EULER CHARACTERISTICS IN THE QUANTUM K-THEORY OF FLAG VARIETIES

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Abstract. We prove that the sheaf Euler characteristic of the product of a Schubert class and an opposite Schubert class in the quantum K-theory ring of a (generalized) flag variety G/P is equal to q^d, where d is the smallest degree of a rational curve joining the two Schubert varieties. This implies that the sum of the structure constants of any product of Schubert classes is equal to 1. Along the way, we provide a description of the smallest degree d in terms of its projections to flag varieties defined by maximal parabolic subgroups.

1. Introduction

The goal of this paper is to relate distances between Schubert varieties in a complex flag variety $X = G/P$ to products of Schubert classes in the quantum K-theory ring $\mathrm{QK}_T(X)$.

The torus-equivariant K-theory ring $K_T(X)$ is an algebra over the ring of virtual representations $\Gamma = K_T(\mathrm{pt})$ of the maximal torus in G. As a module over Γ, the ring $K_T(X)$ has a basis consisting of the classes $\mathcal{O}_v = [\mathcal{O}_{X_v}]$ of the Schubert varieties $X_v \subset X$, and another basis consisting of the classes $\mathcal{O}^u = [\mathcal{O}_{X^u}]$ of the opposite Schubert varieties X^u.

Let $\Gamma[\![q]\!]$ be the ring of formal power series in variables q_β that correspond to the Schubert basis $\{[X_{s_\beta}]\}$ of $H_2(X,\mathbb{Z})$. Given any degree $d = \sum_\beta d_\beta [X_{s_\beta}]$ in $H_2(X,\mathbb{Z})$ we write $q^d = \prod_\beta q_\beta^{d_\beta}$. The (small, equivariant) quantum K-theory ring $\mathrm{QK}_T(X)$ of Givental [10] and Lee [15] is a $\Gamma[\![q]\!]$-algebra, which as a module over $\Gamma[\![q]\!]$ can be defined by $\mathrm{QK}_T(X) = K_T(X) \otimes_\Gamma \Gamma[\![q]\!]$. The product in $\mathrm{QK}_T(X)$ takes the form

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w,d} N_{u,v}^{w,d}\, q^d\, \mathcal{O}^w\,, \qquad (1)$$

Date: March 3, 2019.

2010 *Mathematics Subject Classification.* Primary 14N35; Secondary 19E08, 14N15, 14M15.

The authors acknowledge support from NSF grant DMS-1503662 (Buch), NSFC grants 11771455, 11831017, and Guangdong Introducing Innovative and Enterpreneurial Teams No. 2017ZT07X355 (Li), and the NSA Young Investigator Award H98320-16-1-0013 and a Simons Collaboration Grant (Mihalcea).

where the Schubert structure constants $N_{u,v}^{w,d} \in \Gamma$ are defined in terms of the K-theoretic Gromov-Witten invariants of X.

The sheaf Euler characteristic map $\chi_X : K_T(X) \to \Gamma$ is defined by

$$\chi_X([E]) = \sum_{i \geq 0} (-1)^i \, [H^i(X, E)]$$

for any equivariant vector bundle E. Equivalently, χ_X is the unique Γ-linear map satisfying $\chi_X(\mathcal{O}^u) = \chi_X(\mathcal{O}_v) = 1$. Let $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ denote the $\Gamma[\![q]\!]$-linear extension of χ_X. Our main result is the following theorem.

Theorem. We have $\chi(\mathcal{O}^u \star \mathcal{O}_v) = q^{\mathrm{dist}_X(u,v)}$, where $\mathrm{dist}_X(u,v) \in H_2(X,\mathbb{Z})$ denotes the smallest degree of a rational curve connecting X^u to X_v.

This result was proved earlier in [5] by Buch and Chung when X is a *cominuscule flag variety*, such as a Grassmann variety of Lie type A or a maximal isotropic Grassmannian of type B, C, or D.

