EIGENVALUES OF HERMITIAN MATRICES WITH POSITIVE SUM OF BOUNDED RANK

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Abstract. We give a minimal list of inequalities characterizing the possible eigenvalues of a set of Hermitian matrices with positive semidefinite sum of bounded rank. This answers a question of A. Barvinok.

1. Introduction

The combined work of A. Klyachko [8], A. Knutson, T. Tao [9] and C. Woodward [10], and P. Belkale [1] produced a minimal list of inequalities determining when three (weakly) decreasing n-tuples of real numbers can be the eigenvalues of Hermitian $n \times n$ matrices which add up to zero. The necessity of these inequalities had also been proved by S. Johnson [7] and U. Helmke and J. Rosenthal [6] (see also B. Totaro's paper [11]). We refer to [4] for a description of this work, as well as references to earlier work and applications to a surprising number of other mathematical disciplines.

S. Friedland applied these results to determine when three decreasing n-tuples of real numbers can be the eigenvalues of three Hermitian matrices with positive semidefinite sum, that is, the sum should have non-negative eigenvalues [2]. Friedland's answer included the inequalities of the above named authors, except that a trace equality was changed to an inequality. Friedland's result also needed some extra inequalities. W. Fulton has proved [5] that the extra inequalities are superfluous, and that the remaining ones form a minimal list, i.e. they correspond to the facets of the cone of permissible eigenvalues. All of these results have natural generalizations that work for any number of matrices [6, 4, 10].

In this paper we address the following more general question, which was formulated by A. Barvinok and passed along to us by Fulton. Given weakly decreasing n-tuples of real numbers $\alpha(1), \dots, \alpha(m)$ and an integer $r \leq n$, when can one find Hermitian $n \times n$ matrices $A(1), \dots, A(m)$ such that $\alpha(s)$ is the eigenvalues of $A(s)$ for each s and the sum $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r? The above described problems correspond to the extreme cases $r = 0$ and $r = n$.

Let $\alpha(1), \alpha(2), \dots, \alpha(m)$ be n-tuples of reals, with $\alpha(s) = (\alpha_1(s), \dots, \alpha_n(s))$. The requirement that these n-tuples should be decreasing is equivalent to the inequalities

$$(\dagger) \qquad \alpha_1(s) \geq \alpha_2(s) \geq \cdots \geq \alpha_n(s)$$

for all $1 \leq s \leq m$.

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Given a set $I = \{a_1 < a_2 < \cdots < a_t\}$ of positive integers, we let $s_I = \det(h_{a_i - j})_{t\times t}$ be the Schur function for the partition $\lambda(I) = (a_t - t, \dots, a_2 - 2, a_1 - 1)$. Here h_i denotes the complete symmetric function of degree i. Fulton's result [5] states that the n-tuples $\alpha(1), \dots, \alpha(m)$ can be the eigenvalues of Hermitian matrices with positive semidefinite sum if and only if

$$(\triangleright_n) \qquad \sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) \geq 0$$

for all sequences $(I(1), \dots, I(m))$ of subsets of $[n] = \{1, 2, \dots, n\}$ of the same cardinality t ($1 \leq t \leq n$), such that the coefficient of $s_{\{n-t+1, n-t+2, \dots, n\}}$ in the Schur expansion of the product $s_{I(1)} s_{I(2)} \cdots s_{I(m)}$ is equal to one. Notice that this coefficient is one if and only if the corresponding product of Schubert classes on the Grassmannian $\mathrm{Gr}(t, \mathbb{C}^n)$ equals a point class.

The added condition that the rank of the sum of matrices is at most r results in the additional inequalities

$$(\triangleleft_{n,r}) \qquad \sum_{s=1}^{m} \sum_{p \in P(s)} \alpha_{n+1-p}(s) \leq 0$$

for all sequences $(P(1), \dots, P(m))$ of subsets of $[n-r]$ of the same cardinality t ($1 \leq t \leq n-r$), such that $s_{\{n-r-t+1, \dots, n-r\}}$ has coefficient one in the product $s_{P(1)} s_{P(2)} \cdots s_{P(m)}$. Equivalently, a product of Schubert classes on $\mathrm{Gr}(t, \mathbb{C}^{n-r})$ should be a point class. The necessity of the inequalities $(\triangleleft_{n,r})$ follows from $(\triangleright_n)$ applied to the identity $-A(1) - \cdots - A(m) + B = 0$, by noting that the $n-r$ smallest eigenvalues of the matrix $B = \sum A(i)$ are zero. We remark that without the requirement that a Hermitian matrix is positive semidefinite, rank conditions on the matrix do not correspond to linear inequalities in the eigenvalues. The following theorem is our main result.

