K-THEORETIC GROMOV–WITTEN INVARIANTS OF LINE DEGREES ON FLAG VARIETIES

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ABSTRACT. A homology class $d \in \mathrm{H}_2(X,\mathbb{Z})$ of a complex flag variety $X = G/P$ is called a *line degree* if the moduli space $\overline{\mathcal{M}}_{0,0}(X,d)$ of 0-pointed stable maps to X of degree d is also a flag variety G/P'. We prove a *quantum equals classical* formula stating that any n-pointed (equivariant, K-theoretic, genus zero) Gromov–Witten invariant of line degree on X is equal to a classical intersection number computed on the flag variety G/P'. We also prove an n-pointed analogue of the Peterson comparison formula stating that these invariants coincide with Gromov–Witten invariants of the variety of complete flags G/B. Our formulas make it straightforward to compute the big quantum K-theory ring $\mathrm{QK}^{\mathrm{big}}(X)$ modulo the ideal $\langle Q^d \rangle$ generated by degrees d larger than line degrees.

1. Introduction

In this paper we study the n-pointed genus zero Gromov–Witten invariants of line degrees on a complex flag variety $X = G/P$. Given an effective degree $d \in \mathrm{H}_2(X,\mathbb{Z})$, we let $\overline{\mathcal{M}}_{0,n}(X,d)$ be the Kontsevich moduli space of n-pointed stable maps to X of degree d and genus zero. Given subvarieties $\Omega_1,\dots,\Omega_n \subset X$ in general position, the cohomological *Gromov–Witten invariant* $\langle[\Omega_1],\dots,[\Omega_n]\rangle_d^X$ counts the number of parametrized curves $\mathbb{P}^1 \to X$ of degree d with n marked points on the domain (up to projective transformation), such that the i-th marked point is sent to Ω_i for $1 \leq i \leq n$, assuming that finitely many such curves exist. More generally, the K-theoretic Gromov–Witten invariant $\langle[\mathcal{O}_{\Omega_1}],\dots,[\mathcal{O}_{\Omega_n}]\rangle_d^X$ is defined as the sheaf Euler characteristic $\chi(\mathrm{GW}_d,\mathcal{O}_{\mathrm{GW}_d})$ of the Gromov–Witten subvariety $\mathrm{GW}_d \subset \overline{\mathcal{M}}_{0,n}(X,d)$ of stable maps that send the i-th marked point to Ω_i.

The degree $d \in \mathrm{H}_2(X,\mathbb{Z})$ is called a *line degree* if G acts transitively on the moduli space $\overline{\mathcal{M}}_{0,0}(X,d)$ of zero-pointed stable maps of degree d. This happens when $d = [X_{s_\alpha}]$ is the class of a one-dimensional Schubert variety such that the defining simple root α satisfies combinatorial conditions given in [CC98, LM03, Str02] and discussed in Section 3. All one-dimensional Schubert classes are line degrees if G is simply-laced, if X is the variety G/B of complete flags, or if X is a cominuscule flag variety.

When d is a line degree, the 0 and 1-pointed moduli spaces $M_0 = \overline{\mathcal{M}}_{0,0}(X,d)$ and $M_1 = \overline{\mathcal{M}}_{0,1}(X,d)$ are flag varieties, $M_0 = G/P'$ and $M_1 = G/(P \cap P')$, and the natural projections $p : M_1 \to X$ and $q : M_1 \to M_0$ coincide with the evaluation map and the forgetful map. Our main result (Theorem 3.1) is a *quantum equals classical* formula stating that, for arbitrary (equivariant) K-theory classes

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$\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathrm{K}_T(X)$, the associated Gromov–Witten invariant of line degree d is given by

$$\langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle_d^X \;=\; \chi_{M_0}\left(q_* p^* \mathcal{F}_1 \cdot \ldots \cdot q_* p^* \mathcal{F}_n \right) , \tag{1}$$

where $\chi_{M_0} : \mathrm{K}_T(M_0) \to \mathrm{K}_T(\mathrm{pt})$ is the sheaf Euler characteristic map. The proof uses that $\overline{\mathcal{M}}_{0,n}(X,d)$ has rational singularities and is birational to the n-fold product of M_1 over M_0. We also prove an n-pointed analogue of the Peterson comparison formula, stating that the Gromov–Witten invariant (1) coincides with a Gromov–Witten invariant of the variety of complete flags G/B.

Early results in this direction were proved for 3-pointed Gromov–Witten invariants of classical Grassmannians and cominuscule flag varieties [Buc03, BKT03, BKT09, CMP08, LL12, BM11, CP11]. The comparison formula and a variant of (1) were proved for 3-pointed Gromov–Witten invariants in [LM14]. Our results imply an analogous formula for n-pointed cohomological Gromov–Witten invariants, special cases of which were obtained in [CGH$^+$22] and [PS].

