{s,r} = -T_{r,s}$ whenever $r+s$ is odd. If $r+s = 2m > 2k$ is even, we see that

$$T_{r,s} + T_{s,r} = (-1)^{\frac{r-s}{2}}\, 2\,(\sigma_m^2 - 2\,\sigma_{m+1}\sigma_{m-1} + 2\,\sigma_{m+2}\sigma_{m-2} - \cdots) = 0 \tag{9}$$

using the relations (7) in the cohomology ring of IG.

The previous observations are generalized in the next two lemmas.

Lemma 1.2. *Let $\lambda = (\lambda_1,\ldots,\lambda_{j-1})$ and $\mu = (\mu_{j+2},\ldots,\mu_\ell)$ be integer vectors. Assume that $(j,j+1) \notin D$ and that for each $h < j$, $(h,j) \in D$ if and only if $(h,j+1) \in D$. Then for any integers r and s we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s-1,r+1,\mu}\,.$$

In particular, $T_{\lambda,r,r+1,\mu} = 0$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and the identity follows by induction, because

$$T_{\rho,t} = \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|}.$$

Therefore, we may assume that μ is empty. Set $\ell = j+1$. Then we have

$$\begin{aligned} T_{\lambda,r,s} &= \sum_\alpha (-1)^{|\alpha|}\, 2^{m(D,\alpha,\ell)}\, T_{\lambda+\alpha,r} T_{s-|\alpha|} - \sum_\alpha (-1)^{|\alpha|}\, 2^{m(D,\alpha,\ell)}\, T_{\lambda+\alpha,r+1} T_{s-|\alpha|-1} \\ &= \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|}\, 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)}\, T_{\lambda+\alpha+\beta} T_{r-|\beta|} T_{s-|\alpha|} \\ &\quad - \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|}\, 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)}\, T_{\lambda+\alpha+\beta} T_{r+1-|\beta|} T_{s-1-|\alpha|} \end{aligned}$$

where the sums are over all (D,ℓ)-compatible sequences $\alpha \in \mathbb{N}^{j-1}$ and $(D,\ell-1)$-compatible sequences $\beta \in \mathbb{N}^{j-1}$. The assumptions on D imply that these two sets of sequences coincide, and this proves the lemma. □

Lemma 1.3. *Let $\lambda = (\lambda_1, \dots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \dots, \mu_\ell)$ be integer vectors, assume $(j, j+1) \in D$, and that for each $h > j+1$, $(j,h) \in D$ if and only if $(j+1,h) \in D$. If $r, s \in \mathbb{Z}$ are such that $r+s > 2k$, then we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s,r,\mu}.$$

In particular, $T_{\lambda,r,r,\mu} = 0$ for any $r > k$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and $\rho' = (\lambda, s, r, \tau)$, and the identity follows by induction, because

$$T_{\rho,t} = \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|} = -\sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho'+\alpha} T_{t-|\alpha|} = -T_{\rho',t}.$$

Thus we may assume that μ is empty. Set $\ell = j+1$, and note that $(h,h') \in D$ for all $h < h' \leq \ell$. If $m > 0$ is the least integer such that $2m \geq \ell$, we claim that $T_\rho = T_{\lambda,r,s}$ satisfies the relation

$$T_\rho = \sum_{i=2}^{2m} (-1)^i\, T_{\rho_1,\rho_i}\, T_{\rho_2,\dots,\widehat{\rho_i},\dots,\rho_{2m}}. \tag{10}$$

Equation (10) follows from the formal identity of raising operators

$$\prod_{1\leq h<h'\leq 2m} \frac{1-R_{hh'}}{1+R_{hh'}} = \sum_{i=2}^{2m} (-1)^i\, \frac{1-R_{1i}}{1+R_{1i}} \prod_{\substack{2\leq h<h'\leq 2m \\ h\neq i\neq h'}} \frac{1-R_{hh'}}{1+R_{hh'}},$$

which is equivalent the classical formula

$$\prod_{1\leq h<h'\leq 2m} \frac{x_h - x_{h'}}{x_h + x_{h'}} = \text{Pfaffian}\left(\frac{x_h - x_{h'}}{x_h + x_{h'}}\right)_{1\leq h,h'\leq 2m}$$

due to Schur [S, Sec. IX]. The proof is completed using induction, starting from the base case of $j = 1$, which was obtained in (9). □

During the above discussion the set D has remained fixed, but in subsequent arguments we will need to modify it. For this, we use a simple observation.

Lemma 1.4. *If $(i,j) \notin D$ and $D \cup (i,j)$ is a valid set of pairs, then*

$$T(D,\lambda) = T(D \cup (i,j), \lambda) + T(D \cup (i,j), R_{ij}\lambda).$$

Proof. The assertion follows immediately from the identity

$$1 - R_{ij} = \frac{1-R_{ij}}{1+R_{ij}} + \frac{1-R_{ij}}{1+R_{ij}} R_{ij}. \qquad \square$$

2. From $\mathrm{IG}(n-k,2n)$ to $\mathrm{OG}(n-k,2n+1)$

2.1. For each $k \geq 0$, the odd orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(n-k, 2n+1)$ parametrizes the $(n-k)$-dimensional isotropic subspaces in $\mathbb{C}^{2n+1}$, equipped with a nondegenerate symmetric bilinear form. Our aim is to show that if λ is any k-strict partition, then σ_λ is given by the raising operator expression of Theorem 1. For technical reasons, we will use an isomorphism to transfer this relation to the cohomology ring of OG, and work with the latter space.

The Schubert varieties in OG are indexed by the same set of k-strict partitions $\mathcal{P}(k,n)$ as for $\mathrm{IG}(n-k,2n)$. Given a complete flag $F_\bullet$ of subspaces of $\mathbb{C}^{2n+1}$ such that $F_{n+i} = F_{n+1-i}^{\perp}$ for $1 \leq i \leq n+1$ and $\lambda \in \mathcal{P}(k,n)$, we define the codimension $|\lambda|$ Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where

$$\overline{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}. \tag{11}$$

Let $\tau_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$ be the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$.

For any $\lambda \in \mathcal{P}(k,n)$, let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k. Let $\mathcal{Q}_{\mathrm{IG}}$ and $\mathcal{Q}_{\mathrm{OG}}$ be the universal quotient vector bundles over $\mathrm{IG}(n-k,2n)$ and $\mathrm{OG}(n-k,2n+1)$, respectively. It is known (see e.g. [BS, §3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\mathrm{IG}})$ to $c_p(\mathcal{Q}_{\mathrm{OG}})$ for all p extends to a ring isomorphism $\phi : \mathrm{H}^*(\mathrm{IG},\mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG},\mathbb{Q})$. Moreover, we have $\phi(\sigma_\lambda) = 2^{\ell_k(\lambda)}\tau_\lambda$ for all $\lambda \in \mathcal{P}(k,n)$.

We let $c_p = c_p(\mathcal{Q}_{\mathrm{OG}})$. The Chern classes c_p are related to the special Schubert classes τ_p on OG by the equations

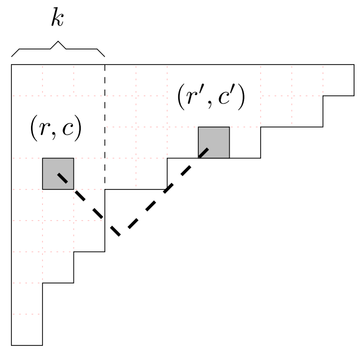
$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

Using the isomorphism ϕ, we can therefore describe the Giambelli formula for $\mathrm{OG}(n-k,2n+1)$ as follows. For any integer sequence α, set $m_\alpha = \prod_i c_{\alpha_i}$; then for every $\lambda \in \mathcal{P}(k,n)$, we have

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \, m_\lambda \tag{12}$$

in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$.

2.2. For λ any k-strict partition, we say that the box $[r,c]$ in row r and column c of λ is *k-related* to the box $[r',c']$ if $|c-k-1| + r = |c'-k-1| + r'$. If $c \leq k < c'$, then this is equivalent to $c + c' = 2k + 2 + r - r'$. For example, in the partition displayed below, the grey box $[r,c]$ is k-related to $[r',c']$. The notion of k-related boxes makes sense also for boxes outside the Young diagram of λ.



Given two Young diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row). For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

Equivalently, $\lambda \to \mu$ means that $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for each j, $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and conditions (1) and (2) are true. Let $\mathbb{A}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2), and define $\mathfrak{N}(\lambda,\mu)$ to be the number of connected components of $\mathbb{A}$. Here two boxes are connected if they share at least a vertex. In [BKT1, Theorem 2.1] we proved that the Pieri rule

$$c_p \cdot \tau_\lambda = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\mathfrak{N}(\lambda,\mu)}\, \tau_\mu \tag{13}$$

holds, for any $p \in [1, n+k]$.

2.3. A comparison of (4) with (11) suggests modifying the definition of valid sets of pairs from §1 to include elements along the diagonal $\{(i,i) \mid i > 0\}$. This convention will make the formalism of our proof of Theorem 1 cleaner, and is in fact crucial in the corresponding proof of Giambelli for even orthogonal Grassmannians.

Set $\Delta = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq j\}$ with the same partial order as in §1.2, and define the notion of a valid set of pairs exactly as before. Given a k-strict partition λ and an integer $t \geq \ell(\lambda)$, we obtain a valid set of pairs $\mathcal{C}_t(\lambda)$ by

$$\mathcal{C}_t(\lambda) = \{(i,j) \in \Delta \mid \lambda_i + \lambda_j > 2k + j - i \text{ and } j \leq t\}.$$

Furthermore, we let $\mathcal{C}(\lambda) = \mathcal{C}_{\ell(\lambda)}(\lambda)$. Notice that $\mathcal{C}(\lambda)$ includes the pairs (i,i) such that $\lambda_i > k$. It is easy to see that when $k > 0$, a set $D \subset \Delta$ is a valid set of pairs if and only if there exists a k-strict partition λ for which $\mathcal{C}(\lambda) = D$.

An *outer corner* of a valid set of pairs $D \subset \Delta$ is a pair $(i,j) \in \Delta \smallsetminus D$ such that $D \cup (i,j)$ is also a valid set of pairs. The *outside rim* ∂D of D is the set of pairs $(i,j) \in \Delta \smallsetminus D$ such that $i = 1$ or $(i-1, j-1) \in D$.

Lemma 2.1. *Let μ be a k-strict partition such that $\lambda \to \mu$. Then for any $t \geq \ell(\lambda)$, we have $\mathcal{C}_t(\lambda) \subset \mathcal{C}_{t+1}(\mu) \subset \mathcal{C}_t(\lambda) \cup \partial\, \mathcal{C}_t(\lambda)$.*

Proof. If $(i,j) \in \mathcal{C}_{t+1}(\mu)$, then $\lambda_{i-1} + \lambda_{j-1} \geq \mu_i + \mu_j > 2k + j - i$. This proves that $\mathcal{C}_{t+1}(\mu) \subset \mathcal{C}_t(\lambda) \cup \partial\mathcal{C}_t(\lambda)$. If there exists a pair $(i,j) \in \mathcal{C}_t(\lambda) \smallsetminus \mathcal{C}_{t+1}(\mu)$, then $\lambda_i + \lambda_j > 2k + j - i \geq \mu_i + \mu_j$, so we must have $\mu_i = \lambda_i$, $\mu_j = \lambda_j - 1$, and $\lambda_i + \lambda_j = 2k + 1 + j - i$. Condition (2) of §2.2 implies that some box $[h, c]$ of $\mu \smallsetminus \lambda$ is k-related to $[j, \lambda_j]$, and $[h, c-1]$ is also in $\mu \smallsetminus \lambda$ since this box is k-related to $[j-1, \lambda_j]$. The equality $\lambda_j + c = 2k + 2 + j - h$ implies that $(h, j) \in \mathcal{C}_{t+1}(\mu)$, and since $\mathcal{C}_{t+1}(\mu)$ is a valid set of pairs, we must have $h < i$. But we also obtain $\lambda_h < c - 1 = 2k + 1 + j - h - \lambda_j = \lambda_i + i - h$, contradicting the fact that λ is k-strict. This proves that $\mathcal{C}_t(\lambda) \subset \mathcal{C}_{t+1}(\mu)$. □

Definition 2.2. For any valid set of pairs $D \subset \Delta$ and any integer sequence λ we define the cohomology class $T(D, \lambda) \in \mathrm{H}^*(\mathrm{OG})$ by

$$T(D, \lambda) = 2^{-\#\{i \,|\, (i,i) \in D\}}\, \phi(T(D \cap \Delta^\circ, \lambda)),$$

where $T(D \cap \Delta^\circ, \lambda) \in \mathrm{H}^*(\mathrm{IG})$ is defined by (8).

To prove (12) and hence also establish Theorem 1, it suffices to show that if λ is a k-strict partition, the Pieri rule

$$c_p \cdot T(\mathcal{C}(\lambda), \lambda) = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\mathfrak{N}(\lambda, \mu)}\, T(\mathcal{C}(\mu), \mu) \tag{14}$$

holds in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$, for all p. To see this, write $\mu \succ \lambda$ if μ strictly dominates λ, i.e., $\mu \neq \lambda$ and $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for each $i \geq 1$. We deduce from (13) and (14) that

$$2^{\ell_k(\lambda)}\, \tau_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, \tau_\mu = c_{\lambda_1} \cdots c_{\lambda_\ell} = 2^{\ell_k(\lambda)}\, T(\mathcal{C}(\lambda), \lambda) + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, T(\mathcal{C}(\mu), \mu),$$

for some constants $a_{\lambda\mu} \in \mathbb{Z}$. By induction on λ, it follows that $\tau_\lambda = T(\mathcal{C}(\lambda), \lambda)$, which is a restatement of (12).

Observe that Lemmas 1.2, 1.3, and 1.4 carry over verbatim to our current setting where $D \subset \Delta$. These lemmas are the main properties of the cohomology classes $T(D, \lambda)$ that we use, and as such constitute the technical core of our proof of Theorem 1. But the non-trivial scheme that puts them to work together is an algorithm with a substitution rule; this is explained in the next section.

3. The Substitution Rule

3.1. Throughout the next two sections we fix $p > 0$, the k-strict partition λ of length ℓ, and choose n sufficiently large so that we can ignore it in the sequel. Set $\mathcal{C} = \mathcal{C}(\lambda)$. For any $d \geq 1$ define the raising operator R_d^λ by

$$R_d^\lambda = \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i < j \,:\, (i,j) \in \mathcal{C}} (1 + R_{ij})^{-1}.$$

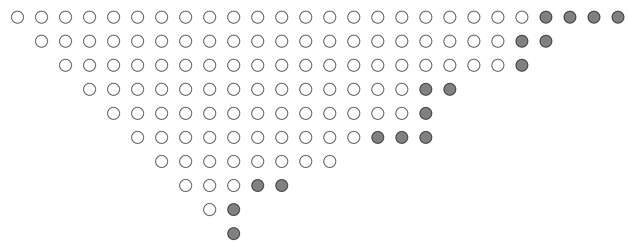


FIGURE 1. A valid set of pairs $\mathcal{C}$ (white dots) and a subset of $\partial\mathcal{C}$ (grey dots).

We compute that

$$c_p \cdot T(\mathcal{C},\lambda) = c_p \cdot 2^{-\ell_k(\lambda)} R^\lambda_\ell \, m_\lambda = 2^{-\ell_k(\lambda)} R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1} \, m_{\lambda,p}$$

$$= 2^{-\ell_k(\lambda)} R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R^2_{i,\ell+1} + \cdots) \, m_{\lambda,p}$$

and therefore

$$c_p \cdot T(\mathcal{C},\lambda) = \sum_{\nu\in\mathcal{N}} T(\mathcal{C},\nu), \tag{15}$$

where $\mathcal{N} = \mathcal{N}(\lambda,p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$. Our strategy for proving Theorem 1 is to show that the right hand side of equation (15) is equal to the right hand side of the Pieri rule (14).

3.2. The following objects will be used as book keeping tools in a delicate process of rewriting the right hand side of (15). Let $m \geq 1$ be minimal such that $\lambda_m \leq k$; we call m the *middle* row of λ. Notice that m is the smallest positive integer for which $(m,m) \notin \mathcal{C}$.

Definition 3.1. A *valid 4-tuple* of *level* h is a 4-tuple $\psi = (D,\mu,S,h)$, such that h is an integer with $0 \leq h \leq \ell + 1$, D is a valid set of pairs containing $\mathcal{C}$, all pairs (i,j) in D satisfy $i \leq m$ and $j \leq \ell + 1$, S is a subset of $D \smallsetminus \mathcal{C}$, and μ is an integer sequence of length at most $\ell + 1$. The evaluation of ψ is defined by $\mathrm{ev}(\psi) = T(D,\mu) \in \mathrm{H}^*(\mathrm{OG},\mathbb{Q})$.

All valid 4-tuples encountered in this paper will also satisfy that $D \subset \mathcal{C} \cup \partial\mathcal{C}$ (see Lemma 4.1), but for technical reasons we do not require this in the definition. We will represent the set Δ as the positions on or above the main diagonal of a matrix, and the various sets of pairs D as sets of entries in this matrix. In Figure 1 the white dots represent a set of pairs $\mathcal{C}$, the grey dots are a subset of the outside rim of $\mathcal{C}$, and we have $m = 10$. The union of the white and grey dots form the set D in a typical valid 4-tuple (D,μ,S,h).

In the following we set $\mu_0 = \infty$ whenever μ is an integer sequence.

Definition 3.2. For any $y \in \mathbb{Z}$ we let $r(y)$ denote the largest integer such that $r(y) \leq \ell + 1$ and $\lambda_{r(y)-1} > 2k + r(y) - y$.

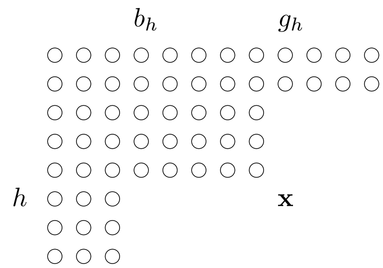


FIGURE 2. The set $\mathcal{C}$ near the pair $\mathbf{x} = (h, g_h) \in \partial\mathcal{C}$.