Theorem 1. *Let $\alpha(1), \dots, \alpha(m)$ be n-tuples of real numbers satisfying (†), and let $r \leq n$ be an integer. There exist Hermitian $n \times n$ matrices $A(1), \dots, A(m)$ with eigenvalues $\alpha(1), \dots, \alpha(m)$ such that the sum $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r, if and only if the inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are satisfied. Furthermore, for $r \geq 1$ and $m \geq 3$ the inequalities (†), $(\triangleright_n)$, and $(\triangleleft_{n,r})$ are independent in the sense that they correspond to facets of the cone of admissible eigenvalues.*

As proved in [10], the minimal set of inequalities in the case $r = 0, m \geq 3$ consists of the inequalities $(\triangleright_n)$ for $t < n$, along with the trace equality $\sum_{s=1}^{m} \sum_{i=1}^{n} \alpha_i(s) = 0$ and, for $n > 2$, also the inequalities (†). The cases $r = 0, m \leq 2$, or $m = 1$ are not interesting. The situation for $m = 2$ and $r > 0$ is described by the following special cases of Weyl's inequalities [12] (see also [4, p. 3]).

Corollary 1. *Let $\alpha(1), \alpha(2)$ be n-tuples satisfying (†), and let $r \leq n$ be an integer. There exist Hermitian $n \times n$ matrices $A(1), A(2)$ with eigenvalues $\alpha(1), \alpha(2)$ such that the sum $A(1) + A(2)$ is positive semidefinite of rank at most r, if and only if $\alpha_i(1) + \alpha_j(2) \geq 0$ for $i + j = n + 1$ and $\alpha_i(1) + \alpha_j(2) \leq 0$ for $i + j = n + r + 1$. These inequalities are independent when $r \geq 1$; they imply (†) for $r = 1$, and are independent of (†) for $r \geq 2$.*

Proof. Given subsets $I, J \subset [n]$ of cardinality t, the coefficient of $s_{\{n-t+1,\ldots,n\}}$ in $s_I \cdot s_J$ is equal to one if and only if $J = \{n+1-i \mid i \in I\}$. This implies that the inequalities $(\rhd_n)$ and $(\lhd_{n,r})$ are consequences of the inequalities of the corollary. The claims about independence of inequalities are left as an easy exercise. $\square$

In the special case $r = 1$ of Corollary 1, the sum $A(1) + A(2)$ may be written as $\mathbf{x}\mathbf{x}^*$ for some (column) vector $\mathbf{x} \in \mathbb{C}^n$. Inspired by a question from the referee, we give an explicit description of the set of all vectors $\mathbf{x}$ that can appear in this way for fixed $\alpha(1)$ and $\alpha(2)$ satisfying the inequalities (see Proposition 1). It shows that this set is always a product of odd dimensional spheres.

Theorem 1 also has the following consequence. Although the statement does not use any inequalities, it appears to be non-trivial to prove without the use of inequalities.

Corollary 2. *Let $\alpha(1), \ldots, \alpha(m)$ be n-tuples of real numbers and let $r \leq n$. There exist Hermitian $n \times n$ matrices $A(1), \ldots, A(m)$ with these eigenvalues such that $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r, if and only if there are Hermitian $n \times n$ matrices with the same eigenvalues and positive semidefinite sum, as well as Hermitian $(n-r) \times (n-r)$ matrices $C(1), \ldots, C(m)$ with negative semidefinite sum, such that the eigenvalues of $C(s)$ are the $n-r$ smallest numbers from $\alpha(s)$.*

Proof. The inequalities $(\lhd_{n,r})$ for n-tuples $\alpha(1), \ldots, \alpha(m)$ are identical to the inequalities $(\rhd_{n-r})$ for $\tilde{\alpha}(1), \ldots, \tilde{\alpha}(m)$, where $\tilde{\alpha}(s) = (-\alpha_n(s) \geq \cdots \geq -\alpha_{r+1}(s))$. $\square$

Our proof of Theorem 1 is by induction on r, where we rely on the above mentioned results of Klyachko, Knutson, Tao, Woodward, and Belkale to cover the base case $r = 0$. To carry out the induction, we use an enhancement of Fulton's methods from [5]. We remark that Theorem 1 remains true if the Hermitian matrices are replaced with real symmetric matrices or even quaternionic Hermitian matrices. This follows because the results for zero-sum matrices hold in this generality [4, Thm. 20].