Implicit in the statement is the claim that, given opposite Schubert varieties X^u and X_v in X, there exists a unique minimal degree $\mathrm{dist}_X(u,v)$ in $H_2(X,\mathbb{Z})$ for which X^u and X_v can be connected by a rational curve of this degree. We will refer to this degree as the *distance* between X^u and X_v. Results of Fulton and Woodward [9] show that this question is equivalent to the existence of a minimal degree d for which q^d occurs in the product $[X^u] \star [X_v]$ in the small quantum cohomology ring $\mathrm{QH}(X)$. Postnikov has proved in [16] that this is true when $X = G/B$ is a variety of complete flags. We deduce the existence of a minimal degree in general from these results. We also show that $\mathrm{dist}_X(u,v)$ can be expressed in terms of distances in flag varieties defined by maximal parabolic subgroups.

Let $\overline{\mathcal{M}}_{0,3}(X,d)$ denote the Kontsevich moduli space of stable maps to X of degree d and genus zero. Any 2-pointed K-theoretic Gromov-Witten invariant $\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d$ can be interpreted as the sheaf Euler characteristic of the Gromov-Witten variety $\mathrm{ev}_1^{-1}(X^u) \cap \mathrm{ev}_2^{-1}(X_v)$ in $\overline{\mathcal{M}}_{0,3}(X,d)$. It was proved in [3] that this Gromov-Witten variety is either empty or unirational with rational singularities. Since it is non-empty if and only if there exists a rational curve of degree d from X^u to X_v, the 2-point Gromov-Witten invariants of X are determined by the formula

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d = \begin{cases} 1 & \text{if } d \geq \mathrm{dist}_X(u,v); \\ 0 & \text{otherwise.} \end{cases}$$

We show that this is equivalent to our identity $\chi_X(\mathcal{O}^u \star \mathcal{O}_v) = q^{\mathrm{dist}_X(u,v)}$ by using the Frobenius property of the product $\star$ of $\mathrm{QK}_T(X)$. Still another equivalent formulation is that the small quantum K-metric of X is given by

$$(\!(\mathcal{O}^u, \mathcal{O}_v)\!) = \frac{q^{\mathrm{dist}_X(u,v)}}{\prod_\beta (1 - q_\beta)} \,.$$

According to the definition of the quantum K-theory ring $\mathrm{QK}_T(X)$, the product (1) of two Schubert classes could potentially have infinitely many non-zero terms. Our main result directly implies that, for all but finitely many degrees d, the sum of structure constants $\sum_w N_{u,v}^{w,d}$ is equal to zero. In addition, the sum of these sums over all degrees d is equal to 1:

$$\sum_d \sum_w N_{u,v}^{w,d} = 1\,.$$

It has recently been established that the quantum K-theory ring $\mathrm{QK}_T(X)$ satisfies *finiteness*, in the sense that only finitely many of the structure constants $N_{u,v}^{w,d}$ are non-zero. Equivalently, the quantum K-theory ring contains a subring defined by $\mathrm{QK}_T^{\mathrm{poly}}(X) = K_T(X) \otimes_\Gamma \Gamma[q]$. This was proved by Buch, Chaput, Mihalcea, and Perrin when the Picard group of X has rank one [6, 3, 4], by Kato [12, 11] for varieties of complete flags G/B, and by Anderson, Chen, and Tseng [1] for arbitrary flag varieties G/P. In addition, Kato's work proves a relation between the quantum K-theory ring $\mathrm{QK}_T(G/B)$ and the equivariant K-homology of affine Grassmannians that was conjectured in [14].

The sheaf Euler characteristic map $\chi_X : K_T(X) \to \Gamma$ is almost never a ring homomorphism, for example because $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_v) = 0 \neq 1 = \chi_X(\mathcal{O}^u) \cdot \chi_X(\mathcal{O}_v)$ whenever X^u and X_v are disjoint Schubert varieties in X. However, our main theorem implies that χ_X lifts to a well-defined ring homomorphism $\widehat{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ defined by $\widehat{\chi}(\mathcal{O}^u) = \widehat{\chi}(q_\beta) = 1$. We consider this as evidence that the quantum K-theory ring is a natural construction.