The (equivariant) big quantum K-theory ring $\mathrm{QK}^{\mathrm{big}}(X)$ introduced in [Giv00, Lee04] is a power series deformation of the K-theory ring $\mathrm{K}(X)$ that encodes the n-pointed K-theoretic Gromov–Witten invariants of all degrees. Degrees of curves are encoded as powers Q^d of Novikov variables, and additional variables t_w dual to Schubert classes $\mathcal{O}^w \in \mathrm{K}(X)$ encode insertions in Gromov–Witten invariants. The small quantum K-theory ring $\mathrm{QK}(X)$ is recovered when all t_w are specialized to 0.

Our formulas make it straightforward to compute the multiplicative structure of $\mathrm{QK}^{\mathrm{big}}(X)$ modulo the ideal $\langle Q^e \rangle$ generated by degrees e larger than line degrees. We provide some examples of products in this ring in the last section. There are other approaches to computing n-pointed K-theoretic Gromov–Witten invariants when $\mathrm{K}(X)$ is multiplicatively generated by line bundles; for example, some computations in type A have been obtained by using the J-function [GL03, LP04, IMT15].

2. Preliminaries

2.1. **Flag varieties.** Let G be a complex reductive linear algebraic group, and fix a maximal torus T, a Borel subgroup B, and a parabolic subgroup P, such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B^- \cap B = T$. Let Φ be the associated root system, with basis of simple roots $\Delta \subset \Phi^+$. Let $W = N_G(T)/T$ be the Weyl group of G, and $W_P = N_P(T)/T$ the Weyl group of P. Then W is generated by the simple reflections $\{s_\alpha \mid \alpha \in \Delta\}$, and W_P is determined by (and determines) the set $\Delta_P = \{\alpha \in \Delta \mid s_\alpha \in W_P\}$.

Let $X = G/P$ be the flag variety defined by P. Any Weyl group element $w \in W$ defines the Schubert varieties $X_w = \overline{BwP/P}$ and $X^w = \overline{B^-wP/P}$ in X. These varieties depend only on the coset wW_P in W/W_P. When w belongs to the subset $W^P \subset W$ of minimal representatives of the cosets in W/W_P, we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$, where $\ell(w)$ denotes the Coxeter length of w.

2.2. **Gromov–Witten invariants.** For any projective T-variety Y, let $\mathrm{K}_T(Y)$ be the equivariant K-theory ring, defined as the Grothendieck ring of T-equivariant algebraic vector bundles. This ring is an algebra over $\mathrm{K}_T(\mathrm{pt})$, the representation ring of T. Let $\chi_Y : \mathrm{K}_T(Y) \to \mathrm{K}_T(\mathrm{pt})$ be the push-forward map along the structure morphism. The equivariant K-theory $\mathrm{K}_T(X)$ of the flag variety $X = G/P$ is a free $\mathrm{K}_T(\mathrm{pt})$-module with basis $\{\mathcal{O}^w \mid w \in W^P\}$, where $\mathcal{O}^w = [\mathcal{O}_{X^w}] \in \mathrm{K}_T(X)$ is the K-theoretic Schubert class defined by the structure sheaf of X^w.

The homology group $\mathrm{H}_2(X,\mathbb{Z})$ is a free abelian group generated by the classes $[X_{s_\alpha}]$ of the 1-dimensional Schubert varieties for $\alpha \in \Delta \smallsetminus \Delta_P$. Given an effective degree $d \in \mathrm{H}_2(X,\mathbb{Z})$ and $n \in \mathbb{N}$, we let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of degree d and genus zero (see [FP97]). This moduli space is non-empty when $d \neq 0$ or $n \geq 3$. In this case, the evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$, defined for $1 \leq i \leq n$, sends a stable map to the image of the i-th marked point in its domain.

Given K-theory classes $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathrm{K}_T(X)$, the corresponding n-pointed (equivariant) K-theoretic Gromov–Witten invariant of degree d and genus zero is defined by

$$\langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle_d^X \;=\; \chi_{\overline{\mathcal{M}}_{0,n}(X,d)}\left(\prod_{i=1}^{n} \mathrm{ev}_i^* \mathcal{F}_i \right) \;\in \mathrm{K}_T(\mathrm{pt})\,.$$

Similarly, the cohomological Gromov–Witten invariant given by $\gamma_1, \dots, \gamma_n \in \mathrm{H}_T^*(X)$ is defined by

$$\langle \gamma_1, \dots, \gamma_n \rangle_d^X \;=\; \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \;\in \mathrm{H}_T(\mathrm{pt})\,.$$

Non-equivariant Gromov–Witten invariants are obtained by replacing T with the trivial group; these Gromov–Witten invariants are integers.