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2. The inequalities are necessary and sufficient

In this section we prove that the inequalities of Theorem 1 are necessary and sufficient. For a subset $I = \{a_1 < a_2 < \cdots < a_t\}$ of $[n]$ of cardinality t, we let $\sigma_I \in H^* \operatorname{Gr}(t, \mathbb{C}^n)$ denote the Schubert class for the partition $\lambda(I) = (a_t - t, \ldots, a_1 - 1)$. The corresponding Schubert variety is the closure of the subset of points $V \in \operatorname{Gr}(t, \mathbb{C}^n)$ for which $V \cap \mathbb{C}^{n-a_i} \subsetneq V \cap \mathbb{C}^{n-a_i+1}$ for all $1 \leq i \leq t$. Let $S^n_t(m)$ denote the set of sequences $(I(1), \ldots, I(m))$ of subsets of $[n]$ of cardinality t, such that the product $\prod_{s=1}^n \sigma_{I(s)}$ is non-zero in $H^* \operatorname{Gr}(t, \mathbb{C}^n)$, and we let $R^n_t(m) \subset S^n_t(m)$ be the subset of sequences such that $\prod_{s=1}^n \sigma_{I(s)}$ equals the point class $\sigma_{\{n-t+1,\ldots,n-1,n\}}$.

The inequalities $(\rhd_n)$ are indexed by all sequences $(I(1), \ldots, I(m))$ which belong to the set $R^n(m) = \bigcup_{1 \leq t \leq n} R^n_t(m)$. Furthermore, it is known [1, 10] that if $\alpha(1), \ldots, \alpha(m)$ are decreasing n-tuples of reals satisfying $(\rhd_n)$, then they also satisfy the larger set of inequalities indexed by sequences from $S^n(m) = \bigcup_{1 \leq t \leq n} S^n_t(m)$, that is $\sum_{s=1}^m \sum_{i\in I(s)} \alpha_i(s) \geq 0$ for all $(I(1),\dots,I(m)) \in S^n(m)$. Similarly, the inequalities of $(\lhd_{n,r})$ are indexed by $R^{n-r}(m)$, and if $\alpha(1),\dots,\alpha(m)$ satisfy these inequalities, then we also have $\sum_{s=1}^m \sum_{p\in P(s)} \alpha_{n+1-p}(s) \leq 0$ for all sequences $(P(1),\dots,P(m)) \in S^{n-r}(m)$.

We first show that the inequalities $(\rhd_n)$ and $(\lhd_{n,r})$ are necessary. Suppose $A(1),\dots,A(m)$ are Hermitian $n\times n$ matrices with eigenvalues $\alpha(1),\dots,\alpha(m)$, such that the sum $B = A(1)+\cdots+A(m)$ is positive semidefinite with rank at most r. Let $\beta = (\beta_1 \geq \cdots \geq \beta_r, 0, \dots, 0)$ be the eigenvalues of B. For any sequence $(I(1),\dots,I(m)) \in R^n_t(m)$ we have that $(J, I(1),\dots,I(m))$ is in $R^n_t(m+1)$ where $J = \{1,2,\dots,t\}$. This is true because $\sigma_J \in H^* \operatorname{Gr}(t,\mathbb{C}^n)$ is the unit. Since $-B + A(1)+\cdots+A(m) = 0$, it follows from [4, Thm. 11] that

$$-\sum_{j\in J} \beta_{n+1-j} + \sum_{s=1}^m \sum_{i\in I(s)} \alpha_i(s) \geq 0\,,$$

which implies $(\rhd_n)$ because each β_j is non-negative.

On the other hand, if $(P(1),\dots,P(m)) \in R^{n-r}_t(m)$, then $(Q,P(1),\dots,P(m)) \in R^n_t(m)$ where $Q = \{r+1, r+2, \dots, r+t\}$. This follows from the Littlewood-Richardson rule, since $\lambda(Q) = (r)^t$ is a rectangular partition with t rows and r columns. Since $B - A(1) - \cdots - A(m) = 0$, [4, Thm. 11] implies that

$$\sum_{q\in Q} \beta_q - \sum_{s=1}^m \sum_{p\in P(s)} \alpha_{n+1-p}(s) \geq 0\,.$$

Since $\beta_q = 0$ for every $q \in Q$, this shows that $(\lhd_{n,r})$ is true.

If $I = \{i_1 < i_2 < \cdots < i_t\}$ is a subset of $[n]$ and P is a subset of $[t]$, we set $I_P = \{i_p \mid p \in P\}$. To prove that the inequalities are sufficient, we need the following generalization of [5, Prop. 1 (i)].

