A LITTLEWOOD-RICHARDSON RULE FOR THE K-THEORY OF GRASSMANNIANS

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ABSTRACT. We prove an explicit combinatorial formula for the structure constants of the Grothendieck ring of a Grassmann variety with respect to its basis of Schubert structure sheaves. We furthermore relate K-theory of Grassmannians to a bialgebra of stable Grothendieck polynomials, which is a K-theory parallel of the ring of symmetric functions.

1. INTRODUCTION

Let $X = \mathrm{Gr}(d, \mathbb{C}^n)$ be the Grassmann variety of d dimensional subspaces of $\mathbb{C}^n$. The goal of this paper is to give an explicit combinatorial description of the Grothendieck ring $K^\circ X$ of algebraic vector bundles on X.

K-theory of Grassmannians is a special case of K-theory of flag varieties, which was studied by Kostant and Kumar [13] and by Demazure [5]. Lascoux and Schützenberger defined Grothendieck polynomials which give formulas for the structure sheaves of the Schubert varieties in a flag variety [16, 14]. The combinatorial understanding of these polynomials was further developed by Fomin and Kirillov [9, 8].

Recall that if $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d)$ is a partition with d parts and $\lambda_1 \leq n-d$, then the Schubert variety in X associated to λ is the subset

$$\Omega_\lambda = \{V \in \mathrm{Gr}(d, \mathbb{C}^n) \mid \dim(V \cap \mathbb{C}^{n-d+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq d\}\,. \tag{1.1}$$

Here $\mathbb{C}^k \subset \mathbb{C}^n$ denotes the subset of vectors whose last $n-k$ components are zero. The codimension of Ω_λ is equal to the weight $|\lambda| = \sum \lambda_i$ of λ. If we identify partitions with their Young diagrams, then a Schubert variety Ω_μ is contained in Ω_λ if and only μ contains λ. From the fact that the open Schubert cells $\Omega_\lambda^\circ = \Omega_\lambda \smallsetminus \cup_{\mu \supsetneq \lambda} \Omega_\mu$ form a cell decomposition of X, one can deduce that the classes of the structure sheaves $\mathcal{O}_{\Omega_\lambda}$ form a basis for the Grothendieck ring of X.

We will study the structure constants for $K^\circ X$ with respect to this basis. These are the unique integers $c_{\lambda\mu}^\nu$ such that

$$[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] = \sum_\nu c_{\lambda\mu}^\nu \, [\mathcal{O}_{\Omega_\nu}]\,. \tag{1.2}$$

These constants depend only on the partitions λ, μ, and ν, not on the Grassmannian where the Schubert varieties for these partitions are realized. Furthermore $c_{\lambda\mu}^\nu$ is non-zero only if $|\nu| \geq |\lambda| + |\mu|$. The constants $c_{\lambda\mu}^\nu$ for which $|\nu| = |\lambda| + |\mu|$ are the usual Littlewood-Richardson coefficients, i.e. the structure constants for the

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cohomology ring $H^*(X)$ with respect to the basis of cohomology classes of Schubert varieties. Another known case is a Pieri formula of Lenart which expresses the coefficients $c^{\nu}_{\lambda,(k)}$ for multiplying with the structure sheaf of a special Schubert variety $\Omega_{(k)}$ as binomial coefficients [18]. Notice that since the duality isomorphism $\mathrm{Gr}(d,\mathbb{C}^n) \longrightarrow \mathrm{Gr}(n-d,\mathbb{C}^n)$ takes Ω_λ to the Schubert variety $\Omega_{\lambda'}$ for the conjugate partition [10, Ex. 9.20], the structure constants must satisfy $c^{\nu}_{\lambda\mu} = c^{\nu'}_{\lambda'\mu'}$.

Our main result is an explicit combinatorial formula stating that the coefficient $c^{\nu}_{\lambda\mu}$ is $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of objects called *set-valued tableaux* which satisfy certain properties. Set-valued tableaux are similar to semistandard Young tableaux, but allow a non-empty set of integers in each box of a Young diagram rather than a single integer. When $|\nu| = |\lambda| + |\mu|$ our formula specializes to the classical Littlewood-Richardson rule.