Since $\lambda_0 = \infty$ we have $r(y) \geq 1$. Notice that for $(i, j) \in \Delta$ and $j \leq \ell$ we have $(i, j) \in \mathcal{C} \Leftrightarrow \lambda_j > 2k + j - i - \lambda_i \Leftrightarrow j < r(i + \lambda_i + 1)$. This gives the relation

$$\mathcal{C} = \{(i, j) \in \Delta \mid j < r(i + \lambda_i + 1)\}\,. \tag{16}$$

The function $r(y)$ is also connected to the notion of k-relatedness of boxes. Assume that some box $[i, c]$ with $c > k$ is k-related to a box $[j, d]$ in the first k columns of λ. Then $\lambda_j \geq d = 2k + 2 + j - i - c$, which implies that $j < r(i + c)$. Furthermore, if $[j + 1, d + 1] \notin \lambda$ then $r(i + c) = j + 1$.

Definition 3.3. Let $h \in \mathbb{N}$ satisfy $1 \leq h \leq m$ and let μ be an integer sequence.

(a) We define $b_h = r(h + \lambda_h + 1)$ and $g_h = b_{h-1}$. By convention we set $g_1 = \ell + 1$.

(b) Set $R(\mu) = \{[i, c] \in \mu \smallsetminus \lambda \mid c > k \text{ and } \mu_{r(i+c)} \leq 2k + r(i + c) - i - c\}$.

(c) Assume that $h \geq 2$ and $\mu_h \geq \lambda_{h-1}$. If $[h, \lambda_{h-1}] \in R(\mu)$ then set $e_h(\mu) = \lambda_{h-1}$. Otherwise choose $e_h(\mu) > \max\{k, \lambda_h\}$ minimal such that $[h, c] \notin R(\mu)$ for $e_h(\mu) \leq c \leq \lambda_{h-1}$. Finally, set $f_h(\mu) = r(h + e_h(\mu))$.

Notice that for $h < m$ we have $b_h = \min\{j \geq m \mid (h, j) \notin \mathcal{C}\}$. The integers b_h and g_h are illustrated in Figure 2. In the definition of $R(\mu)$, suppose that some box $[i, c] \in \mu \smallsetminus \lambda$ with $c > k$ is k-related to a box $[j, d]$ in the first k columns of λ, such that $[j + 1, d + 1] \notin \lambda$. Then we have $r(i + c) = j + 1$ and $d = 2k + 2 + j - i - c$. It follows that $[i, c] \in R(\mu)$ if and only if $\mu_{j+1} < d$. In particular, if μ is a k-strict partition such that $\lambda \to \mu$, then the set $\mathbb{A}$ from §2.2 consists of the boxes of $\mu \smallsetminus \lambda$ in columns $k + 1$ and higher which are not in $R(\mu)$. Notice also that since $h + \lambda_h + 1 \leq h + e_h(\mu) \leq h + \lambda_{h-1}$ and $r(y)$ is a monotone increasing function of y, we always have

$$b_h \leq f_h(\mu) \leq g_h\,. \tag{17}$$

In particular we have $f_h(\mu) \geq m$ and $(h, f_h(\mu)) \notin \mathcal{C}$; when $h = m$ the inequality is true because $\lambda_{m-1} > 2k - e_h(\mu) = 2k + m - h - e_h(\mu)$. Furthermore, if $[h, \lambda_{h-1}] \in R(\mu)$, then $f_h(\mu) = g_h$.

Example 3.4. Let $k = 3$, $\lambda = (9, 7, 3, 2, 1, 1)$, and $\mu = (11, 12, 7, 2, 2)$. Then

$$\mathcal{C}(\lambda) = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}.$$

Figure 3 illustrates λ and μ, with the boxes in $\mu \smallsetminus \lambda$ shaded, and the boxes in $R(\mu)$ marked. Note that there is one box in $\lambda \smallsetminus \mu$. We have $e_2(\mu) = 8$, $f_2(\mu) = 5$, $g_2 = 5$, $e_3(\mu) = 6$, $f_3(\mu) = 4$, and $g_3 = 5$.

Lemma 3.5. *If $2 \leq h \leq m$ then we have $\lambda_{h-1} - \lambda_h \geq g_h - b_h + 1$.*

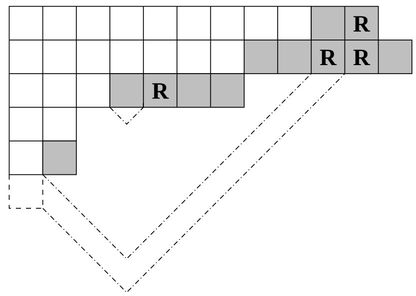


FIGURE 3. The shapes λ and μ, with $\mu \smallsetminus \lambda$ shaded.

Proof. The inequality is clear if $b_h = g_h$, as λ is k-strict and $h \leq m$. If $b_h < g_h$, then since $b_h = r(h+\lambda_h+1)$ and $g_h = r(h+\lambda_{h-1})$ we have $\lambda_{b_h} \leq 2k + b_h - h - \lambda_h$ and $\lambda_{g_h-1} > 2k+g_h-h-\lambda_{h-1}$, which implies that $\lambda_{h-1}-\lambda_h > g_h-\lambda_{g_h-1}-b_h+\lambda_{b_h} \geq g_h - b_h$. □

Lemma 3.6. *Let μ be an integer sequence and $2 \leq h \leq m$. If $\mu_h \geq \lambda_{h-1}$ and $\lambda_{h-1} + \mu_{g_h} \leq 2k + g_h - h$, then $f_h(\mu) = g_h$.*

Proof. We have $\lambda_{h-1} > k$ and $[h, \lambda_{h-1}] \in \mu \smallsetminus \lambda$. Since $g_h = r(h + \lambda_{h-1})$, the inequality $\mu_{g_h} \leq 2k+g_h-h-\lambda_{h-1}$ shows that $[h, \lambda_{h-1}] \in R(\mu)$. This implies that $e_h(\mu) = \lambda_{h-1}$ and $f_h(\mu) = r(h + \lambda_{h-1}) = g_h$, as required. □

Lemma 3.7. *Let $2 \leq h \leq m$ and let μ and μ' be integer sequences such that $\mu_h \geq \lambda_{h-1}$, $\mu'_h \geq \lambda_{h-1}$, and $\mu_j = \mu'_j$ for $\max(m, h+1) \leq j \leq g_h$. Then $[h, c] \in R(\mu)$ if and only if $[h, c] \in R(\mu')$ for all $c \leq \lambda_{h-1}$. In particular, we have $e_h(\mu) = e_h(\mu')$ and $f_h(\mu) = f_h(\mu')$.*

Proof. Let $[h, c] \in \mu \smallsetminus \lambda$ satisfy $k < c \leq \lambda_{h-1}$, and set $j = r(h + c)$. Since k-strictness of λ implies that $k + m + 1 \leq h + c \leq h + \lambda_{h-1}$, we obtain $m \leq j \leq g_h$ and hence $\mu_j = \mu'_j$ provided that $j > h$. It follows that $[h, c] \in R(\mu)$ if and only if $[h, c] \in R(\mu')$, as required. □

If we are given a fixed valid 4-tuple (D, μ, S, h) with $1 \leq h \leq m$, we will use the shorthand notation $b = b_h$, $g = g_h$, $R = R(\mu)$, $e = e_h(\mu)$, and $f = f_h(\mu)$; the values e and f will be used only when $\mu_h \geq \lambda_{h-1}$. The precise value of f will play a crucial role in our proof that the Pieri terms in (14) appear in (15) with the correct multiplicities. For example, it is part of the following definition of a condition X, that will be used to identify undesired valid 4-tuples.

Definition 3.8. Let $(i, j) \in \Delta$ be arbitrary. We define two conditions $\mathrm{W}(i, j)$ and X on a valid 4-tuple (D, μ, S, h) as follows.

$$\mathrm{W}(i,j) \ : \ \mu_i + \mu_j > 2k + j - i \,.$$

Condition X is true if and only if $(h, h) \in D$ and

$$\mu_h \geq \mu_{h-1} \text{ or } \mu_h > \lambda_{h-1} \text{ or } (\mu_h = \lambda_{h-1} \text{ and } (h, f) \notin S)\,.$$

3.3. The following *substitution rule* will be applied iteratively to rewrite the right hand side of (15). It may be applied to any valid 4-tuple of positive level and will result in either a REPLACE statement, indicating that the 4-tuple should be replaced by one or two new 4-tuples, or a STOP statement, indicating that the 4-tuple should not be replaced.

Substitution Rule

Let (D, μ, S, h) be a valid 4-tuple of level $h \geq 1$. Assume first that $(h,h) \notin D$. If

(i) there is an outer corner (i,h) of D with $i \leq m$ such that $\mathrm{W}(i,h)$ holds

then REPLACE (D, μ, S, h) with

$$(D \cup (i,h), \mu, S, h) \quad \text{and} \quad (D \cup (i,h), R_{ih}\mu, S \cup (i,h), h).$$

Otherwise, if

(ii) D has no outer corner in column h and $\mu_h > \lambda_{h-1}$,

then STOP.

Assume now that $(h,h) \in D$. If

(iii) there is an outer corner (h,j) of D with $j \leq \ell + 1$ such that $\mathrm{W}(h,j)$ holds,

then REPLACE (D, μ, S, h) with

$$\begin{cases} (D \cup (h,j), \mu, S, h) \quad \text{and} \quad (D \cup (h,j), R_{hj}\mu, S \cup (h,j), h) & \text{if } \mu_j \leq \mu_{j-1}, \\ (D \cup (h,j), R_{hj}\mu, S \cup (h,j), h) & \text{if } \mu_j > \mu_{j-1}. \end{cases}$$

Otherwise, if

(iv) $\mathrm{W}(h,g)$ or X holds, and D has an outer corner (i,g) with $i \leq h$,

then REPLACE (D, μ, S, h) with

$$(D \cup (i,g), \mu, S, h) \quad \text{and} \quad (D \cup (i,g), R_{ig}\mu, S \cup (i,g), h).$$

Otherwise, if

(v) X holds,

then STOP.

If none of the above conditions hold, REPLACE (D, μ, S, h) with $(D, \mu, S, h-1)$.

Definition 3.9. Let **(x)** be one of the conditions **(i)**–**(v)** of the Substitution Rule. We say that a valid 4-tuple ψ *meets* condition **(x)** if ψ reaches condition **(x)** in the Substitution Rule, and condition **(x)** is satisfied. Whenever the Substitution Rule REPLACES ψ by one or two 4-tuples ψ_i, we refer to ψ as the *parent* term and the ψ_i are its *children*.

3.4. Initially, we define the set $\Psi = \{(\mathcal{C}, \nu, \emptyset, \ell{+}1) \mid \nu \in \mathcal{N}(\lambda, p)\}$; thus $\sum_{\psi\in\Psi} \mathrm{ev}(\psi)$ agrees with the right hand side of (15). We then apply an *algorithm* which will change this set by replacing some 4-tuples with one or two new valid 4-tuples. The algorithm applies the Substitution Rule to each element (D, μ, S, h) of level $h \geq 1$. If the substitution rule results in a REPLACE statement, then the set is changed accordingly; otherwise the substitution rule results in a STOP statement, in which case the 4-tuple (D, μ, S, h) is left untouched. These substitutions are iterated until no further elements can be REPLACED, i.e., until the substitution rule results in a STOP statement when applied to any remaining 4-tuple with $h \geq 1$.

Since the set of pairs D is not allowed to grow beyond column $\ell{+}1$, the algorithm will terminate after a finite number of steps. Notice that if $\psi = (D, \mu, S, h)$ is any 4-tuple produced by the algorithm, then the initial 4-tuple $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell + 1)$ that gave rise to ψ can be recovered by the equation $\nu = \prod_{(i,j)\in S} L_{ij}\mu$. Here L_{ij} denotes the lowering operator which is the inverse of R_{ij}. Furthermore, the sequence of 4-tuples leading from ψ_0 to ψ is uniquely determined by ψ because all choices made along the way are recorded in the set S. In particular, no 4-tuple can be produced multiple times.

Suppose that the 4-tuple $\psi = (D, \mu, S, h)$ occurs in the algorithm. If ψ is REPLACED by two 4-tuples ψ_1 and ψ_2, we deduce from Lemma 1.4 that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi_1){+}\mathrm{ev}(\psi_2)$. Moreover, if ψ meets **(iii)** and is REPLACED by the single 4-tuple $\psi' = (D \cup (h, j), R_{hj}\mu, S \cup (h, j), h)$, then Lemmas 1.2 and 1.4 imply that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi')$. Indeed, it follows from Corollary 4.10 below that $\mu_{j-1} = \mu_j{-}1$ and $D{\cup}(h, j)$ has no outer corner in column j, so Lemma 1.2 shows that $\mathrm{ev}(D{\cup}(h, j), \mu, S, h) = 0$.

When the algorithm terminates, let Ψ_0 (respectively Ψ_1) denote the collection of all 4-tuples (D, μ, S, h) in the final set such that $h = 0$ (respectively $h > 0$). We say that a 4-tuple ψ *survives the algorithm* if at least one of its successors lies in Ψ_0. The above analysis implies that

$$\sum_{\nu\in N} T(\mathcal{C}, \nu) = \sum_{\psi\in\Psi_0} \mathrm{ev}(\psi) + \sum_{\psi\in\Psi_1} \mathrm{ev}(\psi).$$

In the next section, we will prove the following two claims.

Claim 1. For each 4-tuple $\psi = (D, \mu, S, 0)$ in Ψ_0 with $\mu_{\ell+1} \geq 0$, μ is a k-strict partition with $\lambda \to \mu$ and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$. Furthermore, for each such partition μ, there are exactly $2^{\mathfrak{N}(\lambda,\mu)}$ such 4-tuples ψ, in accordance with the Pieri rule.

Claim 2. There exists an involution $\iota : \Psi_1 \to \Psi_1$ of the form $\iota(D, \mu, S, h) = (D, \mu', S', h)$ such that $\mathrm{ev}(\psi) + \mathrm{ev}(\iota(\psi)) = 0$, for every $\psi \in \Psi_1$.

We remark that the 4-tuples $\psi \in \Psi_0$ with $\mu_{\ell+1} < 0$ evaluate to zero trivially, by Definition 1.1. The two claims therefore suffice to prove the Pieri rule (14).

For each initial 4-tuple $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell{+}1)$ of the sum (15), the algorithm produces a tree of 4-tuples with root node given by ψ_0. If the Substitution Rule REPLACES a 4-tuple ψ by one or two other 4-tuples ψ_i, we have a branch in the tree from ψ to the ψ_i. The leaves of the tree are exactly the 4-tuples with $h = 0$ or where the Substitution Rule STOPS. The fate of all the terms of the sum (15) is encoded by the collection of all the trees with root nodes $(\mathcal{C}, \nu, \emptyset, \ell + 1)$ for $\nu \in \mathcal{N}(\lambda, p)$. This collection will be called the *substitution forest*; the sum of the cohomology classes represented by the roots of the substitution forest is equal to the sum of classes given by the leaves.

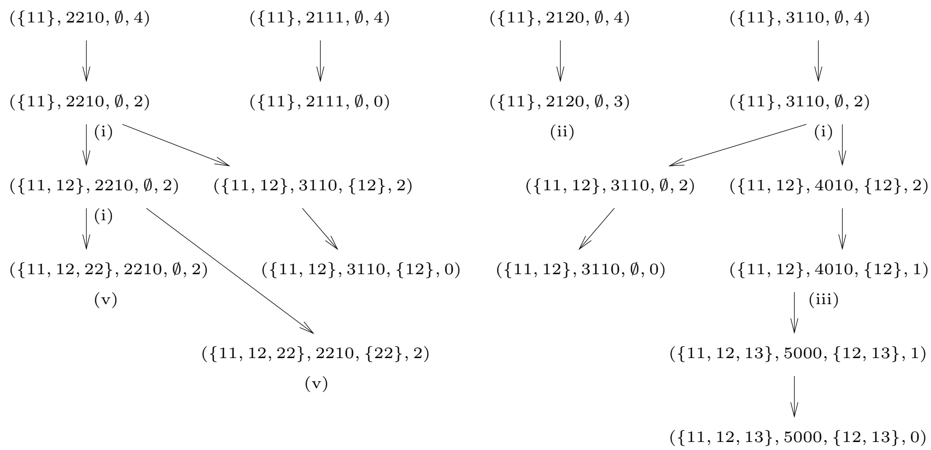


FIGURE 4. The substitution forest for $\lambda = 211$ and $k = p = 1$.

Example 3.10. We discuss an example of the substitution forest in detail. Consider the Grassmannian $\mathrm{OG}(n-1, 2n+1)$ for $n \geq 5$, and the Pieri product

$$c_1 \cdot \tau_{2,1,1} = \tau_{2,1,1,1} + 2\,\tau_{3,1,1} + \tau_5.$$

For simplicity, we will omit the commas in our notation for compositions and pairs. Thus $\lambda = 211$, $k = p = 1$, and we have $\mathcal{C}(\lambda) = \{11\}$ and $\mathcal{N}(\lambda, p) = \{2111, 2120, 2210, 3110\}$. The substitution forest is pictured in Figure 4, except that we have omitted those nodes (D, μ, S, h) which have $(D, \mu, S, h+1)$ as parent and $(D, \mu, S, h-1)$ as child.

Observe that the root $(\{11\}, 2120, \emptyset, 4)$ is the only initial 4-tuple that does not survive the algorithm. We have $\Psi_0 = \{\ (\{11\}, 2111, \emptyset, 0), (\{11, 12\}, 3110, \{12\}, 0), (\{11, 12\}, 3110, \emptyset, 0), (\{11, 12, 13\}, 5000, \{12, 13\}, 0)\ \}$, which corresponds exactly to the terms in the Pieri product $c_1 \cdot \tau_{211}$. Furthermore, each 4-tuple in the set $\Psi_1 = \{\ (\{11, 12, 22\}, 2210, \emptyset, 2), (\{11, 12, 22\}, 2210, \{22\}, 2), (\{11\}, 2120, \emptyset, 3)\ \}$ evaluates to zero in the cohomology ring of OG.

4. Proof of Theorem 1

4.1. Recall the fixed choices of p, λ, ℓ, $\mathcal{C}$, and m from §3.1. In §4.1 through §4.3 we furthermore let $\psi = (D, \mu, S, h)$ denote a 4-tuple which occurs at some step in the algorithm, i.e., a node of the substitution forest. The symbols D, μ, S, h will refer to components of the 4-tuple ψ. We will occasionally work with more than one valid 4-tuple. If (D', μ', S', h') is an additional 4-tuple, then the sets and values that Definition 3.3 associates to this 4-tuple will be called R', e', f', and g'.