Lemma 1. *Let $(I(1),\dots,I(m)) \in S^n_t(m)$ and let $(P(1),\dots,P(m)) \in S^{t-r}_x(m)$, where $0 \leq r \leq t$. Then $(I(1)_{P(1)},\dots,I(m)_{P(m)})$ belongs to $S^{n-r}_x(m)$.*

Proof. The case $r = 0$ of this Lemma is equivalent to part (i) of [5, Prop. 1]. We deduce the lemma from this case using straightforward consequences of the Littlewood-Richardson rule.

Set $Q = \{p + r \mid p \in P(1)\}$. Since $\lambda(Q) = (r)^x + \lambda(P(1))$, it follows that $\sigma_Q \cdot \prod_{s=2}^m \sigma_{P(s)} \neq 0$ on $\operatorname{Gr}(x,t)$. By the $r = 0$ case, this implies that $\sigma_{I(1)_Q} \cdot \prod_{s=2}^m \sigma_{I(s)_{P(s)}} \neq 0$ on $\operatorname{Gr}(x,n)$. Now notice that if $P(1) = \{p_1 < \cdots < p_x\}$ and $I(1) = \{i_1 < \cdots < i_t\}$ then the jth element of $I(1)_Q$ is $i_{p_j+r} \geq i_{p_j} + r$, i.e. $\lambda(I(1)_Q) \supset (r)^x + \lambda(I(1)_{P(1)})$. This means that $\sigma_{(r)^x+\lambda(I(1)_{P(1)})} \cdot \prod_{s=2}^m \sigma_{I(s)_{P(s)}}$ is also non-zero on $\operatorname{Gr}(x,n)$, which implies that $\prod_{s=1}^m \sigma_{I(s)_{P(s)}} \neq 0$ on $\operatorname{Gr}(x,n-r)$. $\square$

We also need the following special case of Corollary 1, which comes from reformulating the Pieri rule in terms of eigenvalues.

Lemma 2. *Let $\alpha = (\alpha_1 \geq \cdots \geq \alpha_n)$ and $\gamma = (\gamma_1 \geq \cdots \geq \gamma_n)$ be weakly decreasing sequences of real numbers. There exist Hermitian $n\times n$ matrices A and C with these eigenvalues such that $C - A$ is positive semidefinite of rank at most one, if and only if $\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \cdots \geq \gamma_n \geq \alpha_n$.*

Proof. Set $\beta = (\beta_1, 0, \dots, 0)$ where $\beta_1 = \sum \gamma_i - \sum \alpha_i$, and assume that $\beta_1 \geq 0$. We must show that there are Hermitian matrices A, B, and C with eigenvalues α, β, and γ such that $A + B = C$ if and only if $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$.

By approximating the eigenvalues with rational numbers and clearing denominators, we may assume that α, β, and γ are partitions. In this case it follows from the work of Klyachko [8] and Knutson and Tao [9] that the matrices A, B, C exist precisely when the Littlewood-Richardson coefficient $c^{\gamma}_{\alpha\beta}$ is non-zero (see [4, Thm. 11]). This is equivalent to the specified inequalities by the Pieri rule. □

The necessity of the inequalities of Lemma 2 also follows from Weyl's inequalities $\alpha_i(A) + \alpha_n(B) \leq \alpha_i(A + B)$ and $\alpha_i(A + B) \leq \alpha_{i-1}(A) + \alpha_2(B)$ with $B = C - A$, where $\alpha_i(A)$ denotes the ith eigenvalue of a Hermitian $n \times n$ matrix A [12]. The existence of the matrices A and C is equivalent to the existence of a (column) vector $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{C}^n$ such that the matrix $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ, where $D = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. We will give an alternative proof that the inequalities are sufficient by explicitly solving this equation in $\mathbf{x}$ when $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$.