In Section 2 we prove the existence of the minimal degree $\mathrm{dist}_X(u,v)$ based on Fulton, Woodward, and Postnikov's results about quantum cohomology. Section 3 then defines the quantum K-theory ring $\mathrm{QK}_T(X)$ and proves the identity $\chi(\mathcal{O}^u \star \mathcal{O}_v) = q^{\mathrm{dist}_X(u,v)}$ and its consequences.

We thank David Anderson for helpful conversations.

2. The distance between Schubert varieties

2.1. **Flag varieties.** Let $X = G/P$ be a flag variety defined by a connected semisimple complex Lie group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = T$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives of the cosets in W/W_P. Let Φ denote the root system of G, with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$. The parabolic subgroup P is determined by the subset $\Delta_P = \{\beta \in \Delta \mid s_\beta \in W_P\}$. Each element $w \in W$ defines a B-stable Schubert variety $X_w = \overline{Bw.P}$ and a B^--stable (opposite) Schubert variety $X^w = \overline{B^-w.P}$. If $w \in W^P$ is a minimal representative, then $\dim(X_w) = \mathrm{codim}(X^w, X) = \ell(w)$.

The group $H_2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module, with a basis consisting of the Schubert classes $[X_{s_\beta}]$ for $\beta \in \Delta \smallsetminus \Delta_P$. Given two elements $d = \sum_\beta d_\beta [X_{s_\beta}]$

and $d' = \sum_\beta d'_\beta [X_{s_\beta}]$ expressed in this basis, we write $d \leq d'$ if and only if $d_\beta \leq d'_\beta$ for each $\beta \in \Delta \smallsetminus \Delta_P$. This defines a partial order on $H_2(X,\mathbb{Z})$.

For any root $\alpha \in \Phi$ that is not in the span Φ_P of Δ_P, there exists a unique irreducible T-invariant curve $X(\alpha) \subset X$ that connects the points $1.P$ and $s_\alpha.P$. An arbitrary irreducible T-invariant curve $C \subset X$ has the form $C = w.X(\alpha)$ for some $w \in W$ and $\alpha \in \Phi^+ \smallsetminus \Phi_P$.

2.2. **Quantum cohomology.** Given an effective degree $d \geq 0$ in $H_2(X,\mathbb{Z})$ we let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of all n-pointed stable maps $f : C \to X$ of arithmetic genus zero and degree $f_*[C] = d$. This space is equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$ for $1 \leq i \leq n$, where ev_i sends a stable map to the image of the i-th marked point in its domain. Given cohomology classes $\gamma_1, \dots, \gamma_n \in H^*(X,\mathbb{Z})$, the corresponding (cohomological) Gromov-Witten invariant of degree d is defined by

$$\langle \gamma_1, \dots, \gamma_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\gamma_1) \wedge \cdots \wedge \mathrm{ev}_n^*(\gamma_n)\,.$$

Let $\mathbb{Z}[q] = \mathbb{Z}[q_\beta : \beta \in \Delta \smallsetminus \Delta_P]$ denote a polynomial ring in variables q_β corresponding to the basis elements of $H_2(X,\mathbb{Z})$. Given any degree $d = \sum_\beta d_\beta [X_{s_\beta}] \in H_2(X,\mathbb{Z})$, we will write $q^d = \prod_\beta q_\beta^{d_\beta}$. The (small) quantum cohomology ring $\mathrm{QH}(X)$ is a $\mathbb{Z}[q]$-algebra which as a $\mathbb{Z}[q]$-module can be defined by $\mathrm{QH}(X) = H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The product is defined by

$$\gamma_1 \star \gamma_2 = \sum_{w,d \geq 0} \langle \gamma_1, \gamma_2, [X_w] \rangle_d \, q^d \, [X^w]$$

for $\gamma_1, \gamma_2 \in H^*(X,\mathbb{Z})$. Here we identify any class $\gamma \in H^*(X,\mathbb{Z})$ with $\gamma \otimes 1 \in \mathrm{QH}(X)$.