3. Gromov–Witten invariants of line degrees

A non-zero homology class $d \in \mathrm{H}_2(X,\mathbb{Z})$ will be called a *line degree* if G acts transitively on the moduli space $\overline{\mathcal{M}}_{0,0}(X,d)$ of 0-pointed stable maps to X of degree d and genus zero. Equivalently, $d = [X_{s_\alpha}]$ is the class of a one-dimensional Schubert variety defined by a simple root $\alpha \in \Delta \smallsetminus \Delta_P$ such that α is a long root within its connected component of $\Delta_P \cup \{\alpha\}$ [CC98, LM03, Str02]. Here we identify the simple roots Δ with the set of nodes in the Dynkin diagram of Φ, so that the connected component of α in $\Delta_P \cup \{\alpha\}$ is an irreducible Dynkin diagram in itself. We further use the convention that all roots of a simply-laced root system are long. In particular, $[X_{s_\alpha}]$ is a line degree if the component of α in $\Delta_P \cup \{\alpha\}$ is simply-laced, even if α is a short root of Φ. All one-dimensional Schubert classes are line degrees if Φ is simply-laced, if X is the variety G/B of complete flags, or if X is a cominuscule flag variety. We note that the definition of line degree depends on the group G. For example, the projective space $\mathbb{P}^{2n-1}$ is a flag variety of both $\mathrm{SL}(2n)$ and $\mathrm{Sp}(2n)$, and $\mathrm{SL}(2n)$ acts transitively on the set of lines in $\mathbb{P}^{2n-1}$ whereas $\mathrm{Sp}(2n)$ does not.

Given a fixed line degree $d = [X_{s_\alpha}]$, we let $P' \subset G$ be the parabolic subgroup defined by

$$\Delta_{P'} = (\Delta_P \cup \{\alpha\}) \smallsetminus \{\beta \in \Delta \mid (\beta, \alpha^\vee) < 0\}\,,$$

that is, $\Delta_{P'}$ is obtained from $\Delta_P \cup \{\alpha\}$ by removing the simple roots adjacent to α in the Dynkin diagram. In this case the moduli spaces of 0 and 1-pointed stable maps to X of degree d and genus zero are the flag varieties $M_0 = \overline{\mathcal{M}}_{0,0}(X,d) = G/P'$ and $M_1 = \overline{\mathcal{M}}_{0,1}(X,d) = G/(P \cap P')$, and the natural projections $p : G/(P \cap P') \to X$ and $q : G/(P \cap P') \to G/P'$ coincide with the evaluation map and the forgetful map [CC98, LM03, Str02].

$$\begin{array}{ccc} M_1 = G/(P \cap P') & \xrightarrow{\mathrm{ev} = p} & X = G/P \\ {\scriptstyle q}\big\downarrow & & \\ M_0 = G/P' & & \end{array}$$

The curve of degree d in X corresponding to $y \in M_0$ is given by

$$L_y \;=\; p\left(q^{-1}(y)\right) \,.$$

Let $\pi_X : G/B \to X$ be the projection. If $d = [X_{s_\alpha}]$ is a line degree of X, we also let d denote the unique line degree $[B s_\alpha B/B]$ of G/B that is mapped to d by push-forward along π_X. This class $[B s_\alpha B/B]$ is the *Peterson lift* of d, see Definition 1 in [Woo05]. In fact, one can check that any class in $\mathrm{H}_2(X;\mathbb{Z})$ is a line degree of X if and only if its Peterson lift is a line degree of G/B.

Our main result is the following theorem. Part (B) and a variant of (A) were proved for 3-pointed Gromov–Witten invariants in [LM14].

Theorem 3.1. *Let $d \in \mathrm{H}_2(X, \mathbb{Z})$ be a line degree of the flag variety X with associated projections $p : M_1 \to X$ and $q : M_1 \to M_0$, and let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathrm{K}_T(X)$ be* K*-theory classes. The following identities hold in $\mathrm{K}_T(\mathrm{pt})$:*

(A) $\langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle_d^X \;=\; \chi_{G/P'}(q_* p^* \mathcal{F}_1 \cdot \ldots \cdot q_* p^* \mathcal{F}_n)$.

(B) $\langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle_d^X \;=\; \langle \pi_X^* \mathcal{F}_1, \dots, \pi_X^* \mathcal{F}_n \rangle_d^{G/B}$.

Proof. To prove part (A), let $M_1^{(n)} = M_1 \times_{M_0} \cdots \times_{M_0} M_1$ be the fiber product of n copies of M_1 over M_0, with projections $e_i : M_1^{(n)} \to M_1$ for $1 \leq i \leq n$. Set $M_n = \overline{\mathcal{M}}_{0,n}(X, d)$, and let $\phi : M_n \to M_1^{(n)}$ be the morphism defined by the n forgetful maps $M_n \to M_1$. We obtain the commutative diagram:

$$\begin{array}{ccccc} M_1^{(n)} & \xrightarrow{e_i} & M_1 & \xrightarrow{q} & M_0 \\ {\scriptstyle \phi}\big\uparrow & & {\scriptstyle p}\big\downarrow & & \\ M_n & \xrightarrow{\mathrm{ev}_i} & X & & \end{array}$$