Let $\widehat{\alpha} = (\widehat{\alpha}_1 \geq \cdots \geq \widehat{\alpha}_k)$ and $\widehat{\gamma} = (\widehat{\gamma}_1 \geq \cdots \geq \widehat{\gamma}_k)$ be the subsequences of α and γ obtained by removing as many equal pairs $\alpha_i = \gamma_j$ as possible. This implies that $\widehat{\gamma}_1 > \widehat{\alpha}_1 > \cdots > \widehat{\gamma}_k > \widehat{\alpha}_k$. For example, if $\alpha = (6, 5, 4, 4, 4, 3, 2, 2, 1)$ and $\gamma = (6, 6, 5, 4, 4, 3, 3, 2, 2)$, then $\widehat{\alpha} = (4, 1)$ and $\widehat{\gamma} = (6, 3)$. Now define real numbers $c_1, \dots, c_k$ by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\widehat{\gamma}_1 - \widehat{\alpha}_1} & \cdots & \frac{1}{\widehat{\gamma}_1 - \widehat{\alpha}_k} \\ \vdots & & \vdots \\ \frac{1}{\widehat{\gamma}_k - \widehat{\alpha}_1} & \cdots & \frac{1}{\widehat{\gamma}_k - \widehat{\alpha}_k} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Notice that the matrix $\left[\frac{1}{\widehat{\gamma}_i - \widehat{\alpha}_j}\right]$ is invertible because its determinant is equal to $(\prod_{i,j}(\widehat{\gamma}_i - \widehat{\alpha}_j))^{-1}(\prod_{i<j}(\widehat{\alpha}_i - \widehat{\alpha}_j)(\widehat{\gamma}_j - \widehat{\gamma}_i))$. The following proposition is inspired by and answers a question from the referee, who suggested that exactly 2^n real solutions $\mathbf{x} \in \mathbb{R}^n$ exist when $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$.

Proposition 1. *Assume that $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$. Then each real number c_p is strictly positive. The matrix $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ if and only if*

$$\sum_{j : \alpha_j = \widehat{\alpha}_p} |x_j|^2 = c_p$$

for each $1 \leq p \leq k$, and $x_j = 0$ whenever $\alpha_j \notin \{\widehat{\alpha}_1, \dots, \widehat{\alpha}_k\}$.

Proof. The characteristic polynomial of the matrix $D + \mathbf{x}\mathbf{x}^*$ is given by $P(T) = \left(\prod_j(\alpha_j - T)\right)\left(1 + \sum_j \frac{|x_j|^2}{\alpha_j - T}\right)$. Suppose $\alpha_j \notin \{\widehat{\alpha}_p\}$ and let m be the number of occurrences of α_j in α. Since α_j occurs at least m times in γ, it must be a root of $P(T)$ of multiplicity at least m, which is possible only if $x_i = 0$ whenever $\alpha_i = \alpha_j$. It is enough to prove the proposition after removing all occurrences of α_j from α and equally many occurrences of α_j from γ. We may therefore assume that if an eigenvalue γ_i is also found in α, then α contains more copies of γ_i than γ.

It follows from the expression for $P(T)$ that the requirement that γ is the list of roots of $P(T)$ is equivalent to a system of linear equations in $|x_1|^2, \dots, |x_n|^2$. If $\alpha_{p-1} > \alpha_p = \cdots = \alpha_q > \alpha_{q+1}$, then each of these equations has the same coefficient in front of $|x_p|^2, \dots, |x_q|^2$, so this group of unknowns can be replaced with its sum. We do this explicitly by discarding $\alpha_{p+1}, \dots, \alpha_q$ from α and $\gamma_{p+1}, \dots, \gamma_q$ from γ, which replaces $|x_p|^2 + \cdots + |x_q|^2$ with $|x_p|^2$ in the equations. This reduces to the situation where $\alpha = \widehat{\alpha}$ and $\gamma = \widehat{\gamma}$, in which case $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ if and only if $|x_i|^2 = c_i$ for each i. It remains to show that $c_i > 0$.

We first note that this is true for at least one choice of eigenvalues γ. In fact, if $\mathbf{x} \in \mathbb{C}^n$ is any vector with non-zero coordinates and $\alpha_1 > \cdots > \alpha_n$, then the list γ of eigenvalues of the matrix $D + \mathbf{x}\mathbf{x}^*$ contains none of the numbers α_j. By Weyl's inequalities, we must therefore have $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$, and the numbers c_j defined by γ are strictly positive because $c_j = |x_j|^2$. If some choice of eigenvalues γ with $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$ results in non-positive real numbers c_j, then by continuity one may also choose γ such that $c_1, \dots, c_n \geq 0$ and $c_j = 0$ for some j. But then for any vector $\mathbf{x}$ with $|x_i|^2 = c_i$ for each i, α_j is in the list of eigenvalues γ of the matrix $D + \mathbf{x}\mathbf{x}^*$, a contradiction. This shows that $c_j > 0$ for each j and finishes the proof. □

Finally, we need the following statement, which is equivalent to the Claim proved in [5, p. 30].