In the following, the image of a stable map to X will be called a stable curve in X. Given a stable curve $C \subset X$, we let $[C]$ denote the degree in $H_2(X;\mathbb{Z})$ defined by C. In particular, we set $[C] = 0$ if C is a single point. The following theorem is a subset of [9, Thm. 9.1].

Theorem 1 (Fulton and Woodward)**.** *Let $u, v \in W^P$ and $d \in H_2(X,\mathbb{Z})$. The Schubert varieties X^u and X_v can be connected by a stable curve of degree d if and only if $q^{d'}$ occurs in the product $[X^u] \star [X_v]$ for some $d' \leq d$.*

Let $Y = G/B$ denote the variety of complete flags. In this case the following was proved in [16, Cor. 3].

Theorem 2 (Postnikov)**.** *Let $u, v \in W$. There is a unique minimal degree $d_{\min}(u,v)$ in $H_2(Y,\mathbb{Z})$ for which $q^{d_{\min}(u,v)}$ occurs in $[Y^u] \star [Y^v]$.*

2.3. **Distance.** In this section we extend Postnikov's result to arbitrary flag varieties $X = G/P$. For each simple root $\beta \in \Delta$ we set $Z_\beta = G/P_\beta$, where $P_\beta \subset G$ is the unique maximal parabolic subgroup containing B for which $s_\beta \notin W_{P_\beta}$. The group $H_2(Z_\beta,\mathbb{Z})$ is free of rank one, generated by $[(Z_\beta)_{s_\beta}]$. To simplify notation we identify $H_2(Z_\beta,\mathbb{Z})$ with $\mathbb{Z}$, by identifying $[(Z_\beta)_{s_\beta}]$

with 1. Let $\pi_\beta : X \to Z_\beta$ denote the projection. Any degree $d \in H_2(X,\mathbb{Z})$ is then given by $d = \sum_\beta d_\beta [X_{s_\beta}]$, where $d_\beta = (\pi_\beta)_*(d)$.

Given $u, v \in W$ we let $\mathrm{dist}_\beta(u,v) \in \mathbb{Z}$ denote the smallest degree of a stable curve in Z_β connecting the Schubert varieties $(Z_\beta)^u$ and $(Z_\beta)_v$. This is well defined since the natural numbers are well ordered. Define the *distance* between X^u and X_v to be the class in $H_2(X,\mathbb{Z})$ given by

$$\mathrm{dist}_X(u,v) = \sum_{\beta \in \Delta \smallsetminus \Delta_P} \mathrm{dist}_\beta(u,v)\,[X_{s_\beta}]\,. \tag{2}$$

This terminology is justified by Theorem 5 below. In the case where both X^u and X_v are single points, the identity (2) can also be found in [2].

Lemma 3. *Let $u, v \in W$ and $\beta \in \Delta \smallsetminus \Delta_P$. Then X^u and X_v can be connected by a stable curve $C \subset X$ for which $(\pi_\beta)_*[C] = \mathrm{dist}_\beta(u,v)$.*

Proof. Since the intersection $\mathrm{ev}_1^{-1}((Z_\beta)^u) \cap \mathrm{ev}_2^{-1}((Z_\beta)_v)$ is a closed subvariety of $\overline{\mathcal{M}}_{0,3}(Z_\beta, \mathrm{dist}_\beta(u,v))$ that is invariant under the action of T, we may find a T-invariant stable curve $\widetilde{C} \subset Z_\beta$ of degree $\mathrm{dist}_\beta(u,v)$ that connects $(Z_\beta)^u$ and $(Z_\beta)_v$. This implies that $\widetilde{C}$ is a chain of irreducible T-invariant curves, that is, there exist $\kappa_0, \kappa_1, \dots, \kappa_m \in W$ and $\alpha_1, \dots, \alpha_m \in \Phi^+ \smallsetminus \Phi_P$ such that $\kappa_0.P_\beta \in (Z_\beta)^u$, $\kappa_m.P_\beta \in (Z_\beta)_v$, $\kappa_i = \kappa_{i-1} s_{\alpha_i}$ for $1 \le i \le m$, and $\widetilde{C} = \kappa_1.Z_\beta(\alpha_1) \cup \dots \cup \kappa_m.Z_\beta(\alpha_m)$. Since we have $\kappa_0 w' \ge u$ for some $w' \in W_{P_\beta}$, there exists a stable curve $C' \subset X$ connecting $\kappa_0.P$ to a point in X^u, such that $(\pi_\beta)_*[C'] = 0$. Similarly we can find $C'' \subset X$ connecting $\kappa_m.P$ to a point in X_v such that $(\pi_\beta)_*[C''] = 0$. We can therefore take $C \subset X$ to be the union of C', C'', and the curves $\kappa_i.X(\alpha_i)$ for $1 \le i \le m$. □