Since q is a locally trivial fibration with non-singular base and fiber, it follows that $M_1^{(n)}$ is a non-singular projective variety. Using that any morphism $\mathbb{P}^1 \to X$ of line degree is an isomorphism onto its image, it follows that ϕ is birational. The variety M_n has rational singularities by Theorem 2 (ii) in [FP97] and Proposition 5.15 in [KM98]. We obtain $\phi_*[\mathcal{O}_{M_n}] = [\mathcal{O}_{M_1^{(n)}}]$ in $\mathrm{K}_T(M_1^{(n)})$, and therefore

$$\begin{aligned} \langle \mathcal{F}_1, \dots, \mathcal{F}_n \rangle_d^X \;&=\; \chi_{M_n}\left(\textstyle\prod_{i=1}^n \mathrm{ev}_i^* \mathcal{F}_i\right) \;=\; \chi_{M_n}\left(\phi^* \textstyle\prod_{i=1}^n e_i^* \, p^* \mathcal{F}_i\right) \\ &=\; \chi_{M_1^{(n)}}\left(\textstyle\prod_{i=1}^n e_i^* \, p^* \mathcal{F}_i\right) \;=\; \chi_{M_0}\left(\textstyle\prod_{i=1}^n q_* p^* \mathcal{F}_i\right) \,, \end{aligned}$$

where the last two equalities follow from the projection formula and Lemma 3.5 in [BM11].

For part (B), let $P_\alpha \subset G$ be the minimal parabolic subgroup given by $\Delta_{P_\alpha} = \{\alpha\}$, where $\alpha \in \Delta \smallsetminus \Delta_P$ is defined by $d = [X_{s_\alpha}]$. We obtain a commutative diagram

$$\begin{array}{ccccc} G/B & \xrightarrow{\pi} & M_1 & \xrightarrow{p} & X \\ {\scriptstyle\pi_\alpha}\big\downarrow & & {\scriptstyle q}\big\downarrow & & \\ G/P_\alpha & \xrightarrow{q_\alpha} & M_0 & & \end{array}$$

where π, π_α, and q_α are the natural projections of flag varieties. Using that the square is Cartesian and $\pi_X = p \circ \pi$, we obtain

$$\begin{aligned} \langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle_d^X \;&=\; \chi_{M_0}\left(\textstyle\prod_{i=1}^n q_* p^* \mathcal{F}_i\right) \;=\; \chi_{G/P_\alpha}\left(\textstyle\prod_{i=1}^n q_\alpha^*\, q_* p^* \mathcal{F}_i\right) \\ &=\; \chi_{G/P_\alpha}\left(\textstyle\prod_{i=1}^n \pi_{\alpha *} \pi^* p^* \mathcal{F}_i\right) \;=\; \chi_{G/P_\alpha}\left(\textstyle\prod_{i=1}^n \pi_{\alpha *} \pi_X^* \mathcal{F}_i\right) \\ &=\; \langle \pi_X^* \mathcal{F}_1, \ldots, \pi_X^* \mathcal{F}_n \rangle_d^{G/B}\,, \end{aligned}$$

where the last equality follows from part (A) applied to G/B, noting that we have $\overline{\mathcal{M}}_{0,0}(G/B, d) = G/P_\alpha$ and $\overline{\mathcal{M}}_{0,1}(G/B, d) = G/B$. $\square$

Theorem 3.1 implies the analogous identities for cohomological Gromov–Witten invariants in the following corollary by the equivariant Hirzebruch formula [EG00], see section 4.1 in [BM11]. A similar statement is proved in [PS] when P is maximal and used to study semisimplicity of big quantum cohomology; when X is a Grassmannian, the statement is proved and applied in [CGH+22]. Part (B) for 3-pointed Gromov–Witten invariants is a special case of Peterson's comparison formula, proved in Woodward's paper [Woo05].

Corollary 3.2. *Let $d \in \mathrm{H}_2(X, \mathbb{Z})$ be a line degree of X, and let $\gamma_1, \ldots, \gamma_n \in \mathrm{H}_T^*(X)$ be cohomology classes. The following identities hold in $\mathrm{H}_T^*(\mathrm{pt})$:*

(A) $\langle \gamma_1, \ldots, \gamma_n \rangle_d^X \;=\; \int_{G/P'} q_* p^* \gamma_1 \cdot \ldots \cdot q_* p^* \gamma_n$.

(B) $\langle \gamma_1, \ldots, \gamma_n \rangle_d^X \;=\; \langle \pi_X^* \gamma_1, \ldots, \pi_X^* \gamma_n \rangle_d^{G/B}$.

Remark 3.3. Theorem 3.1 and Corollary 3.2 imply that non-equivariant Gromov–Witten invariants of line degree are enumerative for Schubert classes in the following sense. Let $\Omega_1, \ldots, \Omega_n \subset X$ be Schubert varieties in general position and let

$$Y \;=\; \bigcap_{i=1}^n q\left(p^{-1}(\Omega_i)\right) \;=\; \{y \in M_0 \mid L_y \cap \Omega_i \neq \emptyset \;\; \forall\, 1 \leq i \leq n\}$$

be the subvariety of $M_0 = G/P'$ parametrizing lines meeting $\Omega_1, \ldots, \Omega_n$. Then

$$\langle [\mathcal{O}_{\Omega_1}], \ldots, [\mathcal{O}_{\Omega_n}] \rangle_d^X \;=\; \chi(Y, \mathcal{O}_Y)\,. \tag{2}$$

This follows from Theorem 3.1 (A) since the projection $q : p^{-1}(\Omega_i) \to q(p^{-1}(\Omega_i))$ of the Schubert variety $p^{-1}(\Omega_i)$ is cohomologically trivial [Ram85]. The left hand side of (2) is the sheaf Euler characteristic $\chi(\mathrm{GW}_d, \mathcal{O}_{\mathrm{GW}_d})$ of the Gromov–Witten variety $\mathrm{GW}_d = \bigcap_{i=1}^n \mathrm{ev}_i^{-1}(\Omega_i) \subset \overline{\mathcal{M}}_{0,n}(X, d)$ of stable maps that send the i-th marked point to Ω_i (see section 4.1 in [BM11]).