Lemma 3 (Fulton)**.** *Let $\alpha(1), \dots, \alpha(m)$ be weakly decreasing n-tuples of real numbers which satisfy ($\triangleright_n$). Suppose that for some sequence $(I(1), \dots, I(m)) \in S^n_t(m)$ we have $\sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) = 0$. For $1 \leq s \leq m$ we let $\alpha'(s)$ be the sequence of $\alpha_i(s)$ for $i \in I(s)$ and let $\alpha''(s)$ be the sequence of $\alpha_i(s)$ for $i \notin I(s)$, both in weakly decreasing order. Then $\{\alpha'(s)\}$ satisfy ($\triangleright_t$) and $\{\alpha''(s)\}$ satisfy ($\triangleright_{n-t}$).*

We prove that the inequalities ($\triangleright_n$) and ($\triangleleft_{n,r}$) are sufficient by a 'lexicographic' induction on (n, r). As the starting point we take the cases where $r = 0$, which are already known [8, 1, 10], [4, Thm. 17]. For the induction step we let $1 \leq r \leq n$ be given and assume that the inequalities are sufficient in all cases where n is smaller, as well as the cases with the same n and smaller r. Using this hypothesis, we start by proving the following fact. Given two decreasing n-tuples α and β, we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i.

Lemma 4. *Let β, γ, and $\alpha(2), \dots, \alpha(m)$ be weakly decreasing n-tuples with $\beta \geq \gamma$, such that $\beta, \alpha(2), \dots, \alpha(m)$ satisfy ($\triangleright_n$) and $\gamma, \alpha(2), \dots, \alpha(m)$ satisfy ($\triangleleft_{n,r}$). There exists a decreasing n-tuple $\alpha(1)$ such that $\beta \geq \alpha(1) \geq \gamma$ and $\alpha(1), \dots, \alpha(m)$ satisfy both ($\triangleright_n$) and ($\triangleleft_{n,r}$).*

Proof. We start by decreasing some entries of β in the following way. First decrease β_n until an inequality ($\triangleright_n$) becomes an equality, or until $\beta_n = \gamma_n$. If the latter happens, then we continue by decreasing β_{n-1} until an inequality ($\triangleright_n$) becomes an equality, or until $\beta_{n-1} = \gamma_{n-1}$. If the latter happens we continue by decreasing β_{n-2}, etc. If we are able to decrease all entries in β so that $\beta = \gamma$, then we can use $\alpha(1) = \gamma$.

Otherwise we may assume that for some sequence $(I(1), \dots, I(m)) \in R^n_t(m)$ we have an equality $\sum_{i \in I(1)} \beta_i + \sum_{s=2}^m \sum_{i \in I(s)} \alpha_i(s) = 0$. For each $s \geq 2$ we let $\alpha'(s)$ be the decreasing t-tuple of numbers $\alpha_i(s)$ for $i \in I(s)$, and we let $\alpha''(s)$ be the decreasing $(n-t)$-tuple of numbers $\alpha_i(s)$ for $i \notin I(s)$. Similarly we define decreasing tuples $\beta' = (\beta_i)_{i \in I(1)}$, $\beta'' = (\beta_i)_{i \notin I(1)}$, and $\gamma'' = (\gamma_i)_{i \notin I(1)}$. By Lemma 3 we know that $\beta', \alpha'(2), \dots, \alpha'(m)$ satisfy ($\triangleright_t$) and that $\beta'', \alpha''(2), \dots, \alpha''(m)$ satisfy ($\triangleright_{n-t}$). In particular, since the entries of the t-tuples add up to zero, we can find Hermitian $t \times t$ matrices $A'(1), \dots, A'(m)$ with eigenvalues $\gamma', \alpha'(2), \dots, \alpha'(m)$ such that $\sum A'(s) = 0$.

We claim that the $(n-t)$-tuples $\gamma'', \alpha''(2), \dots, \alpha''(m)$ satisfy $(\triangleleft_{n-t,r})$. This is clear if $n-t \le r$. Otherwise set $J(s) = \{n+1-i \mid i \notin I(s)\}$. Since $\lambda(J(s))$ is the conjugate partition of $\lambda(I(s))$, it follows that $(J(1), \dots, J(m)) \in R^n_{n-t}(m)$. For any sequence $(P(1), \dots, P(m)) \in R_x^{n-t-r}(m)$, we obtain from Lemma 1 that the sequence $(J(1)_{P(1)}, \dots, J(m)_{P(m)})$ belongs to $S_x^{n-r}(m)$. Notice that if $J(s) = \{j_1 < j_2 < \dots < j_{n-t}\}$, then $\alpha''_{n-t+1-p}(s) = \alpha_{n+1-j_p}(s)$. The claim therefore follows because

$$\sum_{p \in P(1)} \gamma''_{n-t+1-p} + \sum_{s=2}^{m} \sum_{p \in P(s)} \alpha''_{n-t+1-p}(s) = \sum_{j \in J(1)_{P(1)}} \gamma_{n+1-j} + \sum_{s=2}^{m} \sum_{j \in J(s)_{P(s)}} \alpha_{n+1-j}(s) \le 0 \,.$$