Recall that $[Y_v] = [Y^{w_0 v}]$, where w_0 is the longest element of W. It therefore follows from Theorem 2 that $q^{d_{\min}(u, w_0 v)}$ is the unique minimal power of q that occurs in the product $[Y^u] \star [Y_v] \in \mathrm{QH}(Y)$.

Lemma 4. *We have $\mathrm{dist}_Y(u,v) = d_{\min}(u, w_0 v)$.*

Proof. Write $d = d_{\min}(u, w_0 v) = \sum_{\beta \in \Delta} d_\beta [Y_{s_\beta}]$ and let $\beta \in \Delta$ be given. Since q^d occurs in the quantum product $[Y^u] \star [Y_v]$, there exists a stable curve $C \subset Y$ of degree d from Y^u to Y_v by Theorem 1. Since $\pi_\beta(C) \subset Z_\beta$ is a curve of degree at most d_β from $(Z_\beta)^u$ to $(Z_\beta)_v$, we have $d_\beta \ge \mathrm{dist}_\beta(u,v)$. On the other hand, according to Lemma 3 we can find a stable curve $C \subset Y$ from Y^u to Y_v such that $(\pi_\beta)_*[C] = \mathrm{dist}_\beta(u,v)$. Theorem 1 then implies that the product $[Y^u] \star [Y_v]$ contains a power $q^{d'}$ for which $d' \le [C]$, and the minimality of $d_{\min}(u, w_0 v)$ implies that $d_{\min}(u, w_0 v) \le d'$. We deduce that $d_\beta \le \mathrm{dist}_\beta(u,v)$, which completes the proof. □

Theorem 5. *Let $u, v \in W$ and $d \in H_2(X,\mathbb{Z})$. There exists a stable curve of degree d from X^u to X_v if and only if $d \ge \mathrm{dist}_X(u,v)$.*

Proof. Write $d = \sum_\beta d_\beta [X_{s_\beta}]$. The implication 'only if' follows because $(\pi_\beta)_*(d) = d_\beta$, as in the proof of Lemma 4. Assume that $d \ge \mathrm{dist}_X(u,v)$

and define $d' = \sum_{\beta\in\Delta} d'_\beta[Y_{s_\beta}] \in H_2(Y,\mathbb{Z})$ by $d'_\beta = d_\beta$ for $\beta \in \Delta \smallsetminus \Delta_P$ and $d'_\beta = \mathrm{dist}_\beta(u,v)$ for $\beta \in \Delta_P$. Since $d' \geq \mathrm{dist}_Y(u,v)$, it follows from Theorem 1 that there exists a stable curve $C \subset Y$ of degree d' from Y^u to Y_v. The image of C under the projection $Y \to X$ is a curve from X^u to X_v of degree at most d. By possibly attaching some extra components, we obtain the desired stable curve of degree d. □

3. Quantum K-theory

3.1. K-theory. The equivariant K-theory ring $K_T(X)$ is the Grothendieck ring of T-equivariant algebraic vector bundles on X, equipped with the product coming from the tensor product of vector bundles. This ring is an algebra over the ring $\Gamma = K_T(\mathrm{pt})$ of virtual representations of T, with a basis consisting of the Schubert classes $\mathcal{O}_v = [\mathcal{O}_{X_v}]$ for $v \in W^P$; the opposite Schubert classes $\mathcal{O}^u = [\mathcal{O}_{X^u}]$ for $u \in W^P$ form another basis. The ring Γ can be identified with a ring of Laurent polynomials in as many variables as the rank of T.