When the Schubert varieties satisfy the condition

$$\sum_{i=1}^n \operatorname{codim}(\Omega_i, X) \;=\; \dim \overline{\mathcal{M}}_{0,n}(X, d) \;=\; \dim M_0 + n\,,$$

we also have

$$\langle[\Omega_1],\dots,[\Omega_n]\rangle_d^X \;=\; \int_{G/P'}[Y] \;=\; \#Y\,, \tag{3}$$

that is, the cohomological Gromov–Witten invariant $\langle[\Omega_1],\dots,[\Omega_n]\rangle_d^X$ is the number of lines in X of degree d meeting $\Omega_1,\dots,\Omega_n$.

Example 3.4. Let $\gamma \in \mathrm{H}^*(\mathbb{P}^3)$ be the class of a line. Then Remark 3.3 implies that $\langle\gamma,\gamma,\gamma,\gamma\rangle_1^{\mathbb{P}^3}$ counts the number of lines that meet 4 given lines in general position in $\mathbb{P}^3$, and this number can be computed as a classical intersection number on $M_0 = \mathrm{Gr}(2,4)$.

4. Applications to big quantum K-theory

4.1. **Definitions.** Set $\Gamma = \mathrm{K}_T(\mathrm{pt}) \otimes \mathbb{Q}$. Given a fixed flag variety $X = G/P$, we let

$$\Gamma[\![Q,t]\!] \;=\; \Gamma[\![Q_\alpha, t_w \mid \alpha \in \Delta \smallsetminus \Delta_P,\, w \in W^P]\!]$$

be the ring of formal power series over Γ, in Novikov variables Q_α dual to the Schubert basis of $\mathrm{H}_2(X,\mathbb{Z})$, and formal variables t_w dual to the Schubert basis of $\mathrm{K}_T(X)$. The big equivariant quantum K-theory ring of X is a $\Gamma[\![Q,t]\!]$-algebra defined by

$$\mathrm{QK}_T^{\mathrm{big}}(X) = \mathrm{K}_T(X,\mathbb{Q}) \otimes_\Gamma \Gamma[\![Q,t]\!]$$

as a module. We proceed to define the multiplicative structure on $\mathrm{QK}_T^{\mathrm{big}}(X)$ following [Giv00, Lee04].

For $d = \sum_{\alpha\in\Delta\smallsetminus\Delta_P} d_\alpha [X_{s_\alpha}] \in \mathrm{H}_2(X,\mathbb{Z})$ we write

$$Q^d = \prod_{\alpha\in\Delta\smallsetminus\Delta_P} Q_\alpha^{d_\alpha}\,,$$

and for any function $h : W^P \to \mathbb{N}$ we define

$$t^h = \prod_{w\in W^P} t_w^{h(w)} \quad , \qquad h! = \prod_{w\in W^P} h(w)! \quad , \text{ and } \qquad |h| = \sum_{w\in W^P} h(w)\,.$$

Given K-theory classes $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3 \in \mathrm{K}_T(X)$, we let $\langle\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{O}^h\rangle_d^X$ denote the $(|h|+3)$-pointed Gromov–Witten invariant of degree d with $h(w)$ insertions of $\mathcal{O}^w$, for $w \in W^P$, in addition to the first three insertions. We then define

$$(\!(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)\!) \;=\; \sum_{d,h} \langle\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{O}^h\rangle_d^X \,\frac{t^h}{h!}\, Q^d \;\in \Gamma[\![Q,t]\!]\,,$$

with the sum over all effective degrees $d \in \mathrm{H}_2(X,\mathbb{Z})$ and functions $h : W^P \to \mathbb{N}$. We extend this by linearity to a symmetric 3-form on the $\Gamma[\![Q,t]\!]$-module $\mathrm{QK}_T^{\mathrm{big}}(X)$. The *quantum metric* on $\mathrm{QK}_T^{\mathrm{big}}(X)$ is then defined by $(\!(\mathcal{F}_1,\mathcal{F}_2)\!) = (\!(\mathcal{F}_1,\mathcal{F}_2,1)\!)$, and the *quantum product* $\mathcal{F}_1 \star \mathcal{F}_2 \in \mathrm{QK}_T^{\mathrm{big}}(X)$ is the unique class defined by

$$(\!(\mathcal{F}_1 \star \mathcal{F}_2, \mathcal{F}_3)\!) = (\!(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)\!)$$

for all $\mathcal{F}_3 \in K_T(X)$. The small quantum K-theory ring is the quotient $\mathrm{QK}_T(X) = \mathrm{QK}_T^{\mathrm{big}}(X)/\langle t\rangle$ by the ideal generated by t_w for $w \in W^P$.