By induction on n there exists a decreasing $(n-t)$-tuple $\alpha''(1)$ such that $\beta'' \ge \alpha''(1) \ge \gamma''$ and $\alpha''(1), \dots, \alpha''(m)$ satisfy both of $(\triangleright_{n-t})$ and $(\triangleleft_{n-t,r})$. By the cases of Theorem 1 that we assume are true by induction, we can find Hermitian $(n-t) \times (n-t)$ matrices $A''(1), \dots, A''(m)$ with eigenvalues $\alpha''(1), \dots, \alpha''(m)$ and with positive semidefinite sum of rank at most r. We can finally take $\alpha(1)$ to be the eigenvalues of $A'(1) \oplus A''(1)$. $\square$

We can now finish the proof that the inequalities of Theorem 1 are sufficient. Let $\gamma = (\alpha_2(1), \alpha_3(1), \dots, \alpha_n(1), M)$ for some large negative number $M \ll 0$. We claim that when M is sufficiently small, the n-tuples $\gamma, \alpha(2), \dots, \alpha(m)$ satisfy $(\triangleleft_{n,r-1})$. In fact, let $(P(1), \dots, P(m)) \in R_t^{n-r+1}(m)$. If $1 \in P(1)$ then the inequality for this sequence holds by choice of M. Otherwise we have that $(Q, P(2), \dots, P(m)) \in R_t^{n-r}(m)$ where $Q = \{p-1 \mid p \in P(1)\}$, and the required inequality follows because

$$\sum_{q \in Q} \alpha_{n+1-q}(1) + \sum_{s=2}^{m} \sum_{p \in P(s)} \alpha_{n+1-p}(s) \le 0 \,.$$

By Lemma 4 we may now find a decreasing n-tuple $\tilde{\alpha}(1)$ with $\alpha(1) \ge \tilde{\alpha}(1) \ge \gamma$, such that $\tilde{\alpha}(1), \alpha(2), \dots, \alpha(m)$ satisfy $(\triangleright_n)$ and $(\triangleleft_{n,r-1})$. By induction on r there exist Hermitian $n \times n$ matrices $\tilde{A}(1), A(2), \dots, A(m)$ with eigenvalues $\tilde{\alpha}(1)$, $\alpha(2), \dots, \alpha(m)$, such that $\tilde{A}(1) + A(2) + \dots + A(m)$ is positive semidefinite of rank at most $r-1$. Finally, using Lemma 2 and the choice of γ we may find a Hermitian matrix $A(1)$ with eigenvalues $\alpha(1)$ such that $A(1) - \tilde{A}(1)$ is positive semidefinite of rank at most 1. The matrices $A(1), A(2), \dots, A(m)$ now satisfy the requirements.

3. Minimality of the inequalities

In this section we prove that when $r \ge 1$ and $m \ge 3$, the inequalities (†), $(\triangleright_n)$, and $(\triangleleft_{n,r})$ are independent, thereby proving the last statement of Theorem 1. It is enough to show that for each inequality among $(\triangleright_n)$ or $(\triangleleft_{n,r})$, there exist strictly decreasing n-tuples $\alpha(1), \dots, \alpha(m)$ such that the given inequality is an equality and all other inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict. In addition we must show that for each $1 \le i \le n-1$ there exist $\alpha(1) = (\alpha_1(1) > \dots > \alpha_i(1) = \alpha_{i+1}(1) > \dots > \alpha_n(1))$ and strictly decreasing n-tuples $\alpha(2), \dots, \alpha(m)$, such that all inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict.