The sheaf Euler characteristic map $\chi_X : K_T(X) \to \Gamma$ is defined as the pushforward map along the structure morphism $X \to \{\mathrm{pt}\}$; that is,

$$\chi_X([E]) = \sum_{i\geq 0} (-1)^i \, [H^i(X,E)] \,,$$

where the sheaf cohomology group $H^i(X,E)$ is regarded as a representation of T. Equivalently, χ_X is the unique Γ-linear map defined by $\chi_X(\mathcal{O}^w) = \chi_X(\mathcal{O}_w) = 1$ [17, 18]. More generally, if $Z \subset X$ is any closed T-invariant subvariety that is unirational and has rational singularities, then $\chi_X([\mathcal{O}_Z]) = 1$ [8, Cor. 4.18].

3.2. Gromov-Witten invariants. Given classes $\gamma_1,\dots,\gamma_n \in K_T(X)$ and a degree $d \in H_2(X,\mathbb{Z})$, the corresponding (equivariant, K-theoretic, n-pointed, genus zero) Gromov-Witten invariant of X is defined by

$$\langle \gamma_1,\dots,\gamma_n\rangle_d = \chi_{\overline{\mathcal{M}}_{0,n}(X,d)}(\mathrm{ev}_1^*(\gamma_1)\cdot \ldots \cdot \mathrm{ev}_n^*(\gamma_n)) \ \in \Gamma \,.$$

Since the moduli space $\overline{\mathcal{M}}_{0,n}(X,d)$ is empty for $d=0$ and $n \leq 2$, we will use the convention that $\langle \gamma_1,\dots,\gamma_n\rangle_0 = \chi_X(\gamma_1\cdot\ldots\cdot\gamma_n)$ for any $n \geq 0$. This is consistent with the above definition since the general fibers of the forgetful map $\overline{\mathcal{M}}_{0,n+1}(X,d) \to \overline{\mathcal{M}}_{0,n}(X,d)$ are projective lines and all evaluation maps on $\overline{\mathcal{M}}_{0,n}(x,0)$ are identical (see [13, Thm. 7.1] or [6, Thm. 3.1]).

The two-point Gromov-Witten invariant $\langle \mathcal{O}^u, \mathcal{O}_v\rangle_d$ is equal to the sheaf Euler characteristic of the Gromov-Witten variety $\mathrm{ev}_1^{-1}(X^u) \cap \mathrm{ev}_2^{-1}(X_v)$, which by [3, Cor. 3.3] is either empty or unirational with rational singularities. Since the question of non-emptiness is determined by Theorem 5, we obtain the following identity, interpreting the formula for 2-pointed K-theoretic Gromov-Witten invariants from [7, Remark 7.5] in terms of the distance function.

Proposition 6. *For $u, v \in W^P$ and $d \in H_2(X,\mathbb{Z})$ we have*

$$\langle \mathcal{O}^u, \mathcal{O}_v \rangle_d = \begin{cases} 1 & \text{if } d \geq \mathrm{dist}_X(u,v); \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6 for $d = 0$ recovers the well known fact that $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_v) = 1$ whenever $u \leq v$ in the Bruhat order on W^P, and $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_v) = 0$ otherwise. In particular, the bilinear map $K_T(X) \times K_T(X) \to \Gamma$ defined by $(\gamma_1, \gamma_2) \mapsto \chi_X(\gamma_1 \cdot \gamma_2)$ is a perfect pairing.