The algorithm has two phases. A 4-tuple ψ is in Phase 1 if $(h, h) \notin D$, and in Phase 2 if $(h, h) \in D$. The level h is always used to index an entry of the integer sequence μ in ψ; it begins at $h = \ell + 1$ and decreases as the 4-tuple proceeds through the algorithm. In Phase 1 we have $h \geq m$, while $h \leq m$ in Phase 2. Throughout the algorithm we have $i \leq m \leq j$ for each $(i, j) \in S$, so μ is obtained from the initial composition ν by removing boxes from rows weakly below the middle row of λ and adding them to rows weakly above the middle row.

The set D is initially equal to $\mathcal{C}$ and grows when REPLACE statements are encountered. Lemma 4.1 below shows that all pairs added to D come from the outer rim $\partial\mathcal{C}$. In Phase 1, pairs are added by rule **(i)** to column h, so as the level h decreases from $\ell+1$ to m, these pairs are added along vertical columns of $\partial\mathcal{C}$, proceeding from top (row 1) to bottom (row m) and right to left. In Phase 2, the set D mainly grows when rule **(iii)** adds pairs to row h, in which case the pairs are added in horizontal rows of $\partial\mathcal{C}$, from left to right and bottom to top. In some cases rule **(iv)** will add extra pairs (i,g) to D, where $i \leq h$. Lemma 4.6 implies that if ψ meets **(iv)**, then it will not survive the algorithm, and only pairs from column g of $\partial\mathcal{C}$ can be added to its successors. In particular, all 4-tuples in Ψ_0 are produced from the the initial 4-tuples by applications of rules **(i)** and **(iii)**.

Our proof of Theorem 1 occupies the remainder of this section. In §4.2 we prove some properties satisfied by 4-tuples that occur in the algorithm. Additional properties for 4-tuples in Ψ_0 are proved in §4.3. The proof of Claim 1 is then given in §4.4, while Claim 2 is justified in §4.5.

4.2. We prove some lemmas that reveal what can happen to the 4-tuple $\psi = (D,\mu,S,h)$ during the algorithm.

Lemma 4.1. *We have* $D \subset \mathcal{C} \cup \partial\mathcal{C}$.

Proof. It is enough to show that if the substitution rule adds the pair (i,j) to D, then $(i,j) \in \partial\mathcal{C}$. Notice first that $i \leq h \leq j$. If $(i,j) \notin \partial\mathcal{C}$, then $i > 1$ and $(i-1,j-1) \notin \mathcal{C}$. Let $\psi' = (D',\mu',S',h')$ be the most recent predecessor of ψ such that $(i-1,j) \notin D'$. Then ψ' meets rule **(i)**, **(iii)**, or **(iv)**, which adds the pair $(i-1,j)$ to D'. Since the pair $(i-1,j-1) \in D' \smallsetminus \mathcal{C}$ was added to a predecessor of ψ' of level smaller than j, it follows that $i \leq h \leq h' \leq j-1$, so ψ' does not meet **(i)** or **(iii)**. But ψ' also does not meet **(iv)** because $g' \leq j-1$, a contradiction. $\square$

Lemma 4.2. *If* $j > h$ *and* $(j,j) \notin D$*, then* $\mu_j \leq \lambda_{j-1}$.

Proof. Assume that $\mu_j > \lambda_{j-1}$ and let $\psi' = (D',\mu',S',j)$ be the most recent predecessor of ψ of level j. Then $\mu'_j \geq \mu_j$, and since ψ' does not meet **(ii)**, it follows that D' has an outer corner (i,j) in column j. If $i < j$, then since $(i,j-1) \in \mathcal{C}$ we obtain $\mu'_j + \mu'_i > \lambda_{j-1} + \lambda_i > 2k + (j-1) - i$, and otherwise we have $i = j = m$ and $\mu'_j > \lambda_{m-1} > k$. In both cases ψ' satisfies $\mathrm{W}(i,j)$. But then ψ' meets **(i)** and is not the most recent predecessor of ψ of level j, a contradiction. $\square$

Lemma 4.3. *If* $2 \leq h \leq m$*,* $\mu_h \geq \lambda_{h-1}$*, and* $f < g$*, then* $\mu_g = \lambda_{g-1}$ *and* $(h,g) \notin S$.

Proof. By assumption we have $h \leq m \leq f < g$, and Lemma 4.2 implies that $\mu_g \leq \lambda_{g-1} \leq \lambda_m \leq k$. Set $x = 2k + g - h - \mu_g$. Since $(h,f) \notin \mathcal{C}$ we get $(h,g-1) \notin \mathcal{C}$ which implies $\lambda_h < x$. Since $[h,\lambda_{h-1}] \notin R$ we also obtain $x < \lambda_{h-1}$. Assume that $\mu_g < \lambda_{g-1}$. Then we get $\lambda_{g-1} > 2k + g - h - x$, which implies that $g \leq r(h+x) \leq r(h+\lambda_{h-1}) = g$. The definition of x now shows that $[h,x] \in R$, from which we deduce that $x < e$ and $g = r(h+x) \leq r(h+e) = f$, a contradiction. Finally assume that $(h,g) \in S$. Since $(h,g-1) \notin \mathcal{C}$ we deduce that the parent of ψ is $\psi' = (D \smallsetminus (h,g), \mu', S \smallsetminus (h,g), h)$, where $\mu' = L_{hg}\mu$. But Lemma 4.2 then implies that $\mu_g < \mu'_g \leq \lambda_{g-1}$, a contradiction. $\square$

We next make some observations concerning condition X and rules **(iv)** and **(v)**.

Lemma 4.4. *If condition* X *holds for* ψ*, then* X *also holds for the children of* ψ*. In particular,* ψ *does not survive the algorithm, and all its successors have level* h*.*

Proof. Assume that $\psi = (D, \mu, S, h)$ satisfies condition X and let $\psi' = (D', \mu', S', h)$ be a child of ψ. If $S' = S$ then $\mu' = \mu$ and X also holds for ψ', so we may assume that $S' \smallsetminus S = \{(i, j)\}$ and $\mu' = R_{ij}\mu$, where $i \leq h$. We can also assume that $\lambda_{h-1} = \mu_h = \mu'_h < \mu'_{h-1}$. Since $(h, h) \in D$ and $(i, j) \notin D$, we get $i < h < j$. In particular, ψ meets **(iv)** and $j = g$. Let $\overline{\psi} = (\overline{D}, \overline{\mu}, \overline{S}, g)$ be the most recent predecessor of ψ of level g, and let (a, g) be the outer corner of $\overline{D}$ in column g. Then $a \leq h - 1$, and W(a, g) fails for $\overline{\psi}$ since it does not meet **(i)**. Using this and k-strictness of λ, we obtain $\lambda_{h-1} + \mu'_g < \lambda_{h-1} + \mu_g \leq (\lambda_a + a - h + 1) + \overline{\mu}_g \leq \overline{\mu}_a + \overline{\mu}_g + 1 + a - h \leq 2k + 1 + g - h$. Lemma 3.6 now implies that $f' = g$. We conclude that ψ' satisfies condition X as $\mu'_h = \lambda_{h-1}$ and $(h, f') = (h, g) \notin S'$. □

Lemma 4.5. *Assume that* $(h, h) \in D$ *and* ψ *satisfies* X *or* W(h, g)*.*

(a) *If* $h < g$ *and* $(h, g - 1) \notin D$*, then* ψ *meets* **(iii)***.*

(b) *If* ψ *meets* **(v)***, then* $(h, g) \in D$*.*

Proof. Assume that $h < g$ and $(h, g - 1) \notin D$, and let (h, d) be the outer corner of D in row h. Then $d \leq g - 1$ and $(h - 1, d) \in \mathcal{C}$. Using Lemma 4.2 and the fact that $\mathcal{C}$ and D have equally many pairs in column d, we obtain $\mu_g \leq \lambda_{g-1} \leq \lambda_d \leq \mu_d$. If ψ satisfies condition X, then $\mu_h \geq \lambda_{h-1}$ and it follows that $\mu_h + \mu_d \geq \lambda_{h-1} + \lambda_d > 2k + d - h + 1$. If ψ satisfies W(h, g), then $\mu_h + \mu_d \geq \mu_h + \mu_g > 2k + g - h \geq 2k + d - h$. This shows that ψ satisfies W(h, d) and (a) follows. If ψ meets **(v)** and $(h, g) \notin D$, then part (a) implies that D has an outer corner in column g, so ψ meets **(iii)** or **(iv)**. This contradiction proves (b). □

The following result implies that no term meeting **(iv)** survives the algorithm, and also that applications of **(iv)** happen in an uninterrupted sequence.

Lemma 4.6. *Assume that* ψ *meets* **(iv)** *and let* $\psi' = (D', \mu', S', h)$ *be any successor of* ψ *of level* h*.*

(a) *We have* $(h, g - 1) \in D$*.*

(b) *If* $(h, g) \notin D'$ *then* ψ' *meets* **(iii)** *or* **(iv)***.*

(c) *If* $(h, g) \in D'$ *and* ψ' *does not meet* **(v)***, then* $S' = S$*, the child* $(D', \mu, S, h - 1)$ *of* ψ' *meets* **(v)***, and* $g_{h-1} = g$*.*

Proof. Part (a) follows from Lemma 4.5. If ψ satisfies condition X then assertions (b) and (c) follow from Lemma 4.4, so we may assume that X fails and W(h, g) holds for ψ. Without loss of generality we can also replace ψ with the most recent predecessor whose parent did not meet **(iv)**. Let (i, g) be the outer corner of D in column g. Since ψ does not meet **(iii)** we have $i < h$. Furthermore W(i, g) fails for ψ, since otherwise the most recent predecessor of ψ of level g meets **(i)**. We obtain $\mu_h + \mu_g \leq \lambda_{h-1} + \mu_g \leq \lambda_i - h + 1 + i + \mu_g \leq \mu_i + \mu_g + 1 + i - h \leq 2k + 1 + g - h$. Since ψ satisfies W(h, g) we deduce that $\lambda_i - h + 1 + i = \lambda_{h-1} = \mu_h = 2k + 1 + g - h - \mu_g$.

If $S' \supsetneq S$, then $\mu'_g < \mu_g$, and Lemma 3.6 implies that $f' = g$. In addition, we have either $(h, g) \notin S'$ or $\mu'_h > \mu_h = \lambda_{h-1}$. Since both possibilities imply that ψ' satisfies condition X, it follows that assertions (b) and (c) are true for ψ'.

Otherwise, we have $S' = S$ and $\mu' = \mu$. Then ψ' satisfies W(h, g) and (b) is true. Assume that $(h, g) \in D'$ and ψ' does not meet **(v)**. Then X fails for

ψ', so we must have $\lambda_{h-2} - 1 = \lambda_{h-1} = \mu_h < \mu_{h-1}$. Since $(h-1, g) \notin S$, we deduce that $(D', \mu, S, h-1)$ satisfies condition X. Furthermore, since $(h-1, g) \notin \mathcal{C}$ and $\lambda_{h-2} = \lambda_{h-1} + 1$, we obtain $(h-2, g) \notin \mathcal{C}$, so $g_{h-1} = g$. We conclude that $(D', \mu, S, h-1)$ meets **(v)**, which completes the proof of (c). □

Definition 4.7. Let $\partial^1 \mathcal{C} = \{(i,j) \in \partial\mathcal{C} \mid (i, j-1) \in \mathcal{C} \text{ or } i = j = m\}$.

Corollary 4.8. *Let $(i,j) \in D \smallsetminus \mathcal{C}$, and if $2 \leq h \leq m$ then assume that $j \neq g$. If $(i,j) \in \partial^1\mathcal{C}$ then this pair was added to D by rule* **(i)**, *and otherwise it was added by rule* **(iii)**.

Proof. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $(i,j) \notin D'$. Then (i,j) is an outer corner of D'. If $\psi'' = (D'', \mu'', S'', h'')$ is any predecessor of ψ' meeting **(iv)**, then Lemma 4.6 implies that that $h'' = h'$, $g_{h'} = g$, and $(h', g-1) \in D'$, so we must have $j = g$, a contradiction. It follows that no predecessor of ψ' meets **(iv)**. If ψ' meets **(i)** then $h' = j$, and since $D' \smallsetminus \mathcal{C}$ contains no pairs in column $j-1$ we deduce that $(i,j) \in \partial^1\mathcal{C}$. Finally assume that ψ' meets **(iii)** and $(i,j) \in \partial^1\mathcal{C}$. Let $\widetilde{\psi} = (\widetilde{D}, \widetilde{S}, \widetilde{\mu}, j)$ be the most recent predecessor of ψ' of level j. Then $(i,j) = (h', j)$ is an outer corner of $\widetilde{D}$, since otherwise $(h'-1, j) \in D' \smallsetminus \widetilde{D}$ was added to a 4-tuple on the path from $\widetilde{\psi}$ to ψ', which is impossible. But then the inequality $\widetilde{\mu}_i + \widetilde{\mu}_j = \mu'_i + \mu'_j > 2k + j - i$ implies that $\widetilde{\psi}$ meets **(i)**. This contradiction finishes the proof. □

We now prove some results that will be used later to show that surviving 4-tuples $\psi = (D, \mu, S, 0) \in \Psi_0$ satisfy $\lambda \to \mu$.

Lemma 4.9. *Let $\psi = (D, \mu, S, h)$ and let $j \leq \ell$ be a positive integer.*

(a) *If $h \geq 1$ and $(h,j) \notin \mathcal{C}$ and $(h+1, j) \in D$, then we have $\mu_j \geq \lambda_j$.*

(b) *If $h \leq 1$ or $(h-1, j) \in \mathcal{C}$, then we have $\mu_j \geq \lambda_j - 1$. Moreover, if $\mu_j = \lambda_j - 1$, then $D \smallsetminus \mathcal{C}$ contains exactly one pair in column j, and this pair is also in S.*

Proof. Suppose that $\mu_j < \lambda_j$ and choose $i > h$ maximal such that $(i,j) \in D$. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ with $(i,j) \notin D'$. Then ψ' meets rule **(i)**, **(iii)**, or **(iv)**, which adds the pair (i,j) to D'. Let $\overline{\psi} = (D' \cup (i,j), \overline{\mu}, \overline{S}, h')$ be the child of ψ' that is a predecessor of ψ. Notice that $\mu'_t \leq \overline{\mu}_t \leq \lambda_{t-1}$ for all integers t such that $h < t \leq h'$ and $(t,t) \in D' \cup (i,j)$, since otherwise condition X holds for every successor of $\overline{\psi}$ of level t. We also have $\mu'_j \leq \lambda_j$, and if $\mu'_j = \lambda_j$ then $(i,j) \in \overline{S}$, $i < j$, $\overline{\mu}_i > \mu'_i$, and $\overline{\mu}_j < \mu'_j$.

If ψ' meets **(i)**, then $h' = j$. Since $(i-1, j) \notin \mathcal{C}$ we have $\mu'_i + \mu'_j \leq \lambda_{i-1} + \lambda_j \leq 2k + j - i + 1$. As $\mathrm{W}(i,j)$ holds for ψ', it follows that $\mu'_i = \lambda_{i-1}$ and $\mu'_j = \lambda_j$. But this implies that $\overline{\mu}_i > \lambda_{i-1}$, a contradiction.

Therefore ψ' meets **(iii)** with $h' = i$, or it meets **(iv)** with $h' \geq i$. In either case we have $g' = j$, and since ψ' does not satisfy condition X, it must satisfy $\mathrm{W}(h', j)$. Since $(h'-1, j) \notin \mathcal{C}$ and thus $\mu'_{h'} + \mu'_j \leq \lambda_{h'-1} + \lambda_j \leq 2k + j - h' + 1$, it follows that $\mu'_{h'} = \lambda_{h'-1}$ and $\overline{\mu}_j < \mu'_j = \lambda_j$. We obtain $\lambda_{h'-1} + \overline{\mu}_j \leq 2k + j - h'$, so Lemma 3.6 shows that $\overline{f} = j$. Since $\overline{\mu}_i > \mu'_i$, we must also have $i < h'$, so $(h', \overline{f}) \notin \overline{S}$ and $\overline{\psi}$ satisfies condition X. This contradiction completes the proof of part (a).