Remark 4.1. Let 0 denote the identity element of W, so that t_0 is dual to $1 \in \mathrm{K}_T(X)$. The product $\mathcal{F}_1 \star \mathcal{F}_2 \in \mathrm{QK}_T^{\mathrm{big}}(X)$ is known to be independent of t_0 for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{K}_T(X)$; this follows from [Giv00] (see also Prop. 2.10 in [IMT15]). In fact, since the general fibers of the forgetful map $\overline{\mathcal{M}}_{0,n+1}(X,d) \to \overline{\mathcal{M}}_{0,n}(X,d)$ are isomorphic to $\mathbb{P}^1$, we have $\langle 1, \mathcal{O}^h \rangle_d^X = \langle \mathcal{O}^h \rangle_d^X$ for all effective $d \in \mathrm{H}_2(X,\mathbb{Z})$ and $h : W^P \to \mathbb{N}$ with $|h| \geq 3$, and therefore $(\!(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\!) = e^{t_0}\, (\!(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\!)|_{t_0=0}$ for all $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathrm{K}_T(X)$. We obtain

$$\begin{aligned}(\!((\mathcal{F}_1 \star \mathcal{F}_2)|_{t_0=0}, \mathcal{F}_3)\!) \;&=\; e^{t_0}\, (\!((\mathcal{F}_1 \star \mathcal{F}_2)|_{t_0=0}, \mathcal{F}_3)\!)|_{t_0=0} \;=\; e^{t_0}\, (\!(\mathcal{F}_1 \star \mathcal{F}_2, \mathcal{F}_3)\!)|_{t_0=0} \\ &=\; e^{t_0}\, (\!(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\!)|_{t_0=0} \;=\; (\!(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\!)\,.\end{aligned}$$

It follows that $\mathcal{F}_1 \star \mathcal{F}_2 \in \mathrm{QK}_T^{\mathrm{big}}(X)$ is the unique class that is independent of t_0 and satisfies

$$(\!(\mathcal{F}_1 \star \mathcal{F}_2, \mathcal{F}_3)\!)|_{t_0=0} \;=\; (\!(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\!)|_{t_0=0} \tag{4}$$

for all $\mathcal{F}_3 \in \mathrm{K}_T(X)$. Notice that the product $\mathcal{F}_1 \star \mathcal{F}_2$ can be constructed from the *quantum potential* $\mathcal{G}(Q,t) = (\!(1,1,1)\!) \in \Gamma[\![Q,t]\!]$ specialized at $t_0 = 0$,

$$\mathcal{G}_0 \;=\; (\!(1,1,1)\!)|_{t_0=0} \;=\; \sum_{d,h:\, h(0)=0} \langle 1,1,1,\mathcal{O}^h \rangle_d^X \, \frac{t^h}{h!}\, Q^d \;\in \Gamma[\![Q,t]\!]\,,$$

by observing that $(\!(\mathcal{O}^u, \mathcal{O}^v, \mathcal{O}^w)\!)|_{t_0=0} = \partial_{t_u} \partial_{t_v} \partial_{t_w} \mathcal{G}_0$ holds for $u, v, w \in W^P \smallsetminus \{0\}$.

Remark 4.2. The product in the equivariant big quantum cohomology ring

$$\mathrm{QH}_T^{\mathrm{big}}(X) = \mathrm{H}_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[\![Q,t]\!]$$

is defined by

$$[X^u] \star [X^v] = \sum_{w,d,h} \langle [X^u], [X^v], [X_w], [X]^h \rangle_d^X \, Q^d\, \frac{t^h}{h!}\, [X^w]\,,$$

where $\langle [X^u], [X^v], [X_w], [X]^h \rangle_d^X$ is the cohomological Gromov–Witten invariant with $h(w)$ insertions of $[X^w]$ in addition to the first three insertions.

4.2. **Examples.** In this section we apply Theorem 3.1 to compute some examples of big quantum K-theory products modulo powers of the Novikov variables of degrees larger than line degrees. In each case there is a unique line degree, and only one Novikov variable which will be denoted by Q. Congruence $\equiv$ is always modulo Q^2. Our examples are compatible with a positivity property of big quantum K-theory that we plan to discuss elsewhere.