3.3. **Quantum K-theory.** The (small, equivariant) quantum K-theory ring $\mathrm{QK}_T(X)$ of Givental [10] and Lee [15] is an algebra over the ring of formal power series $\Gamma[\![q]\!] = \Gamma[\![q_\beta : \beta \in \Delta \smallsetminus \Delta_P]\!]$. As a module over $\Gamma[\![q]\!]$ it is defined by $\mathrm{QK}_T(X) = K_T(X) \otimes_\Gamma \Gamma[\![q]\!]$. The *quantum K-metric* is the $\Gamma[\![q]\!]$-bilinear pairing $\mathrm{QK}_T(X) \times \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ determined by

$$(\!(\gamma_1, \gamma_2)\!) \;=\; \sum_{d \geq 0} q^d \langle \gamma_1, \gamma_2 \rangle_d$$

for $\gamma_1, \gamma_2 \in K_T(X)$. The *quantum product* $\star$ is the unique $\Gamma[\![q]\!]$-bilinear product $\mathrm{QK}_T(X) \times \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ determined by

$$(\!(\gamma_1 \star \gamma_2, \gamma_3)\!) \;=\; \sum_{d \geq 0} q^d \langle \gamma_1, \gamma_2, \gamma_3 \rangle_d$$

for all $\gamma_1, \gamma_2, \gamma_3 \in K_T(X)$. It was proved by Givental [10] that this product $\star$ is associative. The symmetry of Gromov-Witten invariants implies that the Frobenius property $(\!(\gamma_1 \star \gamma_2, \gamma_3)\!) = (\!(\gamma_1, \gamma_2 \star \gamma_3)\!)$ holds for all $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{QK}_T(X)$. The string identity $\langle \gamma_1, \ldots, \gamma_n, 1 \rangle_d = \langle \gamma_1, \ldots, \gamma_n \rangle_d$ implies that $1 \in K_T(X)$ is a multiplicative unit in $\mathrm{QK}_T(X)$.

Remark 7. Consider the formal linear combination $t = \sum_{u \in W^P} t_u \mathcal{O}^u$, where the coefficients t_u are independent commuting variables. The *quantum K-potential* of X is the generating function

$$G(t,q) = \sum_{n \geq 0} \sum_{d \geq 0} \frac{q^d}{n!} \langle t, \ldots, t \rangle_{d,n}\,,$$

where $\langle t, \ldots, t \rangle_{d,n} = \langle t, \ldots, t \rangle_d$ denotes an n-pointed Gromov-Witten invariant with t repeated n times. The quantum K-metric can be obtained from $G(t,q)$ as

$$(\!(\mathcal{O}^u, \mathcal{O}^v)\!) \;=\; \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_v} G(t,u) \Big|_{t=0}\,,$$

and the quantum product $\star$ is determined by

$$(\!(\mathcal{O}^u \star \mathcal{O}^v, \mathcal{O}^w)\!) \;=\; \frac{\partial}{\partial t_u} \frac{\partial}{\partial t_v} \frac{\partial}{\partial t_w} G(t,u) \Big|_{t=0}\,.$$

If we do not specialize the variables t_u to zero, then we arrive at the big quantum K-theory ring of X.

3.4. **Sums of structure constants.** Let $\chi : \mathrm{QK}_T(X) \to \Gamma[\![q]\!]$ denote the $\Gamma[\![q]\!]$-linear extension of the sheaf Euler characteristic map χ_X. The following identity is our main result. It was known in the special case where X is a cominuscule flag variety [5].

Theorem 8. *For $u, v \in W^P$ we have $\chi(\mathcal{O}^u \star \mathcal{O}_v) = q^{\mathrm{dist}_X(u,v)}$.*

Proof. It follows from Proposition 6 that

$$(\!(\mathcal{O}^u, \mathcal{O}_v)\!) = \frac{q^{\mathrm{dist}_X(u,v)}}{\prod_\beta (1-q_\beta)}\,,$$

where the product is over $\beta \in \Delta \smallsetminus \Delta_P$. This identity implies that

$$\chi(\gamma) = (\!(\gamma, 1)\!) \textstyle\prod_\beta (1-q_\beta)$$

for any class $\gamma \in \mathrm{QK}_T(X)$. We obtain

$$\begin{aligned} \chi(\mathcal{O}^u \star \mathcal{O}_v) &= (\!(\mathcal{O}^u \star \mathcal{O}_v, 1)\!) \textstyle\prod_\beta (1-q_\beta) \;=\; (\!(\mathcal{O}^u, \mathcal{O}_v \star 1)\!) \textstyle\prod_\beta (1-q_\beta) \\ &= (\!(\mathcal{O}^u, \mathcal{O}_v)\!) \textstyle\prod_\beta (1-q_\beta) \;=\; q^{\mathrm{dist}_X(u,v)}\,, \end{aligned}$$

as required. □

The Schubert structure constants of the quantum K-theory ring $\mathrm{QK}_T(X)$ are the classes $N_{u,v}^{w,d} \in \Gamma$, indexed by $u, v, w \in W^P$ and $d \in H_2(X, \mathbb{Z})$, defined by