If $\mu_j \leq \lambda_j - 2$, then $D \smallsetminus \mathcal{C}$ contains at least two pairs in column j, say $(a+1, j)$ and (a, j), and the assumptions in (b) imply that $a \geq h$. Let $\psi' = (D', \mu', S', a)$ be the most recent predecessor of ψ of level a. Part (a) applied to ψ' implies that $\mu'_j \geq \lambda_j$, a contradiction since $\mu'_j = \mu_j$. □

Corollary 4.10. *Assume that ψ meets* **(iii)** *and let (h,j) be the outer corner of D in row h. If $\mu_j > \mu_{j-1}$, then $\mu_{j-1} = \mu_j - 1$ and $D \cup (h,j)$ has no outer corner in column j.*

Proof. Lemma 4.2 implies that $\mu_{j-1} < \mu_j \leq \lambda_{j-1}$. Since $(h-1, j-1) \in \mathcal{C}$, it follows from Lemma 4.9(b) that $\mu_j = \lambda_{j-1} = \mu_{j-1}+1$ and $D \smallsetminus \mathcal{C}$ contains a unique pair $(i, j-1)$ in column $j-1$. Since $i < j-1$, it is enough to show that $i = h$. Lemma 4.6 implies that $(i, j-1)$ was added by **(i)** or **(iii)**, so ψ satisfies $\mathrm{W}(i, j-1)$, and we obtain $\mu_i + \mu_j = \mu_i + \mu_{j-1} + 1 > 2k + j - i$. Assume that $i > h$ and let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i. Then $(i, j-1) \in D'$ and $g' = j$. Since $\mu'_i = \mu_i$ and $\mu'_j \geq \mu_j$, $\mathrm{W}(i,j)$ holds for ψ'. But then ψ' meets **(iv)** and is not the most recent predecessor, a contradiction. □

Lemma 4.11. *Assume that $j > m$. If $h = 0$ or $(h,j) \in D$, then $\mu_j \leq \mu_{j-1}$.*

Proof. Assume that $\mu_j > \mu_{j-1}$. Then Lemmas 4.2 and 4.9(b) imply that $\mu_j = \lambda_{j-1} = \mu_{j-1} + 1$, and $D \smallsetminus \mathcal{C}$ contains a unique pair $(i, j-1)$ in column $j-1$, with $i \geq h$. Let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i for which $(i,j) \notin D'$. The assumptions of the lemma then imply that $\psi' \neq \psi$. Lemma 4.6 shows that $(i, j-1)$ was added to D by **(i)** or **(iii)**, so ψ' satisfies $\mathrm{W}(i, j-1)$. Since $\mu'_j \geq \mu_j > \mu_{j-1} = \mu'_{j-1}$, ψ' also satisfies $\mathrm{W}(i,j)$, so ψ' meets **(iii)** or **(iv)**. The choice of ψ' implies that (i,j) must be the outer corner added to D', so in fact ψ' meets **(iii)**. Now the statement of rule **(iii)** implies that $(i,j) \in S$, so $\mu'_j > \mu_j > \mu'_{j-1}$. This contradicts Corollary 4.10. □

Lemma 4.12. *Assume $h < m$. Then $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $h < j \leq m$.*

Proof. If the statement is false, then choose $i > h$ minimal such that $\mu_i > \lambda_{i-1}$ or $\mu_i \geq \mu_{i-1}$, and let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i. Since $\mu'_i > \lambda_{i-1}$ or $\mu'_i \geq \mu'_{i-1}$ and condition X fails for ψ', we have $(i,i) \notin D'$, so $i = m$ and $\mu'_m \geq \lambda_{m-1} > k$. But then $\mu'_t + \mu'_m \geq \lambda_t + \mu'_m > (k + m - t) + k$ for $1 \leq t < m$, so ψ' meets **(i)**, a contradiction. □

Lemma 4.13. *Assume that $(h,h) \in D$, $\mu_h = \lambda_{h-1}$, and $[h, \lambda_{h-1}] \in R$. Then ψ satisfies condition* X *and does not survive the algorithm.*

Proof. Since $[h, \lambda_{h-1}] \in R$ and $r(h + \lambda_{h-1}) = g$, we have $\mu_g \leq 2k + g - h - \lambda_{h-1}$, hence $\mathrm{W}(h,g)$ fails for ψ. Lemma 4.12 shows that $\mu_h > \mu_{h+1} > \cdots > \mu_m$, so $\mathrm{W}(d,g)$ fails for $h \leq d \leq m$. If $(h,g) \notin D$ then condition X holds because $(h,f) = (h,g) \notin S$. Otherwise choose $i \geq h$ maximal such that $(i,g) \in D$, and let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $(i,g) \notin D'$. Then ψ' meets **(iv)** and ψ is a successor of the child $\psi'' = (D, \mu, S, h')$ of ψ'. If $h' > h$, then Lemma 4.4 implies that X fails and $\mathrm{W}(h', g)$ holds for ψ'. Since $\mathrm{W}(h', g)$ fails for ψ'', it follows that $i < h'$ and $\mu_{h'} + \mu_g = 2k + g - h'$. We also have $\lambda_{h'-1} + \mu_g \leq \lambda_{h-1} + \mu_g + h - h' \leq 2k + g - h'$, so $\mu_{h'} \geq \lambda_{h'-1}$, and using Lemma 3.6 we get $(h', f'') = (h', g) \notin S$. But then ψ'' satisfies condition X, a contradiction. We therefore have $h' = h$, $\mathrm{W}(h,g)$ fails for ψ', X holds for ψ', and the result follows from Lemma 4.4. □

In our applications of Lemma 4.13 we only need the fact that ψ does not survive the algorithm, so it is enough to know that a predecessor of ψ meets **(iv)**. The last six lines of the above proof could therefore be omitted.

4.3. In this section we will study a 4-tuples $\psi = (D, \mu, S, 0) \in \Psi_0$ with $\mu_{\ell+1} \geq 0$. For such a 4-tuple, Corollary 4.8 implies that each pair $(i, j) \in D \smallsetminus \mathcal{C}$ was added by **(i)** or **(iii)**. More precisely, the pair (i, j) was added by **(i)** if $(i, j) \in \partial^1\mathcal{C}$, and otherwise the pair was added by **(iii)**.

Proposition 4.14. *Suppose that $\psi = (D, \mu, S, 0)$ and $\mu_{\ell+1} \geq 0$. Then μ is a k-strict partition with $|\mu| = |\lambda| + p$, satisfying $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for every $j \geq 1$, and $\lambda_j \leq \mu_j$ when $\lambda_j > k$. Furthermore, we have $D = \mathcal{C}_{\ell+1}(\mu)$.*

Proof. By Lemma 4.12 we have $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $1 \leq j \leq m$, and Lemmas 4.2 and 4.11 show that $\mu_j \leq \min(\lambda_{j-1}, \mu_{j-1})$ for $j > m$. We deduce that μ is a k-strict partition. Lemma 4.9(b) implies that $\lambda_j - 1 \leq \mu_j$ for every j. Clearly $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and $|\mu| = |\lambda| + p$.

It remains to show that the set $\mathcal{C}_{\ell+1}(\mu)$ is equal to D. If $D \not\subset \mathcal{C}_{\ell+1}(\mu)$, then since $\mathcal{C}_{\ell+1}(\mu)$ and D are both valid sets of pairs, we can find an inner corner $(i, j) \in D \smallsetminus \mathcal{C}_{\ell+1}(\mu)$ such that $(i+1, j) \notin D$ and $(i, j+1) \notin D$. Since $(i, j) \notin \mathcal{C}$ by Lemma 2.1, the pair (i, j) was added by **(i)** or **(iii)**, and $\mathrm{W}(i, j)$ holds since μ_i and μ_j did not change since this event.

On the other hand, if $\mathcal{C}_{\ell+1}(\mu) \not\subset D$, then we can find an outer corner (i, j) of D such that $(i, j) \in \mathcal{C}_{\ell+1}(\mu)$. If $(i, j) = (m, m)$ or if $(i, j-1) \in \mathcal{C}$, then the most recent predecessor of ψ of level j meets **(i)**, and otherwise we deduce from Lemma 2.1 that the most recent predecessor of ψ of level i meets **(iii)**. This contradiction finishes the proof. □

Lemma 4.15. *Assume that $\psi = (D, \mu, S, 0)$ and j are such that $\mu_j = \lambda_j - 1 \geq 0$, and let (i, j) be the unique pair in column j of $D \smallsetminus \mathcal{C}$. Then the removed box $[j, \lambda_j]$ and the above box $[j-1, \lambda_j]$ are k-related to the boxes $[i, c]$ and $[i, c-1]$, respectively, where $c = 2k + 2 + j - i - \lambda_j$, and these latter boxes belong to R.*

Proof. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $(i, j) \notin D'$. Since $\mu'_j = \lambda_j$ and ψ' satisfies $\mathrm{W}(i, j)$, we obtain $\mu_i \geq \mu'_i + 1 \geq c$, and since $(i, j) \notin \mathcal{C}$ we similarly have $\lambda_i \leq c - 2$. The boxes $[i, c]$ and $[i, c-1]$ belong to R because $\mu_{j+1} < \lambda_j$ and $\mu_j < \lambda_j$ (see §3.2). □

Lemma 4.16. *Assume that $\psi = (D, \mu, S, 0)$, $\mu_{\ell+1} \geq 0$, and let $i \leq m$ be any integer such that $\mu_i = \lambda_{i-1}$. Then $[i, \lambda_{i-1}] \notin R$ and $(i, f_i(\mu)) \in S$.*

Proof. Let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i. Then $\mu'_i = \mu_i$ and $(i, i) \in D'$; if $i = m$ this follows because $\mu'_m = \lambda_{m-1} > k$. Lemma 4.13 implies that $[i, \lambda_{i-1}] \notin R'$. Since all pairs $(c, d) \in D \smallsetminus D'$ were added by **(iii)** and satisfy $d > g'$, it follows that $\mu'_j = \mu_j$ for $m \leq j \leq g'$, so Lemma 3.7 shows that $[i, \lambda_{i-1}] \notin R$ and $f' = f_i(\mu)$. Finally, we must have $(i, f') \in S' \subset S$ since ψ' does not satisfy condition X. □

Proposition 4.17. *If $\psi = (D, \mu, S, 0)$ and $\mu_{\ell+1} \geq 0$, then we have $\lambda \to \mu$.*

Proof. By Proposition 4.14, it suffices to check that conditions (1) and (2) of §2.2 are true. Condition (1) follows from Lemma 4.16 since $[i, \lambda_{i-1}] \notin R$ for each i. Suppose that $\mu_j + 1 = \lambda_j = d$ for $j_1 \leq j \leq j_2$. According to Lemma 4.15, each removed box $[j, d]$ for $j_1 \leq j \leq j_2$ is k-related to some box $[i_j, c_j] \in \mu \smallsetminus \lambda$, and the box $[i_j, c_j - 1]$ is also in $\mu \smallsetminus \lambda$. Condition (1) implies that each box $[j, d]$ is k-related to at most one box of $\mu \smallsetminus \lambda$. It follows that if $j < j_2$, then $[i_j, c_j] = [i_{j+1}, c_{j+1} - 1]$, so all the boxes $[i_j, c_j]$ lie in the same row of $\mu \smallsetminus \lambda$. Condition (2) follows from this since we also know that the box $[j_1 - 1, d]$ is k-related to $[i_{j_1}, c_{j_1} - 1]$. □

4.4. Propositions 4.14 and 4.17 tell us that if $\psi = (D, \mu, S, 0)$ is any 4-tuple in Ψ_0 with $\mu_{\ell+1} \geq 0$, then μ is a k-strict partition with $\lambda \to \mu$, $D = \mathcal{C}_{\ell+1}(\mu)$ is uniquely determined by μ, and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$ is a term appearing in the Pieri rule (14). To account for the multiplicities, we give an explicit construction of the possible sets S in these 4-tuples. In this section we fix an arbitrary k-strict partition μ such that $\lambda \to \mu$ and $|\mu| = |\lambda| + p$.

A *component* means an (edge or vertex) connected component of the set $\mathbb{A}$ of §2.2. We say that a box B of $\mathbb{A}$ is *distinguished* if the box directly to the left of B does not lie in $\mathbb{A}$. We say that B is *optional* if it is the rightmost distinguished box in its component. Notice that $\mathfrak{N}(\lambda, \mu)$ is equal to the number of optional distinguished boxes in $\mathbb{A}$. To each distinguished box $B = [i, c]$ we associate the pair $(i, j) = (i, r(i + c))$. The inequality $\lambda_{i-1} > 2k + i - (i + c)$ implies that $i \leq j$, so $(i, j) \in \Delta$. Let E (respectively F) be the set of pairs associated to optional (respectively non-optional) distinguished boxes. We furthermore let G be the set of all pairs $(i, j) \in \Delta$ for which some box in row i of $\mu \smallsetminus \lambda$ is k-related to a box in row j of $\lambda \smallsetminus \mu$.

Lemma 4.18. (a) *We have $E \cup F \cup G \subset \mathcal{C}_{\ell+1}(\mu) \cap \partial\mathcal{C}$.*

(b) *Each pair in $E \cup F$ is associated to exactly one distinguished box of $\mathbb{A}$.*

(c) *The sets E, F, and G are pairwise disjoint.*

(d) *If $(i, j) \in F$, then $j = f_i(\mu)$.*

(e) *If $(i, j) \in E \cup F \cup G$, $i < j$, and $(i, j - 1) \notin \mathcal{C}$, then $\mu_j < \lambda_{j-1}$.*

Proof. Let $(i, j) \in G$. Then $\mu_j = \lambda_j - 1$ and the boxes $[j, \lambda_j]$ and $[j - 1, \lambda_j]$ are k-related to $[i, d]$ and $[i, d - 1]$, where $d = 2k + 2 + j - i - \lambda_j$. We also have $\lambda_i + 1 < d \leq \mu_i$. Therefore $\lambda_i + \lambda_j < d + \lambda_j - 1 = 2k + 1 + j - i$ and $\mu_i + \mu_j \geq d + \mu_j = 2k + 1 + j - i$, so $(i, j) \in \mathcal{C}_{\ell+1}(\mu) \smallsetminus \mathcal{C}$. Assume that (i, j) is associated to a distinguished box $[i, c] \in \mathbb{A}$. Since $r(i + c) = j < j + 1 = r(i + d)$, we must have $c < d$, hence $\mu_j = 2k + 1 + j - i - d \leq 2k + j - i - c$. But then $[i, c] \in R(\mu)$, a contradiction. It follows that $G \cap (E \cup F) = \emptyset$.

Now let $[i, c] \in \mathbb{A}$ be distinguished and set $j = r(i + c)$. Then $j \geq r(i + \lambda_i + 1)$, so (16) shows that $(i, j) \notin \mathcal{C}$. Since $[i, c] \notin R(\mu)$ we get $\mu_j > 2k + j - i - c \geq 2k + j - i - \mu_i$, hence $(i, j) \in \mathcal{C}_{\ell+1}(\mu)$. Using Lemma 2.1, this establishes (a). If $[i, c'] \in R(\mu)$ is any box with $c' > c$, then we must have $j < r(i + c')$, since otherwise $\mu_j \leq 2k + j - i - c' < 2k + j - i - c$. This proves (b) and finishes the proof of (c). If $(i, j) \in F$, then $\mu_i = \lambda_{i-1}$, $c = e_i(\mu)$, and $j = f_i(\mu)$. This establishes part (d).

In the situation of (e), notice that if $\mu_j = \lambda_{j-1}$, then $(i, j) \in E \cup F$ is associated to a distinguished box $[i, c] \in \mathbb{A}$. We must have $c > k + 1$, since otherwise $\lambda_i \leq k$, $i = m$, and $j = r(m + k + 1) = i$. Since $[i, c] \notin R(\mu)$ and $(i, j - 1) \notin \mathcal{C}$, we also have $c > 2k + j - i - \mu_j = 2k + j - i - \lambda_{j-1} \geq \lambda_i + 1$. It follows that $[i, c - 1] \in R(\mu)$. Set $j' = r(i + c - 1)$. Then $j' \leq j$ and $\mu_{j'} - j' \leq 2k - i - c + 1 \leq \mu_j - j$, which shows that $j' = j$ and $\mu_j = 2k + j - i - c + 1 < \lambda_{j-1}$. This contradiction proves (e). □

To every subset E' of E we associate the set of pairs $S(E') := E' \cup F \cup G$. This is a disjoint union, and there are exactly $2^{\mathfrak{N}(\lambda,\mu)}$ sets of this form. The following proposition therefore completes the proof of Claim 1.

Proposition 4.19. *Let $S \subset \Delta$ be any subset. Then $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$ if and only if $S = S(E')$ for some subset $E' \subset E$.*

Proof. We first assume that $\psi = (\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$. Lemmas 4.16 and 4.18(d) then imply that $F \subset S$. We next show that $G = \{(i,j) \in S \mid \mu_j < \lambda_j\}$. If $\mu_j < \lambda_j$ then S contains a unique pair (i,j) in column j. Lemma 4.15 shows that $[j, \lambda_j]$ is k-related to a box in row i of $\mu \smallsetminus \lambda$, and condition (1) implies that no other box in $\mu \smallsetminus \lambda$ is k-related to $[j, \lambda_j]$. It follows that (i,j) is also the unique pair of G in column j.

Let $(i,j) \in S \smallsetminus G$. We will show that (i,j) is the pair associated to a distinguished box of $\mathbb{A}$. If $i = j = m$, then $\lambda_m \leq k < \mu_m$ and (m,m) is associated to the distinguished box $[m, k+1] \in \mathbb{A}$. We can therefore assume that $i < j$, hence $\mu_i > \lambda_i$. Since $(i,j) \notin G$ we also have $\lambda_j \leq \mu_j$. If $\lambda_i + \mu_j \geq 2k + j - i$, then the inequality $\lambda_{j-1} \geq \mu_j > 2k + j - i - \lambda_i - 1$ implies that $j \leq r(i + \lambda_i + 1)$. Since $(i,j) \notin \mathcal{C}$, it follows from (16) that $j = r(i + \lambda_i + 1)$. We deduce that $[i, \lambda_i + 1] \in \mathbb{A}$ is a distinguished box and (i,j) is the associated pair.

Otherwise we have $\lambda_i + \mu_j < 2k + j - i$. In this case we set $c = 2k + j - i - \mu_j$. Since $(i,j) \in \mathcal{C}_{\ell+1}(\mu)$ we have $\lambda_i < c < \mu_i$. We also have $c > k$; if $i = m$ this follows because $\mu_j \leq \lambda_m \leq k$. We claim that $\mu_j < \lambda_{j-1}$. If $(i, j-1) \in \mathcal{C}$, then this follows because $\mu_j < 2k + j - i - \lambda_i \leq \lambda_{j-1}$, so assume that $(i, j-1) \notin \mathcal{C}$. Then we must have $j > m$, and (i,j) was added to S in Phase 2 of the algorithm. By Lemma 4.2 the first predecessor (D', μ', S', i) of ψ of level i satisfies that $\mu'_j \leq \lambda_{j-1}$. Since $(i,j) \notin S'$, this implies that $\mu_j < \lambda_{j-1}$, as claimed.

The inequality $\lambda_{j-1} > \mu_j = 2k + j - i - c$ implies that $r(i+c) \geq j$, and since $\lambda_j \leq \mu_j$ we also have $r(i + c + 1) \leq j$. We deduce that $r(i+c) = r(i+c+1) = j$, $[i,c] \in R(\mu)$, and $[i, c+1] \in \mathbb{A}$. This shows that $[i, c+1]$ is distinguished and (i,j) is the associated pair. We conclude that the set $E' := S \smallsetminus (F \cup G)$ is a subset of E, hence $S = S(E')$ has the required form.