Example 4.3. Let $X = \mathbb{P}^1 = \mathrm{SL}(2)/B$. The only line degree in $\mathrm{H}_2(\mathbb{P}^1,\mathbb{Z})$ gives $M_1 = \mathbb{P}^1$ and $M_0 = \{\mathrm{pt}\}$. Let $T = \mathbb{C}^*$ act on $\mathbb{P}^1$ and let $P \in \mathbb{P}^1$ be a T-fixed point. Then $\mathrm{K}_T(\mathbb{P}^1)$ has basis $\{1, \mathcal{O}^1\}$, where $\mathcal{O}^1 = [\mathcal{O}_P]$. Set $a = 1 - \mathcal{O}^1|_P \in \mathrm{K}_T(\mathrm{pt})$. Using that $(\mathcal{O}^1)^n = (1-a)^{n-1}\mathcal{O}^1 \in \mathrm{K}_T(\mathbb{P}^1)$ for $n \geq 1$, we compute the specialized potential (modulo Q^2) as

$$\mathcal{G}_0 \;\equiv\; \frac{e^{(1-a)t} - a}{1-a} + Q\, e^t\,,$$

where $t = t_1$ is dual to $\mathcal{O}^1$. If we write $\mathcal{O}^1 \star \mathcal{O}^1 = c_0 + c_1 \mathcal{O}^1$ with $c_0, c_1 \in \Gamma[\![Q,t]\!]$, then the equations

$$c_0 (\!(1, \mathcal{F})\!) + c_1 (\!(\mathcal{O}^1, \mathcal{F})\!) = (\!(c_0 + c_1 \mathcal{O}^1, \mathcal{F})\!) = (\!(\mathcal{O}^1 \star \mathcal{O}^1, \mathcal{F})\!)\,,$$

for $\mathcal{F} \in \{1, \mathcal{O}^1\}$, are equivalent to

$$c_0 \, \frac{\partial^i \mathcal{G}_0}{\partial t^i} + c_1 \, \frac{\partial^{i+1} \mathcal{G}_0}{\partial t^{i+1}} \;=\; \frac{\partial^{i+2} \mathcal{G}_0}{\partial t^{i+2}}$$

for $i \in \{0, 1\}$. By solving for c_0 and c_1 modulo Q^2, we arrive at

$$\mathcal{O}^1 \star \mathcal{O}^1 \;\equiv\; a\, Q\, e^t + \left(1 - a - a\, Q\, \frac{e^t - e^{at}}{1-a} \right) \mathcal{O}^1 \,.$$

Example 4.4. Let $X = \mathbb{P}^2$. Then $M_1 = \mathrm{Fl}(3)$ is a complete flag variety and $M_0 = \mathbb{P}^{2^*}$ is the dual projective plane. The Schubert basis is $\{1, \mathcal{O}^1, \mathcal{O}^2\}$, where $\mathcal{O}^1 = [\mathcal{O}_L]$ is the class of a line and $\mathcal{O}^2 = [\mathcal{O}_P]$ is the class of a point in $\mathbb{P}^2$.

Working non-equivariantly for simplicity, we have

$$\mathcal{G}_0 \;\equiv\; 1 + t_1 + t_2 + \frac{t_1^2}{2} + Q e^{t_1} \left(1 + t_2 + \frac{t_2^2}{2} \right) ,$$

from which we obtain (see also section 4.3 in [IMT15]):

$$\begin{aligned}
\mathcal{O}^1 \star \mathcal{O}^1 \;&\equiv\; \mathcal{O}^2 + Q e^{t_1} \Big(t_2 + \Big(\tfrac{t_2^2}{2} - t_1 t_2 - t_2 \Big) \mathcal{O}^1 + t_2 (t_1 - t_2) \left(\tfrac{t_1}{2} + 1 \right) \mathcal{O}^2 \Big) \;; \\
\mathcal{O}^1 \star \mathcal{O}^2 \;&\equiv\; Q e^{t_1} \left(1 + (t_2 - t_1) \mathcal{O}^1 + \Big(\tfrac{t_1^2}{2} - t_1 t_2 - t_2 \Big) \mathcal{O}^2 \right) \;; \\
\mathcal{O}^2 \star \mathcal{O}^2 \;&\equiv\; Q e^{t_1} \left(\mathcal{O}^1 - t_1 \mathcal{O}^2 \right) \,.
\end{aligned}$$

Example 4.5. Let $X = \mathrm{Gr}(2,4)$ be the Grassmannian of 2-planes in $\mathbb{C}^4$. Here, $M_0 = \mathrm{Fl}(1,3;4)$ is a point-hyperplane incidence variety and $M_1 = \mathrm{Fl}(4)$ is a complete flag variety. The Schubert basis is $\{1, \mathcal{O}^1, \mathcal{O}^{1,1}, \mathcal{O}^2, \mathcal{O}^{2,1}, \mathcal{O}^{2,2}\}$, where $\mathcal{O}^\lambda$ is the Schubert class indexed by the partition λ. Working non-equivariantly, we obtain (by a computation in Maple):

$$\mathcal{O}^{2,2} \star \mathcal{O}^2 \;\equiv\; Q e^{t_1} \Big(\mathcal{O}^{1,1} + (t_{1,1} - t_1)\, \mathcal{O}^{2,1} + \Big(\tfrac{t_1^2}{2} - t_{1,1} - t_1 t_{1,1} \Big) \mathcal{O}^{2,2} \Big)$$