We start with the latter case. If $n = 2$ we can take $\alpha(1) = (0,0)$ and $\alpha(s) = (2,-1)$ for $2 \leq s \leq m$. For $n \geq 3$, it was shown in [3, Lemma 1] that the n-tuples $\beta(1) = \beta(2) = \cdots = \beta(m) = (n-1, n-3, \dots, 3-n, 1-n)$ satisfy that $\sum_{s=1}^m \sum_{i \in I(s)} \beta_i(s) \geq 2$ for all sequences $(I(1), \dots, I(m)) \in R_t^n(m)$ of subsets of cardinality $t < n$. In fact, this follows because $\sum_{s=1}^m \sum_{i \in I(s)} i = t(n-t) + m\binom{t+1}{2}$. Using this fact, one easily checks that both $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict for $\alpha(1) = (n-1, n-3, \dots, n-2i, n-2i, \dots, 3-n, 1-n)$, with $n-2i$ as the ith and $i+1$st entries, and $\alpha(2) = \cdots = \alpha(m) = (n, n-3, n-5, \dots, 3-n, 1-n)$.

Now consider an inequality from $(\triangleright_n)$, given by a sequence $(I(1), \dots, I(m)) \in R_t^n(m)$. By [10, Thm. 9] we can choose strictly decreasing n-tuples $\alpha(1), \dots, \alpha(m)$ such that $\sum_{s=1}^m \sum_{i=1}^n \alpha_i(s) = \sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) = 0$ and all other inequalities $(\triangleright_n)$ are strict. If $(P(1), \dots, P(m)) \in R_x^{n-r}(m)$ then we have $(Q, P(2), \dots, P(m)) \in R_x^n(m)$ where $Q = \{p + r \mid p \in P(1)\}$. Since the negated n-tuples $\tilde\alpha(1), \dots, \tilde\alpha(m)$ given by $\tilde\alpha(s) = (-\alpha_n(s) > \cdots > -\alpha_1(s))$ must satisfy $(\triangleright_n)$, we obtain that $\sum_{s=1}^m \sum_{p \in P(s)} \alpha_{n+1-p}(s) < \sum_{q \in Q} \alpha_{n+1-q}(1) + \sum_{s=2}^m \sum_{p \in P(s)} \alpha_{n+1-p}(s) \leq 0$. This shows that the inequalities $(\triangleleft_{n,r})$ are strict. If $t < n$ we may finally replace $\alpha_{i_0}(1)$ with $\alpha_{i_0}(1) + \epsilon$, where $i_0 \notin I(1)$, to obtain that $\sum_{s=1}^m \sum_{i=1}^n \alpha_i(s) > 0$.

At last we consider an inequality of $(\triangleleft_{n,r})$ given by a sequence $(P(1), \dots, P(m)) \in R_x^{n-r}(m)$. We once more apply [10, Thm. 9] to obtain strictly decreasing $(n-r)$-tuples $\beta(1), \dots, \beta(m)$ such that $\sum_{s=1}^m \sum_{p=1}^{n-r} \beta_p(s) = \sum_{s=1}^m \sum_{p \in P(s)} \beta_p(s) = 0$, and all other inequalities of $(\triangleright_{n-r})$ are strict. Set $\alpha(s) = (N+r, N+r-1, \dots, N+1, -\beta_{n-r}(s), \dots, -\beta_1(s))$ for $1 \leq s \leq m$, where $N \gg 0$ is a large number. Then the n-tuples $\alpha(1), \dots, \alpha(m)$ strictly satisfy all inequalities from $(\triangleleft_{n,r})$, except for the equalities $\sum_{s=1}^m \sum_{p=1}^{n-r} \alpha_{n+1-p}(s) = \sum_{s=1}^m \sum_{p \in P(s)} \alpha_{n+1-p}(s) = 0$. We must show that $\sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) > 0$ for every sequence $(I(1), \dots, I(m)) \in R_t^n(m)$. If $I(1) \cap [r] \neq \emptyset$ then this follows from our choice of N. Otherwise we have $(J, I(2), \dots, I(m)) \in R_t^{n-r}(m)$ where $J = \{i - r \mid i \in I(1)\}$. Since $\alpha_i(s) > -\beta_{n-r+1-i}(s)$ for $i \in [n-r]$, we obtain that $\sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) > \sum_{i \in J}(-\beta_{n-r+1-i}(1)) + \sum_{s=2}^m \sum_{i \in I(s)} (-\beta_{n-r+1-i}(s)) \geq 0$. Finally, if $x \neq n-r$ we replace $\alpha_{n+1-p_0}(1)$ with $\alpha_{n+1-p_0}(1) - \epsilon$, $p_0 \notin P(1)$, to obtain a strict inequality $\sum_{s=1}^m \sum_{p=1}^{n-r} \alpha_{n+1-p}(s) < 0$. This completes the proof that the inequalities are independent.

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