Now let $E' \subset E$ be an arbitrary subset and set $S = S(E')$. We must show that $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$. Set $\nu = \prod_{(i,j) \in S} L_{ij}\, \mu$. The definition of S ensures that $\nu \geq \lambda$, so $\nu \in \mathcal{N}(\lambda, p)$. We now construct a path $\mathcal{P}$ in the substitution forest by applying the substitution rule of §3.3 repeatedly to the initial 4-tuple $(\mathcal{C}, \nu, \emptyset, \ell+1)$. Whenever the substitution rule assigns two children to a 4-tuple ψ' of $\mathcal{P}$, we use the set S to determine which child is to follow ψ' on the path. More precisely, if $\psi' = (D', \mu', S', h')$ meets **(i)** or **(iii)** and has two children, and if (i,j) is the outer corner being added to D', then we choose the child $\psi'' = (D'', \mu'', S'', h')$ for which $S'' \smallsetminus S' = S \cap \{(i,j)\}$. We will show that $\mathcal{P}$ terminates in the 4-tuple $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0)$.

For $h \geq 0$ we set $D_h = \mathcal{C} \cup \{(i,j) \in \mathcal{C}_{\ell+1}(\mu) \mid i > h \text{ or } (j > h \text{ and } (i, j-1) \in \mathcal{C})\}$. Lemma 2.1 implies that this is a valid set of pairs. We will say that the 4-tuple $\psi' = (D', \mu', S', h')$ is *good* if it satisfies $D_{h'} \subset D' \subset D_{h'-1}$ and $S' = S \cap D'$. It is enough to show that if ψ' is any good 4-tuple on $\mathcal{P}$ with $h' > 0$, then ψ' has a good child that also belongs to $\mathcal{P}$.

Let ψ' be a good 4-tuple of $\mathcal{P}$. We then have $\mu = \prod_{(i,j) \in S \smallsetminus D'} R_{ij}\, \mu'$. We first show that if ψ' meets **(i)** or **(iii)**, and (i,j) is the pair being added to D', then (i,j) is also in $D_{h'-1}$. If $\mu_i + \mu_j = \mu'_i + \mu'_j$ then this is true because ψ' satisfies $\mathrm{W}(i,j)$, and otherwise $S \smallsetminus D'$ must contain at least one pair in row i or column j, which implies that $(i,j) \in D_{h'-1}$ by Lemma 4.18(a). On the other hand, assume that $D' \subsetneq D_{h'-1}$. Then $D_{h'-1} \smallsetminus D_{h'}$ contains an outer corner (i,j) of

D'. If we choose $c \geq j$ maximal such that $(i,c) \in \mathcal{C}_{\ell+1}(\mu)$, then the inequalities $\mu'_i + \mu'_j \geq \mu_i + \mu_j - (c-j) \geq \mu_i + \mu_c - c + j > 2k + j - i$ show that ψ' satisfies $\mathrm{W}(i,j)$. If $h' = j$ then ψ' meets **(i)**, and otherwise we have $h' = i < j$ and ψ' meets **(iii)**. Notice also that if ψ' meets **(iii)** and $\mu'_j > \mu'_{j-1}$, then we must have $(h',j) \in S$ since μ is a partition. These observations show that ψ' meets **(i)** or **(iii)** if and only if $D' \subsetneq D_{h'-1}$, and in this case ψ' is succeeded on $\mathcal{P}$ by a good child.

Now consider a good 4-tuple ψ' of $\mathcal{P}$ such that $D' = D_{h'-1}$. It remains to show that the substitution rule simply decreases the level of ψ', i.e. ψ' does not meet **(ii)**, **(iv)**, or **(v)**. Assume that ψ' meets **(ii)** and choose $i \geq 1$ minimal such that $(i,h') \notin D'$. Then $\mu'_{h'} > \lambda_{h'-1}$ and (i,h') is not an outer corner of D'. We have $i < h'$ and $(i,h'-1) \notin \mathcal{C}$. Using Lemma 2.1 we deduce that $(i,h') \in S$ and $\mu_{h'} = \lambda_{h'-1}$; however this contradicts Lemma 4.18(e).

If ψ' satisfies condition X, then $h' \leq m$ and $\mu'_{h'} = \mu_{h'} = \lambda_{h'-1}$. It follows that $\mathbb{A}$ contains a non-optional distinguished box in row h', and Lemma 4.18(d) implies that $(h', f_{h'}(\mu)) \in F$ is the associated pair. But Lemma 3.7 shows that $f' = f_{h'}(\mu)$, so $(h',f') \in S \cap D_{h'-1} = S'$. We conclude that condition X fails for ψ'. In particular, ψ' does not meet **(v)**. Finally, if ψ' meets **(iv)**, then $(h',g') \notin \mathcal{C}_{\ell+1}(\mu)$, and since $\mu'_{h'} = \mu_{h'}$ and $\mu'_{g'} = \mu_{g'}$, we deduce that $\mathrm{W}(h',g')$ fails for ψ'. This contradiction finishes the proof that the level of ψ' is decreased. □

Example 4.20. Consider the partitions $\lambda = (22,21,18,16,14,7,5,4,3,3,1)$ and $\mu = (25,21,19,17,15,14,6,5,3,2,2)$, and set $k=5$. Then $\lambda \to \mu$. The diagrams of λ and μ are displayed in Figure 5, with boxes from $R(\mu)$ labeled with **R** and distinguished boxes labeled with **O** for optional and **N** for non-optional. The figure also shows the pairs in $\mathcal{C}(\lambda)$ and $\mathcal{C}_{\ell+1}(\mu)$, where $\ell = 11$. The pairs from E, F, and G are labeled accordingly, and the one additional pair from $\mathcal{C}_{\ell+1}(\mu) \smallsetminus \mathcal{C}(\lambda)$ is labeled **D**. The skew dotted lines help to identify the pairs in E and F associated to the distinguished boxes. The compositions ν for which some 4-tuple $(\mathcal{C}_{\ell+1}(\mu),\mu,S,0)$ originates from $(\mathcal{C},\nu,\emptyset,\ell+1)$ may or may not include the boxes labelled with question marks, which can be traded for boxes from the rows of corresponding optional distinguished boxes. There are 2^5 such compositions ν, and for each of them there are two sets S, one of which contains the diagonal pair $(7,7) \in E$.

Remark 4.21. The most subtle ingredient of the Substitution Rule is the reference to condition $\mathrm{W}(h,g)$ in rule **(iv)**. In fact, if we modify the algorithm so that 4-tuples can meet **(iv)** only when they satisfy condition X, then Claim 1 still holds but Claim 2 fails. To see this, let $\psi = (D,\mu,S,h)$ be a 4-tuple that meets **(iv)** but does not satisfy condition X, and assume that the parent of ψ does not meet **(iv)**. Then the modified algorithm replaces ψ with $(D,\mu,S,h-1)$, and the arguments in the proofs of Lemmas 4.4 and 4.6 can be used to show that the latter 4-tuple satisfies X and does not survive the modified algorithm. It follows from this that Claim 1 is true. However, if the modified algorithm is applied with $\lambda = (4,3,1,1)$, $p = 6$, and $k = 1$, then the resulting set Ψ_1 of 4-tuples meeting **(ii)** or **(v)** contains 1119 elements, and 543 of these have non-zero evaluations. This implies that there is no involution $\iota : \Psi_1 \to \Psi_1$ such that $\mathrm{ev}(\psi) + \mathrm{ev}(\iota(\psi)) = 0$ for every $\psi \in \Psi_1$.

4.5. In this section we construct a sign-reversing involution $\iota : \Psi_1 \to \Psi_1$ and show that it has the properties required by Claim 2.

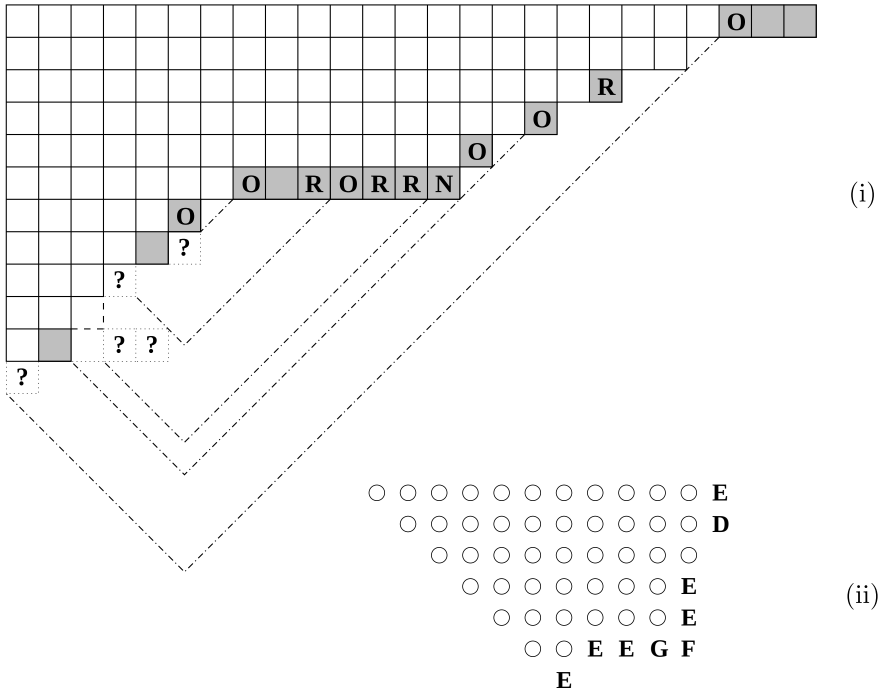


FIGURE 5. (i) λ and μ, (ii) $\mathcal{C}(\lambda)$ and $\mathcal{C}_{\ell+1}(\mu)$ in Example 4.20.

Given a valid 4-tuple $\psi = (D, \mu, S, h)$ with $h \geq 2$ and $\mu_h \geq \lambda_{h-1}$, we define a new 4-tuple $\iota\psi$ as follows. If $(h,h) \notin D$, then set $\iota\psi = (D, \widetilde{\mu}, S, h)$, where the composition $\widetilde{\mu}$ is defined by $\widetilde{\mu}_{h-1} = \mu_h - 1$, $\widetilde{\mu}_h = \mu_{h-1} + 1$, and $\widetilde{\mu}_t = \mu_t$ for $t \notin \{h-1, h\}$. If $(h,h) \in D$ and $\mu_{h-1} = \mu_h$, then set $\iota\psi = \psi$. Assume that $(h,h) \in D$ and $\mu_{h-1} \neq \mu_h$. Let ϖ be the involution of Δ that exchanges $(h-1, g)$ with (h, f), and fixes all other pairs. Then set $\iota\psi = (D, \widetilde{\mu}, \widetilde{S}, h)$, where $\widetilde{S} = \varpi(S)$, and $\widetilde{\mu}$ is the composition obtained from μ by switching the parts μ_{h-1} and μ_h.

Lemma 4.22. *If* $\psi \in \Psi_1$*, then* $\iota(\iota\psi) = \psi$ *and* $\mathrm{ev}(\psi) + \mathrm{ev}(\iota\psi) = 0$.

Proof. Assume that $(h,h) \notin D$. Then $\iota(\iota\psi) = \psi$ is clear, and since ψ meets **(ii)**, it follows from Lemma 1.2 that $\mathrm{ev}(\psi) + \mathrm{ev}(\iota\psi) = 0$. Next, assume that $(h,h) \in D$. Then ψ meets **(v)** and Lemma 4.5 shows that $(h,g) \in D$. It follows that $\widetilde{S} \subset D$ and $\iota\psi$ is a valid 4-tuple. Lemma 3.7 implies that the same values of e, f, and g are assigned to ψ and $\iota\psi$, so we have $\iota(\iota\psi) = \psi$. Finally, since ψ satisfies condition X, we have $\mu_h \geq \lambda_{h-1} > k$, so it follows from Lemma 1.3 that $\mathrm{ev}(\psi) + \mathrm{ev}(\iota\psi) = 0$. $\square$

Given any valid 4-tuple $\psi = (D, \mu, S, h)$, a valid set of pairs $D' \subset D$, and an integer $h' \geq h$, we define the 4-tuple $\psi(D', h') = (D', \prod_{(i,j) \in S \smallsetminus D'} L_{ij}\mu, S \cap D', h')$. Notice that if ψ occurs in the substitution forest, then every predecessor of ψ can be written as $\psi(D', h')$. In particular, the initial 4-tuple leading to ψ is $\psi(\mathcal{C}, \ell+1)$.

To finish the proof of Claim 2, we will show that $\iota(\Psi_1) \subset \Psi_1$. Fix an element $\psi = (D, \mu, S, h) \in \Psi_1$. We will show that the substitution forest has a path leading to $\iota\psi$ and that this 4-tuple meets **(ii)** or **(v)**. Define compositions ν and $\widetilde{\nu}$ by $(\mathcal{C}, \nu, \emptyset, \ell+1) = \psi(\mathcal{C}, \ell+1)$ and $(\mathcal{C}, \widetilde{\nu}, \emptyset, \ell+1) = \iota\psi(\mathcal{C}, \ell+1)$.

The following lemma shows that $\iota\psi \in \Psi_1$ whenever ψ meets **(ii)**. We will say that two valid 4-tuples *meet the same rule*, if both meet the same rule among **(i)**–**(v)** in the substitution rule, or if the substitution rule decreases the level of both 4-tuples.

Lemma 4.23. *Assume that $\psi \in \Psi_1$ meets* **(ii)**.

(a) *We have $\widetilde{\nu} \in \mathcal{N}(\lambda, p)$.*

(b) *Let $\psi' = \psi(D', h')$ be any predecessor of ψ. Then $\iota\psi(D', h')$ meets the same rule as ψ'. In particular, $\iota\psi$ meets* **(ii)**.

Proof. Notice first that $h > m$, as D has no outer corner in column h and $(h,h) \notin D$. Part (a) is true because $\widetilde{\nu}_{h-1} = \widetilde{\mu}_{h-1} = \mu_h - 1 \geq \lambda_{h-1}$, $\widetilde{\nu}_h \geq \widetilde{\mu}_h = \mu_{h-1} + 1 > \lambda_{h-1} \geq \lambda_h$, and $\widetilde{\nu}_t = \nu_t \geq \lambda_t$ for $t \notin \{h-1, h\}$. The inequality $\widetilde{\mu}_h > \lambda_{h-1}$ implies that $\iota\psi$ meets **(ii)**. Let $\psi(D', h')$ be a strict predecessor of ψ. If $h' > h$ and (i, h') is an outer corner of D', then $i \leq m - 1 < h - 1$ and $\mathrm{W}(i, h')$ holds for $\iota\psi(D', h')$ if and only if it holds for $\psi(D', h')$. Part (b) follows from this when $h' > h$, and when $h' = h$ it follows from Lemma 4.2. □

From now on we assume that ψ meets **(v)** and that $\mu_{h-1} \neq \mu_h$, so that $\iota\psi \neq \psi$. We have $(h,h) \in D$, ψ satisfies condition X, and $\mu_h \geq \lambda_{h-1}$. We also have $\widetilde{\mu}_h = \mu_{h-1} \geq \lambda_{h-1}$, and in case of equality $\mu_{h-1} = \lambda_{h-1}$ we must have $(h-1, g) \notin S$ or equivalently $(h, f) \notin \widetilde{S}$. This shows that $\iota\psi$ satisfies condition X.

Lemma 4.24. *We have $\widetilde{\nu} \in \mathcal{N}(\lambda, p)$.*

Proof. Notice that $\nu \geq \lambda$ and $\widetilde{\nu}_i = \nu_i$ for $i \notin \{h-1, h, f, g\}$. Observe also that row $h-1$ of $D \smallsetminus \mathcal{C}$ contains the single pair $(h-1, g)$. If $\widetilde{\mu}_{h-1} > \lambda_{h-1}$ we therefore obtain $\widetilde{\nu}_{h-1} \geq \widetilde{\mu}_{h-1} - 1 \geq \lambda_{h-1}$. Otherwise we have $\mu_h = \widetilde{\mu}_{h-1} = \lambda_{h-1}$, and since $\mu_{h-1} \neq \mu_h$ and ψ satisfies condition X, it follows that $(h, f) \notin S$, so $(h-1, g) \notin \widetilde{S}$ and $\widetilde{\nu}_{h-1} = \lambda_{h-1}$. Using Lemma 3.5 we also obtain $\widetilde{\nu}_h \geq \widetilde{\mu}_h - (g - b + 1) \geq \lambda_h$. Notice that if $h < f = g$, then $\widetilde{\nu}_g = \nu_g \geq \lambda_g$, so we may assume that $f < g$. Lemma 4.3 then implies that $\widetilde{\nu}_g \geq \widetilde{\mu}_g = \mu_g = \lambda_{g-1} \geq \lambda_g$. Using that $[h, e] \notin R$ and the definition of f, we obtain $\mu_f \geq 2k + f + 1 - h - e \geq \lambda_f$. If $f > h$ then we also have $\widetilde{\nu}_f \geq \widetilde{\mu}_f$; when $h < m = f$ this is true because $(h, f) \notin \mathcal{C}$ implies $(f, f+1) \notin D$. We conclude that $\widetilde{\nu}_f \geq \widetilde{\mu}_f = \mu_f \geq \lambda_f$, as required. □

For $t \in \mathbb{N}$ we define the valid set of pairs $D_t = \mathcal{C} \cup \{(i,j) \in D \mid i < t\}$. Let $z \geq 1$ be the smallest positive integer for which $(z, g) \notin \mathcal{C}$ and $\psi(D_z, g)$ does not meet **(i)**, and choose $\widetilde{z} \geq 1$ minimal such that $(\widetilde{z}, g) \notin \mathcal{C}$ and $\iota\psi(D_{\widetilde{z}}, g)$ does not meet **(i)**. We also write $z_1 = \min(z, \widetilde{z})$ and $z_2 = \max(z, \widetilde{z})$, and define the sets $F = \{(i,j) \in D \smallsetminus \mathcal{C} \mid j < g\}$ and $G = \{(i, g) \mid z_1 \leq i < z_2\}$. Let h_0 be maximal such that $(h_0, g) \in D$. Notice that $h \leq h_0$, $D_{z_2} = D_{z_1} \cup G$, and $G \subset \partial^1\mathcal{C}$.

Lemma 4.25. (a) *Both $\psi(D_z, g)$ and $\psi(D_z \cup F, h_0)$ are predecessors of ψ.*

(b) *We have $z_2 \leq h_0 + 1$.*

Proof. All pairs $(i,j) \in D \smallsetminus \mathcal{C}$ with $i < z$ were added to D in Phase 1 of the algorithm, while all pairs $(i, g) \in D \smallsetminus \mathcal{C}$ with $i \geq z$ were added in Phase 2. Lemma 4.6 implies that the latter pairs were added to predecessors of ψ of level h_0, and this happened after all pairs of F were added. Part (a) follows from this.