and

$$\begin{aligned}
&\mathcal{O}^2 \star \mathcal{O}^2 \;\equiv\; \mathcal{O}^{2,2} \,+\, Q e^{t_1} \Big(\\
&\quad t_{1,1}\, \mathcal{O}^1 + \Big(\tfrac{t_{1,1}^2}{2} + t_2 t_{1,1} - t_1 t_{1,1} + t_{2,1} \Big) \mathcal{O}^{1,1} + \Big(\tfrac{t_{1,1}^2}{2} - t_1 t_{1,1} - t_{1,1} \Big) \mathcal{O}^2 \\
&\quad + \Big(\tfrac{t_{1,1}^3}{6} + (t_2 - 2t_1 - 3) \tfrac{t_{1,1}^2}{2} + (t_1^2 - t_1 t_2 + 2t_1 - 2t_2 + t_{2,1}) t_{1,1} - t_1 t_{2,1} - t_{2,1} \Big) \mathcal{O}^{2,1} \\
&\quad + \Big((-t_1 - 3)\, \tfrac{t_{1,1}^3}{6} + \left(t_1^2 - t_1 t_2 + 3t_1 - 3t_2 \right) \tfrac{t_{1,1}^2}{2} \\
&\qquad + \Big(\tfrac{t_1^2 t_2}{2} + 2 t_1 t_2 + t_2 - \tfrac{t_1^3}{3} - t_1^2 - t_1 t_{2,1} - 2 t_{2,1} \Big) t_{1,1} + \tfrac{t_1^2 t_{2,1}}{2} + t_1 t_{2,1} \Big) \mathcal{O}^{2,2} \Big) \,.
\end{aligned}$$

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References

[BKT03] A. S. Buch, A. Kresch, and H. Tamvakis. Gromov-Witten invariants on Grassmannians. *J. Amer. Math. Soc.*, 16(4):901–915, 2003.

[BKT09] A. S. Buch, A. Kresch, and H. Tamvakis. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math.*, 178(2):345–405, 2009.

[BM11] A. S. Buch and L. C. Mihalcea. Quantum K-theory of Grassmannians. *Duke Math. J.*, 156(3):501–538, 2011.

[Buc03] A. S. Buch. Quantum cohomology of Grassmannians. *Compositio Math.*, 137(2):227–235, 2003.

[CC98] A. M. Cohen and B. N. Cooperstein. Lie incidence systems from projective varieties. *Proc. Amer. Math. Soc.*, 126(7):2095–2102, 1998.

[CGH+22] L. Chen, A. Gibney, L. Heller, E. Kalashnikov, H. Larson, and W. Xu. On an equivalence of divisors on $\overline{M}_{0,n}$ from gromov-witten theory and conformal blocks. *Transformation Groups*, 2022.

[CMP08] P.-E. Chaput, L. Manivel, and N. Perrin. Quantum cohomology of minuscule homogeneous spaces. *Transform. Groups*, 13(1):47–89, 2008.

[CP11] P.-E. Chaput and N. Perrin. Rationality of some Gromov-Witten varieties and application to quantum K-theory. *Commun. Contemp. Math.*, 13(1):67–90, 2011.

[EG00] D. Edidin and W. Graham. Riemann-Roch for equivariant Chow groups. *Duke Math. J.*, 102(3):567–594, 2000.

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[Giv00] A. Givental. On the WDVV equation in quantum K-theory. *Michigan Math. J.*, 48:295–304, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[GL03] A. Givental and Y.-P. Lee. Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. *Invent. Math.*, 151(1):193–219, 2003.

[IMT15] H. Iritani, T. Milanov, and V. Tonita. Reconstruction and convergence in quantum K-theory via difference equations. *Int. Math. Res. Not. IMRN*, (11):2887–2937, 2015.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Lee04] Y.-P. Lee. Quantum K-theory. I. Foundations. *Duke Math. J.*, 121(3):389–424, 2004.

[LL12] N. C. Leung and C. Li. Classical aspects of quantum cohomology of generalized flag varieties. *Int. Math. Res. Not. IMRN*, (16):3706–3722, 2012.

[LM03] J. M. Landsberg and L. Manivel. On the projective geometry of rational homogeneous varieties. *Comment. Math. Helv.*, 78(1):65–100, 2003.

[LM14] C. Li and L. C. Mihalcea. K-theoretic Gromov-Witten invariants of lines in homogeneous spaces. *Int. Math. Res. Not. IMRN*, (17):4625–4664, 2014.

[LP04] Y.-P. Lee and R. Pandharipande. A reconstruction theorem in quantum cohomology and quantum K-theory. *Amer. J. Math.*, 126(6):1367–1379, 2004.

[PS] N. Perrin and M. Smirnov. On the big quantum cohomology of coadjoint varieties. arXiv:2112.12436.

[Ram85] A. Ramanathan. Schubert varieties are arithmetically Cohen-Macaulay. *Invent. Math.*, 80(2):283–294, 1985.

[Str02] E. Strickland. Lines in G/P. *Math. Z.*, 242(2):227–240, 2002.

[Woo05] C. T. Woodward. On D. Peterson's comparison formula for Gromov-Witten invariants of G/P. *Proc. Amer. Math. Soc.*, 133(6):1601–1609, 2005.

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