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w, d \geq 0} N_{u,v}^{w,d}\, q^d\, \mathcal{O}^w\,.$$

Corollary 9. *Let $u, v \in W^P$. For all but finitely many degrees $d \in H_2(X, \mathbb{Z})$, the sum $\sum_{w \in W^P} N_{u,v}^{w,d}$ is equal to zero. Moreover, we have*

$$\sum_{d \in H_2(X,\mathbb{Z})} \sum_{w \in W^P} N_{u,v}^{w,d} \;=\; 1\,.$$

Proof. Write $\mathcal{O}^v = \sum_{z \in W^P} f_z\, \mathcal{O}_z$ where $f_z \in \Gamma$. We then have

$$\sum_{w,d \geq 0} N_{u,v}^{w,d}\, q^d = \chi(\mathcal{O}^u \star \mathcal{O}^v) = \sum_z f_z\, \chi(\mathcal{O}^u \star \mathcal{O}_z) = \sum_z f_z\, q^{\mathrm{dist}_X(u,z)}\,.$$

The first claim follows because the last sum has finitely many terms, and the second claim holds because $\sum_z f_z = \chi_X(\mathcal{O}^v) = 1$. □

3.5. **Ring homomorphism.** Let $\Gamma[q] \subset \Gamma[\![q]\!]$ be the subring of polynomials in the variables q_β, and set $\mathrm{QK}_T^{\mathrm{poly}}(X) = K_T(X) \otimes_\Gamma \Gamma[q]$. It has recently been proved that the quantum K-theory ring $\mathrm{QK}_T(X)$ satisfies *finiteness*, that is, this ring contains $\mathrm{QK}_T^{\mathrm{poly}}(X)$ as a subring [6, 3, 4, 12, 11, 1]. While the sheaf Euler characteristic map $\chi_X : K_T(X) \to \Gamma$ is a ring homomorphism only when X is a single point, our last result shows that this changes if $K_T(X)$ is replaced by $\mathrm{QK}_T^{\mathrm{poly}}(X)$.

Corollary 10. *There is a well-defined ring homomorphism* $\widehat{\chi} : \mathrm{QK}_T^{\mathrm{poly}}(X) \to \Gamma$ *defined by* $\widehat{\chi}(\mathcal{O}^u) = \chi(q_\beta) = 1$ *for all* $w \in W^P$ *and* $\beta \in \Delta \smallsetminus \Delta_P$.

Proof. We may consider Γ as an algebra over $\Gamma[q]$ through the ring homomorphism $\mu : \Gamma[q] \to \Gamma$ defined by $\mu(q_\beta) = 1$ for $\beta \in \Delta \smallsetminus \Delta_P$ and $\mu(a) = a$ for $a \in \Gamma$. We can define $\widehat{\chi}$ as a homomorphism of $\Gamma[q]$-modules by setting $\widehat{\chi} = \mu\chi$. Since both of the sets $\{\mathcal{O}^u \mid u \in W^P\}$ and $\{\mathcal{O}_v \mid v \in W^P\}$ are bases for $\mathrm{QK}_T^{\mathrm{poly}}(X)$ over $\Gamma[q]$, it follows from the identity $\widehat{\chi}(\mathcal{O}^u \star \mathcal{O}_v) = 1 = \widehat{\chi}(\mathcal{O}^u) \cdot \widehat{\chi}(\mathcal{O}_v)$ that $\widehat{\chi}$ is a $\Gamma[q]$-algebra homomorphism, as required. $\square$

It would be interesting to know if Corollary 10 can be generalized beyond the setting of flag varieties.

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