For (b) we may assume that $(h_0 + 1, g) \in \partial^1\mathcal{C}$. Since $\widetilde{\mu}_{h_0+1} = \mu_{h_0+1}$ and $\widetilde{\mu}_g = \mu_g$, it is enough to show that $\mathrm{W}(h_0 + 1, g)$ fails for ψ. If $h_0 + 1 < m$, then this follows because the most recent predecessor of ψ of level h_0+1 does not meet **(iii)** or **(iv)**. Otherwise we have $h_0+1=g=m$, in which case the inequality $\mu_m \leq k$ follows because $\mathrm{W}(z,m)$ fails for $\psi(D_z,m)$. □

Given two valid 4-tuples ψ' and ψ'', we write $\psi' \leq \psi''$ if ψ' is a predecessor of ψ'', with $\psi'=\psi''$ allowed. We will write $\psi' < \psi''$ if $\psi' \leq \psi''$ and $\psi' \neq \psi''$. The next proposition implies that $\iota\psi$ appears in the substitution forest; in fact we have $\iota\psi(\mathcal{C},\ell+1) \leq \iota\psi(D_{\widetilde{z}},g) \leq \iota\psi(D_{\widetilde{z}} \cup F, h_0) \leq \iota\psi$.

Proposition 4.26. *Let $\psi'=\psi(D',h')$ be a predecessor of ψ.*

(a) *If $\psi' < \psi(D_{z_1},g)$ then $\iota\psi(D',h')$ meets the same rule as ψ'.*

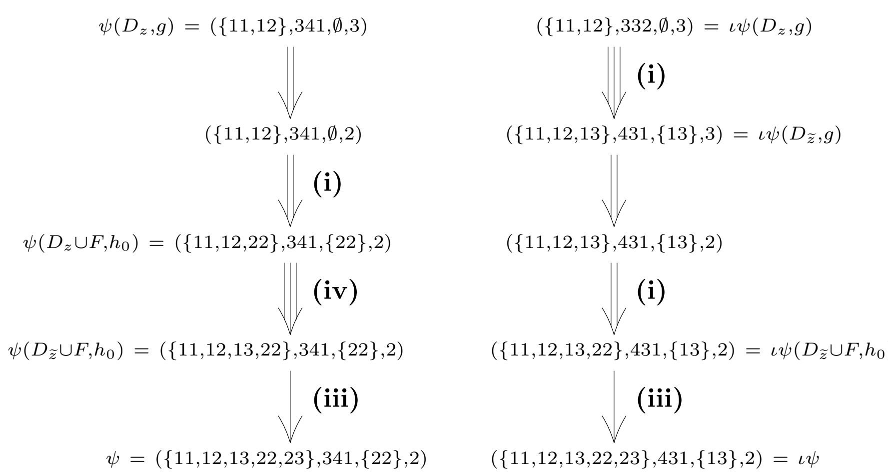
(b) *If $z_1 \leq t < \widetilde{z}$, then $\iota\psi(D_t,g)$ meets* **(i)**.

(c) *If $\psi(D_z,g) \leq \psi' < \psi(D_z \cup F, h_0)$, then $\iota\psi(D' \triangle G, h')$ meets the same rule as ψ'. Here $D' \triangle G = (D' \cup G) \smallsetminus (D' \cap G)$ is the symmetric difference.*

(d) *If $\widetilde{z} \leq t \leq h_0$, then $\iota\psi(D_t \cup F, h_0)$ meets* **(iii)** *or* **(iv)**.

(e) *If $\psi(D,h_0) \leq \psi'$, then $\iota\psi(D',h')$ meets the same rule as ψ'.*

Example 4.27. Let $\lambda=(3,1)$, $p=4$, and $k=1$. Then $\mathcal{C}=\{11,12\}$, and the 4-tuple $\psi=(D,341,\{22\},2) \in \Psi_1$ meets **(v)**, where $D=\{11,12,13,22,23\}$. We compute $f=2$, $g=3$, and $\iota\psi=(D,431,\{13\},2)$. We also have $z=1$, $\widetilde{z}=2$, $h_0=2$, $F=\{22\}$, and $G=\{13\}$. The paths of the substitution forest leading to ψ and $\iota\psi$ are displayed below. Notice how the path from $\psi(D_z,g)$ to $\psi(D_z \cup F,h_0)$ translates to a path from $\iota\psi(D_{\widetilde{z}},g)$ to $\iota\psi(D_{\widetilde{z}} \cup F,h_0)$, and the path from $\psi(D_z \cup F,h_0)$ to $\psi(D_{\widetilde{z}} \cup F,h_0)$ translates to a path from $\iota\psi(D_z,g)$ to $\iota\psi(D_{\widetilde{z}},g)$.



The proof of Proposition 4.26 is based on the following lemmas.

Lemma 4.28. *If $h_0 < h' \leq m$ and $g_{h'}=g$, then $\widetilde{\mu}_{h'}=\mu_{h'}<\lambda_{h'-1}$.*

Proof. Let $\psi'=(D',\mu',S',h')$ be the most recent predecessor of ψ of level h', and notice that $\widetilde{\mu}_{h'}=\mu_{h'}=\mu'_{h'}$. If $(h',h') \notin D'$ then $h'=m$, and since ψ' does not meet **(i)** we obtain $\mu'_m \leq k < \lambda_{m-1}$. We may therefore assume that $(h',h') \in D'$. Since condition X fails for ψ' by Lemma 4.4, we have $\mu'_{h'} \leq \lambda_{h'-1}$. Suppose that $\mu'_{h'}=\lambda_{h'-1}$. Then $(h',f_{h'}(\mu')) \in S'$, and since $(h',g) \notin D$ we must have $f_{h'}(\mu') < g = g_{h'}$. Lemma 4.3 then implies that $\mu'_g=\lambda_{g-1}$, and since

$(h'-1,g-1)\in\mathcal{C}$ we deduce that $\mathrm{W}(h',g)$ holds for ψ'. But then Lemma 4.5(a) implies that ψ' meets **(iii)** or **(iv)**, contradicting the choice of ψ'. We conclude that $\mu'_{h'}<\lambda_{h'-1}$, as required. □

In the next three lemmas, we let $\psi(D',h')=(D',\mu',S',h')$ denote a predecessor of ψ, let $\overline{D}$ be a valid set of pairs such that $D'\smallsetminus G\subset\overline{D}\subset D'\cup G$, and write $\iota\psi(\overline{D},h')=(\overline{D},\overline{\mu},\overline{S},h')$.

Lemma 4.29. *Choose $(i,j)\in\partial\mathcal{C}\smallsetminus\overline{D}$ with $j\neq g$. Assume that* (a) *$(i,j-1)\in\mathcal{C}$ or $i\geq h$, and (i,j) is an outer corner of $\overline{D}$, or* (b) *$(i,j)=(h',g_{h'})\neq(m,m)$. Then $\psi(D',h')$ satisfies $\mathrm{W}(i,j)$ if and only if $\iota\psi(\overline{D},h')$ satisfies $\mathrm{W}(i,j)$.*

Proof. Notice first that $i<g$, since otherwise $i=m=g<j$ and $(i,j)\notin\partial\mathcal{C}$. We also have $j\geq z_2$, as otherwise $z_2-1=m=j$, (m,g) is an outer corner of D_m, and $g=m$. Furthermore, if $z_1\leq i<z_2$, then we obtain $(i,g)\in\partial^1\mathcal{C}$, $j>g$, $i<h$, and $(i,j-1)\in\mathcal{C}$, which is a contradiction. This shows that $i,j\notin\{z_1,z_1+1,\dots,z_2-1,g\}$, so condition $\mathrm{W}(i,j)$ holds for $\iota\psi(\overline{D},h')$ if and only if it holds for $\iota\psi(D',h')$. Henceforth we will work with the latter 4-tuple, which we denote $\iota\psi(D',h')=(D',\widetilde{\mu}',\widetilde{S}',h')$. We then have $\mu'_t=\widetilde{\mu}'_t$ for $t\notin\{h-1,h,f,g\}$.

Assume first that $i=h$. Then we have $m\leq j<g$, and (h,j) is an outer corner of D'. Since $(h,j)\in D$ it follows that $\mathrm{W}(h,j)$ holds for $\psi(D',h')$. We must prove that also $\iota\psi(D',h')$ satisfies $\mathrm{W}(h,j)$. If $j=h$, then we have $i=j=h=m$, and since $(h-1,g-1)\in\mathcal{C}$ we obtain

$$\widetilde{\mu}'_h\geq\widetilde{\mu}_h-g+h=\mu_{h-1}-g+h\geq\lambda_{h-1}-g+h>2k-\lambda_{g-1}\geq k\,.$$

We can therefore assume that $i=h<j$. We have $\widetilde{\mu}'_j=\widetilde{\nu}_j\geq\lambda_j$, and since $\iota\psi$ satisfies condition X we also have $\widetilde{\mu}'_h+g-j+1\geq\widetilde{\mu}_h\geq\lambda_{h-1}$. Since $(h-1,g-1)\in\mathcal{C}$ we obtain

$$\widetilde{\mu}'_h+\widetilde{\mu}'_j\geq\lambda_{h-1}+\lambda_j-g+j-1\geq\lambda_{h-1}+\lambda_{g-1}-g+j-1\geq 2k+j-h\,.$$

If $\mathrm{W}(h,j)$ fails for $\iota\psi(D',h')$, then we must have equality $\widetilde{\mu}'_h+g-j+1=\widetilde{\mu}_h=\lambda_{h-1}$, hence $(h,t)\in\widetilde{S}$ for $j\leq t\leq g$. Since $\lambda_{h-1}\leq\widetilde{\mu}_{h-1}\neq\widetilde{\mu}_h$ we also obtain $\widetilde{\mu}_h<\widetilde{\mu}_{h-1}$. Since $\iota\psi$ satisfies condition X, we deduce that $(h,f)\notin\widetilde{S}$. But then $f<g$, and Lemma 4.3 implies that $(h,g)\notin S$ and hence $(h,g)\notin\widetilde{S}$, a contradiction.

We next show that if $i\neq h$, then the identities $\mu'_i=\widetilde{\mu}'_i$ and $\mu'_j=\widetilde{\mu}'_j$ hold. If $i<h$, then $j>g$ and $(i,j-1)\in\mathcal{C}$. Since this implies that $i<h-1$, the identities are true in this case. Otherwise we have $h<i\leq j\leq b\leq f$, and the identities are clear unless $j=f$. In this case we have $b=f<g$. Notice also that $(h,f)\in D'$; in case (a) this is true because (i,f) is an outer corner of D', and in case (b) it follows from Corollary 4.8 because $h'=i<j=f$ and $(h,f)\in D\cap\partial^1\mathcal{C}$. The sets of pairs $S\smallsetminus D'$ and $\widetilde{S}\smallsetminus D'$ therefore agree in column f, and since $\mu_f=\widetilde{\mu}_f$ by construction, we deduce that $\mu'_f=\widetilde{\mu}'_f$, as required. □

Lemma 4.30. *The 4-tuple $\iota\psi(\overline{D},h')$ does not meet* **(ii)**.

Proof. Suppose that $\iota\psi(\overline{D},h')$ meets **(ii)**. Without loss of generality, we may also assume that $\psi(D',h')$ is the most recent predecessor of ψ of level h'. We have $h'>m$, and since $D\smallsetminus\overline{D}$ contains at most one pair in column h', we obtain $\lambda_{h'-1}\leq\overline{\mu}_{h'}-1\leq\widetilde{\mu}_{h'}=\mu_{h'}\leq\mu'_{h'}$. Lemma 4.2 therefore implies that $\overline{\mu}_{h'}-1=\mu_{h'}=\mu'_{h'}=\lambda_{h'-1}$. In particular, we have $h'\in\{f,g\}$.

Assume that $h' = f$ and $(h, f-1) \notin \mathcal{C}$. Then $[h, \lambda_{h-1}] \notin R(\mu)$, since otherwise $f = r(h + \lambda_{h-1})$ and $\mu_f \leq 2k + f - h - \lambda_{h-1} < \lambda_{f-1}$. We also have $e > k+1$, since otherwise $\lambda_h \leq k$, $h = m$, and $f = r(m+k+1) = m$. Since $[h, e] \notin R(\mu)$ and $(h, f-1) \notin \mathcal{C}$, we obtain $e > 2k + f - h - \mu_f = 2k + f - h - \lambda_{f-1} \geq \lambda_h + 1$. It follows that $[h, e-1] \in R(\mu)$. Set $f_1 = r(h + e - 1)$. Then $f_1 \leq f$ and $\mu_{f_1} - f_1 \leq 2k - h - e + 1 \leq \mu_f - f$. Since Lemma 4.11 implies that $\mu_f \leq \mu_{f_1}$, we therefore obtain $f_1 = f$ and $\mu_f = 2k + f - h - e + 1 < \lambda_{f-1}$, a contradiction.

In view of the above, the absence of an outer corner in column h' of $\overline{D}$ implies that either $h' = f$ and $(h, f) \in \overline{D}$, or $h' = g$ and $(h-1, g) \in \overline{D}$. It follows that the sets of pairs $S \smallsetminus \overline{D}$ and $\widetilde{S} \smallsetminus \overline{D}$ agree in column h', so $\overline{\mu}_{h'} \leq \mu'_{h'}$. This contradiction finishes the proof. □

Lemma 4.31. *If $h' > h_0$, or if $h' > h$ and $\overline{D} = D'$, then $\iota\psi(\overline{D}, h')$ does not satisfy condition* X.

Proof. We first show that $\mu'_h > \lambda_h$ and $\overline{\mu}_h > \lambda_h$. In fact, Lemma 3.5 implies that $\mu_h = \widetilde{\mu}_{h-1} \geq \lambda_{h-1} \geq \lambda_h + g - b + 1$. If $\lambda_h \geq \mu'_h$, then we must have $\widetilde{\mu}_{h-1} = \lambda_{h-1}$ and $(h, j) \in S$ for $b \leq j \leq g$. The equality shows that $(h-1, g) \notin \widetilde{S}$, which in turn implies that $(h, f) \notin S$, a contradiction. The inequality $\overline{\mu}_h > \lambda_h$ is proved by interchanging $\psi(D', h')$ and $\iota\psi(\overline{D}, h')$.

For proving the lemma we may assume that $(h', h') \in \overline{D}$ and $\overline{\mu}_{h'} \geq \lambda_{h'-1}$. The assumptions imply that $\overline{D}$ and D' agree in all rows i with $i \geq h'$. In particular $(h', h') \in D'$. Since $\widetilde{\mu}_{h'} = \mu_{h'}$ and $\widetilde{S}$ agrees with S in row h', we deduce that $\overline{\mu}_{h'} = \mu'_{h'}$. Since $\psi(D', h')$ does not satisfy condition X by Lemma 4.4, we obtain $\overline{\mu}_{h'} = \mu'_{h'} = \lambda_{h'-1} < \mu'_{h'-1}$ and $(h', f_{h'}(\mu')) \in S'$. We claim that $\overline{\mu}_{h'} < \overline{\mu}_{h'-1}$. If $h'-1 = h$, then this is true because $\overline{\mu}_{h'} = \lambda_h < \overline{\mu}_h$. On the other hand, if $h'-1 > h$ and the claim fails, then $\overline{D}$ and D' differ in row $h'-1$, and this implies that $h' > h_0$ and $(h'-1, g) \in G$, hence $h'-1 = h_0 > h$. Since $(h_0, g) \in G \subset \partial^1\mathcal{C}$ we obtain $g_{h'} = b_{h_0} = g$, so Lemma 4.28 implies that $\mu'_{h'} \leq \mu_{h'} < \lambda_{h'-1}$, a contradiction.

We claim that $\overline{\mu}_{g_{h'}} = \mu'_{g_{h'}}$. Notice that $g_{h'} \leq f$, and the claim is clearly true if $g_{h'} < f$. If $g_{h'} = f < g$, then Corollary 4.8 implies that (h, f) was added to D' by rule **(i)**, hence $\overline{\mu}_f = \mu'_f$. If $g_{h'} = f = g$, then Lemma 4.28 implies that $h' \leq h_0$ and $\overline{D} = D'$, so the claim is clear unless $(h-1, g) \in D'$ and $(h, g) \notin D'$. In this case the inclusion $(h', f_{h'}(\mu')) \in D'$ implies that $f_{h'}(\mu') < g$. Lemma 4.3 then implies that $\mu'_g = \lambda_{g-1}$, and since $(h, g-1) \in \mathcal{C}$ we obtain $\mu'_h + \mu'_g > \lambda_h + \lambda_{g-1} \geq 2k + g - h$. Since the outer corner (h, g) of D' was not added by **(i)**, it follows that $(h-1, g)$ was added by **(iv)** when the substitution rule was applied to the parent of $\psi(D', h')$. By applying Lemma 4.6(c) to the parent of $\psi(D', h')$ and using that $h' > h$, we obtain $(h-1, g) \notin S$ and $(h, g) \notin S$. This shows that $\mu'_g = \overline{\mu}_g$, proving the claim. Finally, Lemma 3.7 implies that $f_{h'}(\overline{\mu}) = f_{h'}(\mu')$, therefore $(h', f_{h'}(\overline{\mu})) \in \overline{S}$, and we conclude that $\iota\psi(\overline{D}, h')$ does not satisfy X. □

Proof of Proposition 4.26. Write $\psi' = \psi(D', h') = (D', \mu', S', h')$.

(a). If $\psi' < \psi(D_{z_1}, g)$, then $(h', h') \notin D'$. If $h' > g$, then Lemmas 4.29 and 4.30 imply that $\iota\psi(D', h')$ meets the same rule as ψ', and if $h' = g$, then both ψ' and $\iota\psi(D', h')$ meet **(i)**.

(b). This part follows from the definition of $\widetilde{z}$.

(c). Set $\overline{D} = D' \triangle G$, which is a valid set of pairs. If ψ' meets **(i)** or **(iii)**, then the pair (i, j) that is added to D' belongs to F, so $j < g$, and it follows from

Lemma 4.29 that $\iota\psi(\overline{D},h')$ meets the same rule as ψ'. Otherwise, the substitution rule decreases the level of ψ', so $h' \geq h_0+1 \geq \widetilde{z}$. If $h'=g$, then $D'=D_z$, and it follows from Lemma 4.30 that the level of $\iota\psi(\overline{D},h') = \iota\psi(D_{\widetilde{z}},g)$ is decreased. We may therefore assume that $h' < g$. If $(h',h') \notin D'$, then Lemmas 4.29 and 4.30 imply that the level of $\iota\psi(\overline{D},h')$ is decreased, so assume that $(h',h') \in D'$. Lemma 4.31 implies that $\iota\psi(\overline{D},h')$ does not satisfy condition X.

Write $\iota\psi(\overline{D},h') = (\overline{D},\overline{\mu},\overline{S},h')$. If the level of $\iota\psi(\overline{D},h')$ is not decreased, then this 4-tuple meets **(iii)** or **(iv)**, and Lemma 4.29 implies that a pair from column g is added to $\overline{D}$. It follows that $\overline{D} = D_{\widetilde{z}} \cup F$ and $\iota\psi(\overline{D},h')$ satisfies $\mathrm{W}(h',g)$. Notice that $(\widetilde{z},g) \in \partial^1\mathcal{C}$, since otherwise we obtain $\widetilde{z} = h' = h_0+1$, $\overline{D} = D$, and $\overline{\mu} = \widetilde{\mu}$, hence ψ satisfies $\mathrm{W}(h',g)$, and this contradicts that the most recent predecessor of ψ of level h' does not meet **(iii)** or **(iv)**. The definition of $\widetilde{z}$ therefore implies that $\mathrm{W}(\widetilde{z},g)$ fails for $\iota\psi(D_{\widetilde{z}},g)$, and the same is true for $\iota\psi(\overline{D},h')$. Since the latter 4-tuple satisfies $\mathrm{W}(h',g)$, we deduce that $\overline{\mu}_{\widetilde{z}} - \overline{\mu}_{h'} < h' - \widetilde{z}$. Now notice that $\lambda_{\widetilde{z}} \leq \overline{\mu}_{\widetilde{z}}$; if $\widetilde{z}=m$ then this follows because $h'=m<g$ and hence $(m-1,m) \in \mathcal{C}$. Lemma 4.28 therefore implies that $\lambda_{\widetilde{z}} - \lambda_{h'-1} < \overline{\mu}_{\widetilde{z}} - \widetilde{\mu}_{h'} < h' - \widetilde{z}$, which contradicts the fact that λ is k-strict.

(d). We first show that $(h_0,h_0) \in D_t \cup F$. If this is false, then we must have $h_0 = g = m$, so the pair (m,m) was added to D by rule **(i)**. This implies that $\mu_m > k$ and therefore $\widetilde{\mu}_m > k$. Since $\widetilde{\mu}_i \geq \lambda_i \geq k+m-i$ for all $i<m$, it follows that $\iota\psi$ satisfies $\mathrm{W}(i,m)$ for $1 \leq i \leq m$, hence $\widetilde{z} = m+1 > h_0$, a contradiction.

Write $\iota\psi(D_t \cup F, h_0) = (D_t \cup F, \overline{\mu}, \overline{S}, h_0)$ and assume that both $\mathrm{W}(h_0,g)$ and X fail for this 4-tuple. Then $\mathrm{W}(h_0,g)$ also fails for $\iota\psi$. If $h<h_0$, then $\mathrm{W}(h_0,g)$ fails for ψ as well, so the pair (h_0,g) was added to D by rule **(iv)**. It follows that the predecessor (D,μ,S,h_0) of ψ meets **(v)**, which is a contradiction.

We therefore have $h=h_0$ and $(h,g) \notin \overline{S} \subset D_t \cup F$. Since $\iota\psi$ satisfies X we have $\widetilde{\mu}_h \geq \lambda_{h-1}$, and Lemma 3.6 applied to $\iota\psi$ shows that $f=g$. It follows that $\overline{\mu}_h \geq \lambda_{h-1}$, since otherwise we must have $\overline{\mu}_h < \lambda_{h-1} = \widetilde{\mu}_h$ and $(h,f)=(h,g) \in \widetilde{S}$, contradicting that $\iota\psi$ satisfies X. Now Lemma 3.6 implies that $f_h(\overline{\mu}) = g$, so $\iota\psi(D_t \cup F, h_0)$ satisfies X anyway and meets **(iii)** or **(iv)**.

(e). If $h'>h$, then the substitution rule decreases the level of ψ', and Lemma 4.31 implies that the same thing happens to $\iota\psi(D,h')$. Finally, if $h'=h$, then both $\psi'=\psi$ and $\iota\psi(D',h')=\iota\psi$ satisfy condition X and meet **(v)**. □

This completes the proof of Claim 2, and of Theorem 1.

5. Theta Polynomials

5.1. In this section we develop the theory of theta polynomials systematically; the exposition is influenced by that in Macdonald's text [M, III.8]. We shall show that these polynomials share many common features with the Schur Q-functions. One exception is that when $k>0$, we do not have a natural Hopf algebra structure.

Given any power series $\sum_{i\geq 0} c_i t^i$ in the variable t and an integer sequence $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$, we write $c_\alpha = c_{\alpha_1}c_{\alpha_2}\cdots c_{\alpha_\ell}$ and set $R\,c_\alpha = c_{R\alpha}$ for any raising operator R. We will always work with power series with constant term 1, so that $c_0 = 1$ and $c_i = 0$ for $i<0$. The formal identities (6) imply that the equations

$$\prod_{i<j}(1-R_{ij})\,c_\alpha = \det(c_{\alpha_i+j-i})_{i,j} \tag{18}$$

and

$$\prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, c_\alpha = \text{Pfaffian}(C_{\alpha_i,\alpha_j})_{i<j}$$

where

$$C_{\alpha_i,\alpha_j} = \frac{1-R_{12}}{1+R_{12}}\, c_{\alpha_i,\alpha_j} = c_{\alpha_i}c_{\alpha_j} - 2c_{\alpha_i+1}c_{\alpha_j-1} + 2c_{\alpha_i+2}c_{\alpha_j-2} - \cdots$$

are valid in the polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$.

Let $x = (x_1, x_2, \ldots)$ be a list of commuting independent variables and let $\Lambda = \Lambda(x)$ be the ring of symmetric functions in x. Consider the generating functions

$$E(x\,;t) = \prod_{i=1}^{\infty}(1+x_it) = \sum_{r=0}^{\infty} e_r(x)t^r \quad \text{and} \quad H(x\,;t) = \prod_{i=1}^{\infty}\frac{1}{1-x_it} = \sum_{r=0}^{\infty} h_r(x)t^r$$

for the elementary and complete symmetric functions e_r and h_r, respectively. Fix an integer $k \geq 0$, let $y = (y_1, \ldots, y_k)$, and for each r define $\vartheta_r = \vartheta_r(x\,;y)$ by

$$\vartheta_r = \sum_{i\geq 0} q_{r-i}(x)e_i(y).$$

We let $\Gamma^{(k)}$ be the subring of $\Lambda \otimes \mathbb{Z}[y_1, \ldots, y_k]^{S_k}$ generated by the ϑ_r:

$$\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \ldots].$$

Set $\Theta(t) = \sum_{r\geq 0} \vartheta_r t^r$; we then have

$$\Theta(t) = \prod_{i=1}^{\infty} \frac{1+tx_i}{1-tx_i} \prod_{j=1}^{k}(1+y_jt) = E(x\,;t)H(x\,;t)E(y\,;t)$$

and hence

$$\Theta(t)\Theta(-t) = E(y\,;t)E(y\,;-t) = \sum_{m=0}^{2k}(-1)^m e_m(y^2)t^{2m}\,,$$

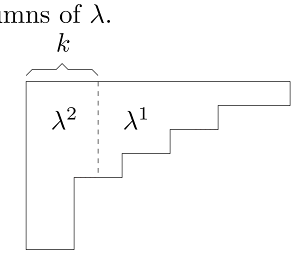
where y^2 denotes $(y_1^2, \ldots, y_k^2)$. It follows that

$$\sum_{i+j=r}(-1)^i\vartheta_i\vartheta_j = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (-1)^{r/2}e_{r/2}(y^2) & \text{if } r \text{ is even.} \end{cases} \tag{19}$$

In particular, when $r = 2m > 2k$, equation (19) gives

$$\vartheta_m^2 = 2\sum_{i=1}^{m}(-1)^{i+1}\vartheta_{m+i}\vartheta_{m-i}. \tag{20}$$

Definition 5.1. Given any k-strict partition λ, we let λ^1 be the strict partition obtained by removing the first k columns of λ, and let λ^2 be the partition of boxes contained in the first k columns of λ.



We say that a partition λ is *k-odd* if all its parts which are greater than $2k$ are odd.

Proposition 5.2. (a) *The ϑ_λ for λ k-strict form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$.*

(b) *The ϑ_λ for λ k-odd form a $\mathbb{Q}$-basis of $\Gamma^{(k)}_{\mathbb{Q}} := \Gamma^{(k)} \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. It follows from (20) that for any partition λ, either λ is k-strict, or ϑ_λ is a $\mathbb{Z}$-linear combination of the ϑ_μ such that μ is k-strict and $\mu \succ \lambda$ (dominance order). Furthermore, we have

$$\vartheta_\lambda(x\,;y) = \sum_\alpha q_{\lambda-\alpha}(x) e_\alpha(y),$$

the sum over all compositions α with $0 \leq \alpha_i \leq k$ for all i. If λ is k-strict, we deduce that the homogeneous summand of ϑ_λ of lowest x-degree is equal to $q_{\lambda^1}(x)e_{\lambda^2}(y)$. Part (a) follows because the set of all products $q_{\lambda^1}(x)e_{\lambda^2}(y)$, given by k-strict partitions λ, is linearly independent over $\mathbb{Z}$.

Equation (19) implies that $\vartheta_{2m} \in \mathbb{Q}[\vartheta_1,\ldots,\vartheta_{2m-1}]$ for $m > k$. By induction on m it follows that $\vartheta_{2m} \in \mathbb{Q}[\vartheta_1,\ldots,\vartheta_{2k},\vartheta_{2k+1},\vartheta_{2k+3},\ldots,\vartheta_{2m-1}]$ for all $m > k$, hence the monomials ϑ_λ indexed by k-odd partitions λ span $\Gamma^{(k)}_{\mathbb{Q}}$ as a vector space over $\mathbb{Q}$. Finally, for each $d \in \mathbb{N}$, the number of k-odd partitions of d is equal to the number of k-strict partitions of d, as verified by the equality of generating functions

$$\begin{aligned}\sum_{\lambda\ k\text{-odd}} t^{|\lambda|} &= \prod_{r=1}^{2k} \frac{1}{1-t^r} \prod_{r>k} \frac{1}{1-t^{2r-1}} = \prod_r \frac{1}{1-t^r} \prod_{r>k} (1-t^{2r}) \\ &= \prod_{r=1}^{k} \frac{1}{1-t^r} \prod_{r>k} (1+t^r) = \sum_{\lambda\ k\text{-strict}} t^{|\lambda|}\,.\end{aligned}$$

This completes the proof of the proposition. □

5.2. Recall the raising operator R^λ from Definition 1.

Definition 5.3. For any k-strict partition λ and formal power series $c = \sum_{i\geq 0} c_i t^i$, the *theta polynomial* $\Theta_\lambda(c)$ is defined by $\Theta_\lambda(c) = R^\lambda c_\lambda$. The theta polynomial $\Theta_\lambda(x\,;y)$ in $\Gamma^{(k)}$ is defined by $\Theta_\lambda = R^\lambda \vartheta_\lambda$.

The $\Theta_\lambda(c)$ are Giambelli polynomials for both the classical and quantum cohomology of isotropic Grassmannians (Theorem 1 and [BKT2]). Definition 5.3 and equation (20) imply that the polynomial $\Theta_\lambda = \Theta_\lambda(x\,;y)$ can be written in the form

$$\Theta_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu} \vartheta_\mu$$

where the sum is over k-strict partitions $\mu \succ \lambda$ and $a_{\lambda\mu} \in \mathbb{Z}$. We deduce from Proposition 5.2(a) that the polynomials Θ_λ indexed by k-strict partitions λ form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$.

Let

$$\mathbb{H}(\mathrm{IG}_k) = \varprojlim \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$$

be the stable cohomology ring of IG; that is, the inverse limit in the category of *graded* rings of the system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{IG}(n+1-k,2n+2),\mathbb{Z}) \leftarrow \cdots$$

From the presentation of $\mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$ given in [BKT1, Thm. 1.2], we deduce that $\mathbb{H}(\mathrm{IG}_k)$ is isomorphic to the polynomial ring $\mathbb{Z}[\sigma_1,\sigma_2,\ldots]$ modulo the relations

$$\sigma_m^2 + 2\sum_{i=1}^{m}(-1)^i\sigma_{m+i}\sigma_{m-i} = 0$$

for all $m > k$. Since the generators ϑ_r of $\Gamma^{(k)}$ satisfy (20), we have a surjective ring homomorphism $\phi : \mathbb{H}(\mathrm{IG}_k) \to \Gamma^{(k)}$ sending σ_r to ϑ_r for each r. Theorem 1 implies that $\phi(\sigma_\lambda) = \Theta_\lambda$ for any k-strict partition λ. Since the Θ_λ form a basis of $\Gamma^{(k)}$, we conclude that ϕ is an isomorphism. This completes the proof of Theorem 2.

5.3. Consider the analogues of the polynomials ϑ_r when the $e_r(y)$ are replaced by complete symmetric functions $h_r(y)$. Define for each r a function $\widehat{\vartheta}_r = \widehat{\vartheta}_r(x\,;y)$ by

$$\widehat{\vartheta}_r = \sum_i q_{r-i}(x)h_i(y)$$

and set $\widehat{\Theta}(t) = \sum_{r\geq 0}\widehat{\vartheta}_r t^r$. We then have $\Theta(t)\widehat{\Theta}(-t) = 1$, or equivalently,

$$\sum_{r=0}^{n}(-1)^r\vartheta_r\widehat{\vartheta}_{n-r} = 0, \qquad n \geq 1. \tag{21}$$

As in [M, I.2, (2.9′)], the equations (21) imply that for any partition λ,

$$\det\left(\vartheta_{\lambda_i+j-i}\right) = \det\left(\widehat{\vartheta}_{\lambda'_i+j-i}\right). \tag{22}$$

Here λ' is the partition conjugate to λ, i.e., $\lambda'_i = \#\{h \mid \lambda_h \geq i\}$ for all i.

If $k = 0$, then $\widehat{\vartheta}_r = \vartheta_r = q_r$ for every $r \geq 0$. Let (1^r) denote the partition $(1,1,\ldots,1)$ of length r.

Proposition 5.4. *Assume that $k \geq 1$ and $r \in \mathbb{N}$. Then $\widehat{\vartheta}_r(x\,;y) = \Theta_{(1^r)}(x\,;y)$.*

Proof. Observe that $\mathcal{C}(1^r) = \emptyset$. It follows from this, the identity (18), and equation (22) that

$$\Theta_{(1^r)} = \prod_{i<j}(1-R_{ij})\,\vartheta_{(1^r)} = \det(\vartheta_{1+j-i})_{1\leq i,j\leq r} = \widehat{\vartheta}_r. \qquad \square$$

Equation (21) and the Whitney sum formula prove that the polynomials $\widehat{\vartheta}_r = \Theta_{(1^r)}$ map to the Chern classes of the dual of the tautological subbundle $\mathcal{S} \to \mathrm{IG}$ under the isomorphism ϕ of §5.2. A Pieri rule for the products $\widehat{\vartheta}_r \cdot \Theta_\lambda$ was obtained by Pragacz and Ratajski [PR].

Proposition 5.5. *The $\widehat{\vartheta}_\lambda$ for λ k-strict form a $\mathbb{Q}$-basis of $\Gamma^{(k)}_{\mathbb{Q}}$.*

Proof. It is clear from the equations (22) that $\Gamma^{(k)} = \mathbb{Z}[\widehat{\vartheta}_1,\widehat{\vartheta}_2,\widehat{\vartheta}_3,\ldots]$. Since $\widehat{\vartheta}_\lambda(x\,;y) = \sum_{\alpha\geq 0} q_{\lambda-\alpha}(x)h_\alpha(y)$, we deduce that if λ is k-strict, the homogeneous summand of $\widehat{\vartheta}_\lambda$ of lowest x-degree is equal to $q_{\lambda^1}(x)h_{\lambda^2}(y)$. Moreover, the set of products $q_{\lambda^1}(x)h_{\lambda^2}(y)$ for all k-strict partitions λ is linearly independent over $\mathbb{Q}$. The result now follows by a dimension count. $\square$

Example 5.6. When $k = 1$, we have

$$3\,\Theta_{31} = 2\,\widehat{\vartheta}_4 - 5\,\widehat{\vartheta}_{31} + 4\,\widehat{\vartheta}_{211} - \widehat{\vartheta}_{1111}.$$

We deduce that the $\widehat{\vartheta}_\lambda$ for λ k-strict do *not* form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$. Furthermore, the transition matrix between the $\mathbb{Q}$-bases $\{\widehat{\vartheta}_\lambda\}$ and $\{\Theta_\lambda\}$ of $\Gamma^{(k)}_{\mathbb{Q}}$ is not triangular with respect to the dominance order.

5.4. We next introduce an analogue of the Schur S-functions in the ring $\Gamma^{(k)}$.

Definition 5.7. For any two finite integer sequences λ, μ, define the function $S^{(k)}_{\lambda/\mu} \in \Gamma^{(k)}$ by setting

$$S^{(k)}_{\lambda/\mu}(x\,;y) = \det(\vartheta_{\lambda_i-\mu_j+j-i}(x\,;y))_{i,j}.$$

Assume that λ and μ are two partitions. Then, arguing as in [M, I.5], the skew function $S^{(k)}_{\lambda/\mu}(x\,;y)$ is zero unless $\lambda_i \geq \mu_i$ for each i. The functions $S_{\lambda/\mu}(x) := S^{(0)}_{\lambda/\mu}(x\,;y)$ are well known (see [M, III.8, Ex. 7] and [W, Sec. 2.7]). We also let

$$s_{\lambda'/\mu'}(y) = \det(e_{\lambda_i-\mu_j+j-i}(y))_{i,j}$$

denote the (ordinary) skew Schur polynomial in the variables y. We have that $s_{\lambda'/\mu'}(y) = 0$ unless $0 \leq \lambda_i - \mu_i \leq k$ for each i. The functions $S_{\lambda/\mu}(x)$ (respectively, $s_{\lambda'/\mu'}(y)$) are known to be linear combinations of Schur Q-functions $Q_\nu(x)$ (respectively, Schur S-polynomials $s_{\nu'}(y)$) with positive integer coefficients.

Proposition 5.8. *For any partitions λ, μ with $\mu \subset \lambda$, we have*

$$S^{(k)}_{\lambda/\mu}(x\,;y) = \sum_\nu S_{\lambda/\nu}(x)s_{\nu'/\mu'}(y) = \sum_\nu S_{\nu/\mu}(x)s_{\lambda'/\nu'}(y) \tag{23}$$

summed over all partitions ν such that $\mu \subset \nu \subset \lambda$.

Proof. Let $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots)$ be another infinite list of variables and define the ring $\widetilde{\Lambda} = \mathbb{Z}[e_1(\widetilde{x}), e_2(\widetilde{x}), \ldots] \otimes \mathbb{Z}[e_1(y), \ldots, e_k(y)]$. According to [M, I.(5.10)], we have

$$s_{\lambda'/\mu'}(\widetilde{x}, y) = \sum_\nu s_{\lambda'/\nu'}(\widetilde{x})s_{\nu'/\mu'}(y) = \sum_\nu s_{\nu'/\mu'}(\widetilde{x})s_{\lambda'/\nu'}(y)$$

in $\widetilde{\Lambda}$. This is mapped to (23) under the ring homomorphism $\widetilde{\Lambda} \to \Gamma^{(k)}$ defined by sending $e_i(\widetilde{x})$ to $q_i(x)$ and $e_j(y)$ to $e_j(y)$. □

The definition of $S^{(k)}_\lambda$ implies that $S^{(k)}_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} d_{\lambda\mu}\vartheta_\mu$ for some integers $d_{\lambda\mu}$, and therefore that the set of $S^{(k)}_\lambda$ for λ k-strict forms another $\mathbb{Z}$-basis of $\Gamma^{(k)}$. For any integer sequence α and raising operator R, set $R\,S^{(k)}_\alpha = S^{(k)}_{R\alpha}$. The next result follows from the identity $S^{(k)}_\lambda = \prod_{i<j}(1-R_{ij})\,\vartheta_\lambda$, which is derived from (18).

Proposition 5.9. *For any k-strict partition λ, we have*

$$\Theta_\lambda(x\,;y) = \prod_{(i,j)\in\mathcal{C}(\lambda)} (1 - R_{ij} + R_{ij}^2 - \cdots)\, S^{(k)}_\lambda(x\,;y).$$

5.5. In this section, we give the proof of Theorem 3. Let λ be a k-strict partition of length ℓ. Note that if $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, then $\mathcal{C}(\lambda) = \emptyset$, and we deduce from (18) that $\Theta_\lambda = S^{(k)}_\lambda$. Part (a) of the theorem then follows by setting $\mu = 0$ in (23).

Suppose now that $\lambda_i + \lambda_j > 2k + j - i$ for all $i < j \leq \ell$. Then λ is a *strict* partition. For any strict partition $\mu \subset \lambda$ with $\ell(\mu) \geq \ell - 1$, we define the shifted skew shape

$$\mathcal{S}(\lambda/\mu) = (\lambda + \epsilon_\ell)/(\mu + \epsilon_\ell)\,,$$

where $\epsilon_\ell = (0, 1, \ldots, \ell - 1)$.

Following [HH, Chp. 9] and [M, III.8, (8.8) and Ex. 8(c)], for any integer sequence γ of length ℓ, the (generalized) Schur Q-function Q_γ is defined by

$$Q_\gamma = \prod_{1\leq i<j\leq \ell} \frac{1-R_{ij}}{1+R_{ij}}\, q_\gamma = R^\lambda q_\gamma.$$

Given any raising operator R, we have

$$R\,\vartheta_\lambda(x\,;y) = \vartheta_{R\lambda}(x\,;y) = \sum_\alpha e_\alpha(y)\, q_{R\lambda-\alpha}(x) = \sum_\alpha e_\alpha(y)\, R\, q_{\lambda-\alpha}(x).$$

It follows that

$$\Theta_\lambda = \sum_\alpha e_\alpha(y)\, R^\lambda q_{\lambda-\alpha}(x) = \sum_\alpha Q_{\lambda-\alpha}(x) e_\alpha(y). \tag{24}$$

where the sums run over all compositions α with $0 \leq \alpha_i \leq k$ for each i.

Since λ is strict and $\lambda_{\ell-1} + \lambda_\ell > 2k+1$, we see that $\lambda_i > \alpha_i$ for all compositions α indexing the sum (24) and every i except possibly $i = \ell$. We deduce from [M, III.8, Ex. 8(c)] or Lemma 1.3 that Q_γ is skew symmetric in γ for all integer vectors $\gamma = \lambda - \alpha$ which appear. It follows that we may rewrite (24) as

$$\Theta_\lambda = \sum_\mu \sum_{w\in S_\ell} (-1)^w Q_\mu(x) e_{\lambda - w(\mu)}(y)$$

summed over strict partitions μ with $\ell(\mu) \in \{\ell-1, \ell\}$. Part (b) follows from this because

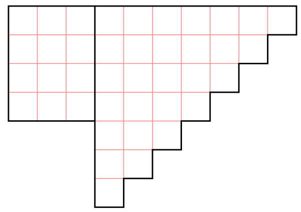
$$\sum_{w\in S_\ell} (-1)^w e_{\lambda-w(\mu)}(y) = \det(e_{\lambda_i - \mu_j}(y))_{1\leq i,j\leq \ell} = s_{\mathcal{S}(\lambda/\mu)'}(y)\,.$$

6. Schubert Polynomials for Isotropic Grassmannians

6.1. The polynomials $\Theta_\lambda(x\,;y)$ fall within the Billey-Haiman theory of type C Schubert polynomials $\mathfrak{C}_w(x,z)$. We will prove and discuss this in detail in this section. Let W_n be the hyperoctahedral group of signed permutations on the set $\{1,\ldots,n\}$, and define $W_\infty = \bigcup_n W_n$. The group W_∞ is generated by the simple transpositions $s_i = (i, i+1)$ for $i \geq 1$, and the sign change s_0 defined by $s_0(1) = \overline{1}$ and $s_0(p) = p$ for $p > 1$. Let $w \in W_\infty$. A *reduced factorization* of w is a product $w = uv$ in W_n such that $\ell(w) = \ell(u) + \ell(v)$. We say that w has a *descent* at position i if $\ell(ws_i) < \ell(w)$; this is equivalent to the inequality $w(i) > w(i+1)$ if we set $w(0) = 0$. The signed permutation w is called *k-Grassmannian* if $w = 1$ or k is the only descent position for w.

The elements of W_n index the Schubert classes in the cohomology ring of the flag variety Sp_{2n}/B, which contains $\mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ as the subring spanned by Schubert classes given by k-Grassmannian elements. In particular, each k-strict partition λ in $\mathcal{P}(k,n)$ corresponds to a k-Grassmannian element $w_\lambda \in W_n$ which we proceed to describe; more details and relations to other indexing conventions can be found in [T1, §4].

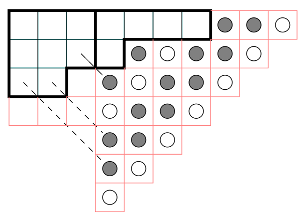
Notice that a k-strict partition λ belongs to $\mathcal{P}(k,n)$ if and only if its Young diagram fits inside the shape Π obtained by attaching an $(n-k)\times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this shape looks as follows.

$\Pi =$ 

The boxes of the staircase partition that are outside λ are organized into south-west to north-east diagonals. Such a diagonal is called *related* if it is k-related to one of the bottom boxes in the first k columns of λ, or to any box $[1, i+1]$ for which $\lambda_1 < i \leq k$; the remaining diagonals are *non-related*. The k-Grassmannian element for λ is defined by

$$w_\lambda = (r_1, \ldots, r_k, \overline{(\lambda^1)_1}, \ldots, \overline{(\lambda^1)_p}, u_1, \ldots, u_{n-k-p}),$$

where $r_1 < \cdots < r_k$ are the lengths of the related diagonals, $p = \ell(\lambda^1) = \ell_k(\lambda)$, and $u_1 < \cdots < u_{n-k-p}$ are the lengths of the non-related diagonals. For example, the partition $\lambda = (7, 4, 2) \in \mathcal{P}(3, 7)$ corresponds to the element $w_\lambda = (2, 5, 6, \overline{4}, \overline{1}, 3, 7)$.

$\lambda =$ 

The signed permutation $w_\lambda \in W_\infty$ depends on λ and k, but is independent of n. Furthermore, if $\lambda_1 \leq k$, then $w_\lambda \in S_\infty$ is the type A Grassmannian permutation for the conjugate partition λ' with descent at position k.

6.2. A sequence $a = (a_1, \ldots, a_m)$ is called *unimodal* if for some index r we have

$$a_1 > a_2 > \cdots > a_{r-1} > a_r < a_{r+1} < \cdots < a_m.$$

A *subsequence* of a is any sequence $(a_{i_1}, \ldots, a_{i_p})$ with $1 \leq i_1 < \cdots < i_p \leq m$.

Let $w \in W_\infty$ and let λ be a strict partition such that $|\lambda| = \ell(w)$. A *Kraśkiewicz tableau* [Kr] for w of shape λ is a filling T of the boxes of λ with nonnegative integers such that, if T_i is the sequence of integers in row i from left to right, then (a) the row word $T_{\ell(\lambda)} \ldots T_1$ is a reduced word for w; and (b) for each i, T_i is a unimodal subsequence of maximum length in the word $T_{\ell(\lambda)} \ldots T_{i+1} T_i$.

For each $w \in W_\infty$ one has a *type C Stanley symmetric function* $F_w(x)$, which is a positive linear combination of Schur Q-functions [BH, FK, L]. There exist several combinatorial interpretations for the coefficients in this expression. We will use a result of Lam [L] stating that

$$F_w(x) = \sum_\lambda e_w^\lambda \, Q_\lambda(x) \tag{25}$$

where e_w^λ equals the number of Kraśkiewicz tableaux for w of shape λ.

Example 6.1. Assume that $k = 0$ and let w_λ be the 0-Grassmannian element defined by a strict partition λ. In this case there exists a unique Kraśkiewicz tableau T_λ for w_λ. This tableau has shape λ and its ith row contains the integers between 0 and $\lambda_i - 1$ in decreasing order. For example, we have

$$T_{(6,5,2)} = \begin{array}{|c|c|c|c|c|c|} \hline 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 4 & 3 & 2 & 1 & 0 & \\ \cline{1-5} 1 & 0 & & & & \\ \cline{1-2} \end{array} .$$

To see this, one checks that every reduced word for w_λ can be obtained from the row word of T_λ by using the commuting relations $s_i s_j = s_j s_i$ for $|i-j| > 2$; condition (b) above then implies that any Kraśkiewicz tableau for w_λ has the same top row as T_λ, and the remaining rows are determined by induction on $\ell(\lambda)$. We deduce that $F_{w_\lambda}(x) = Q_\lambda(x)$.

6.3. Following Billey and Haiman, each $w \in W_\infty$ defines a type C Schubert polynomial $\mathfrak{C}_w(x,z)$. Here $z = (z_1, z_2, \ldots)$ is another infinite set of variables and each $\mathfrak{C}_w$ is a polynomial in the ring $A = \mathbb{Z}[q_1(x), q_2(x), \ldots; z_1, z_2, \ldots]$. The polynomials $\mathfrak{C}_w$ for $w \in W_\infty$ form a $\mathbb{Z}$-basis of A, and their algebra agrees with the Schubert calculus on symplectic flag varieties Sp_{2n}/B, when n is sufficiently large. According to [BH, Thm. 3], for any $w \in W_n$ we have

$$\mathfrak{C}_w(x,z) = \sum_{uv=w} F_u(x)\mathfrak{S}_v(z)\,, \tag{26}$$

summed over all reduced factorizations $w = uv$ in W_n for which $v \in S_n$. Here $\mathfrak{S}_v(z)$ denotes the type A Schubert polynomial of Lascoux and Schützenberger [LS].

We next show that each theta polynomial ϑ_r agrees with the Billey-Haiman Schubert polynomial indexed by the k-Grassmannian element $w_{(r)} \in W_\infty$ corresponding to $\lambda = (r)$. It is easy to see that $w_{(r)}$ has a unique reduced expression, given by $w_{(r)} = s_{k-r+1}s_{k-r+2}\cdots s_k$ when $1 \leq r \leq k$, and by $w_{(r)} = s_{r-k-1}s_{r-k-2}\cdots s_1 s_0 s_1 \cdots s_k$ when $r \geq k+1$. It follows that if $w_{(r)} = uv$ is any reduced factorization of $w_{(r)}$ with $v \in S_\infty$, then $v = w_{(i)}$ for some integer i with $0 \leq i \leq k$. The type A Schubert polynomial for $w_{(i)}$ is given by $\mathfrak{S}_{w_{(i)}}(z) = e_i(z_1, \ldots, z_k)$, and (25) implies that the type C Stanley symmetric function for $u = w_{(r)}w_{(i)}^{-1}$ is $F_u(x) = q_{r-i}(x)$. We conclude from (26) that

$$\mathfrak{C}_{w_{(r)}}(x,z) = \sum_{i=0}^{k} q_{r-i}(x_1, x_2, \ldots)e_i(z_1, \ldots, z_k) = \vartheta_r(x\,;z)\,, \tag{27}$$

as required. Since the Schubert polynomials $\mathfrak{C}_w$ multiply like the Schubert classes on symplectic flag varieties, Theorem 1 and (27) imply the following result.

Proposition 6.2. *The ring $\Gamma^{(k)}$ of theta polynomials is, by the identification of y_i with z_i for $i = 1, \ldots, k$, a subring of the ring of Billey-Haiman Schubert polynomials of type C. For every k-strict partition λ we have $\Theta_\lambda(x\,;z_1, \ldots, z_k) = \mathfrak{C}_{w_\lambda}(x,z)$.*

Remark 6.3. It is important to note that the equality in Proposition 6.2 is taking place in the ring A, where there are relations among the q_r, and these relations are used crucially in its proof. Observe furthermore that Proposition 6.2 may be used to get a different proof of Theorem 2.

Proposition 6.2 and (26) imply that for every k-strict partition λ we have

$$\Theta_\lambda(x\,;y) = \sum_{uv=w_\lambda} F_u(x)\mathfrak{S}_v(y)\,, \tag{28}$$

where the sum is over all reduced factorizations $w_\lambda = uv$ in W_∞ with $v \in S_\infty$. The right factor v in every such factorization must be the identity or a Grassmannian permutation with descent at position k. In fact, it is not hard to check that the right reduced factors of w_λ that belong to S_∞ are exactly the permutations w_ν given by partitions $\nu \subset \lambda^2$. Since the Schubert polynomial $\mathfrak{S}_{w_\nu}(y)$ is equal to the Schur polynomial $s_{\nu'}(y)$, we deduce from (25) that

$$\Theta_\lambda(x;y) = \sum_{\mu,\nu} e^\lambda_{\mu\nu} Q_\mu(x) s_{\nu'}(y)\,, \tag{29}$$

where the sum is over partitions μ and ν such that μ is strict and $\nu \subset \lambda^2$, and $e^\lambda_{\mu\nu}$ is the number of Kraśkiewicz tableaux for $w_\lambda w_\nu^{-1}$ of shape μ. This completes the proof of Theorem 4.

Corollary 6.4. *Let λ be a k-strict partition.*

(a) *The homogeneous summand of $\Theta_\lambda(x\,;y)$ of highest x-degree is the type C Stanley symmetric function $F_{w_\lambda}(x)$, and satisfies $F_{w_\lambda}(x) = R^\lambda q_\lambda(x)$.*

(b) *The homogeneous summand of $\Theta_\lambda(x\,;y)$ of lowest x-degree is $Q_{\lambda^1}(x)\, s_{(\lambda^2)'}(y)$.*

Proof. Part (a) is deduced by setting $y = 0$ in (28) and also in the raising operator expression $\Theta_\lambda(x\,;y) = R^\lambda \vartheta_\lambda(x\,;y)$. Part (b) follows from (29), Example 6.1, and the observation that $w_\lambda w_{\lambda^2}^{-1}$ is the 0-Grassmannian Weyl group element corresponding to the strict partition λ^1. □

Example 6.5. Let $k = 1$ and $\lambda = (3,2,1)$, with corresponding Weyl group element $w_\lambda = (4,\overline{2},\overline{1},3) \in W_4$. Then we have

$$\Theta_{321} = (Q_{42} + Q_{321}) + (Q_{41} + 2\,Q_{32})\, s_{1'} + 2\,Q_{31}\, s_{11'} + Q_{21}\, s_{111'}$$

(with the variables x and y omitted). The terms in this expansion are accounted for by the Kraśkiewicz tableaux in the following table.

ν	$w_\lambda w_\nu^{-1}$	Kraśkiewicz tableaux for $w_\lambda w_\nu^{-1}$
$\emptyset$	$(4,\overline{2},\overline{1},3)$	3 2 0 1 0 1 ; 3 2 1 1 0 0
(1)	$(\overline{2},4,\overline{1},3)$	3 1 0 2 0 ; 3 2 0 0 1 ; 3 0 2 0 1
$(1,1)$	$(\overline{2},\overline{1},4,3)$	3 1 0 0 ; 1 0 3 0
$(1,1,1)$	$(\overline{2},\overline{1},3,4)$	1 0 0

Remark 6.6. The polynomials $2^{-\ell_k(\lambda)}\Theta_\lambda$ given by k-strict partitions λ multiply like the Schubert classes on odd orthogonal Grassmannians $\mathrm{OG}(n-k, 2n+1)$ and agree with the Billey-Haiman Schubert polynomials of type B indexed by k-Grassmannian elements w_λ. For the even orthogonal Grassmannians $\mathrm{OG}(n-k, 2n)$, both the Giambelli formula and the corresponding family of polynomials are more involved; this theory is developed in [BKT3].

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