Our formula implies that if $c^{\nu}_{\lambda\mu}$ is not zero, then ν is contained in the union of all partitions ρ of weight $|\rho| = |\lambda| + |\mu|$ such that the Littlewood-Richardson coefficient $c^{\rho}_{\lambda\mu}$ is non-zero. Geometrically this says that if a structure sheaf $[\mathcal{O}_{\Omega_\nu}]$ occurs in the product $[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}]$, then Ω_ν must contain the intersection of all Schubert varieties Ω_ρ which appear in the product of the cohomology classes of Ω_λ and Ω_μ. Our formula furthermore implies that there exists a flag of Schubert varieties $\Omega_\nu = \Omega_{\nu(0)} \subset \Omega_{\nu(1)} \subset \cdots \subset \Omega_{\nu(k)} = \Omega_\rho$ such that the dimension jumps by one at each step, each structure sheaf $[\mathcal{O}_{\Omega_{\nu(i)}}]$ occurs in the K-theory product $[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}]$, and Ω_ρ occurs in the cohomology product of Ω_λ and Ω_μ. We do not know any geometric reasons for these facts or for the alternating signs of the structure constants.

As a particular consequence of the above, note that for fixed λ and μ there are finitely many partitions ν which give a non-zero coefficient $c^{\nu}_{\lambda\mu}$. This is already surprising since one might conceivably get arbitrarily many such constants by realizing the product $[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}]$ in larger and larger Grassmannians. This observation allows us to define a commutative ring $\Gamma = \bigoplus \mathbb{Z} \cdot G_\lambda$ with a formal basis $\{G_\lambda\}$ indexed by partitions and multiplication defined by $G_\lambda \cdot G_\mu = \sum_\nu c^{\nu}_{\lambda\mu} G_\nu$. The Grothendieck ring $K^\circ \mathrm{Gr}(d,\mathbb{C}^n)$ is then the quotient of this ring by the ideal spanned by the basis elements G_λ for partitions that do not fit in a rectangle with d rows and $n-d$ columns. The pullbacks of Grothendieck rings defined by the natural embeddings $\mathrm{Gr}(d_1,\mathbb{C}^{n_1}) \times \mathrm{Gr}(d_2,\mathbb{C}^{n_2}) \subset \mathrm{Gr}(d_1+d_2,\mathbb{C}^{n_1+n_2})$ furthermore define a coproduct on Γ which makes it a bialgebra.

This bialgebra Γ can be seen as a K-theory parallel of the ring of symmetric functions [20, 10], which in a similar way describes the cohomology of Grassmannians, in addition to representation theory of symmetric and general linear groups and numerous other areas. Furthermore, if we define a filtration of Γ by ideals $\Gamma_p = \bigoplus_{|\lambda| \geq p} \mathbb{Z} \cdot G_\lambda$, then the associated graded bialgebra is naturally the ring of symmetric functions. This filtration corresponds to Grothendieck's γ-filtration of the K-ring of any non-singular algebraic variety. In general the associated graded ring is isomorphic to the Chow ring of the variety after tensoring with $\mathbb{Q}$.

We will realize the algebra Γ as the linear span of all stable Grothendieck polynomials, which we show has a basis indexed by Grassmannian permutations. The Littlewood-Richardson rule is proved by defining stable Grothendieck polynomials in non-commutative variables, and showing that these polynomials multiply exactly like those in commutative variables. In order to carry out this construction we will define a *jeu de taquin* algorithm for set-valued tableaux.

In Section 2 we fix the notation concerning Grothendieck polynomials. In Section 3 we then prove a formula for stable Grothendieck polynomials of 321-avoiding permutations in terms of set-valued tableaux. This formula uses the skew diagram associated to a 321-avoiding permutation [1] and is derived from a more general formula for Grothendieck polynomials of Fomin and Kirillov [9]. Section 4 develops a column bumping algorithm for set-valued tableaux, which in Section 5 is used to prove the Littlewood-Richardson rule for the structure constants $c^{\nu}_{\lambda\mu}$ in Γ. In Section 6 we derive similar Littlewood-Richardson rules for the coproduct in Γ and for writing the stable Grothendieck polynomial of any 321-avoiding permutation as a linear combination of the basis elements of Γ. In Section 7 we deduce a number of consequences of these rules, including a Pieri formula for the coproduct in Γ, a result about multiplicity free products, and the above described bound on partitions ν for which $c^{\nu}_{\lambda\mu}$ is not zero. In Section 8 the relationship between Γ and K-theory of Grassmannians is established. In addition we use the methods developed in this paper to give simple proofs of some unpublished results of A. Knutson regarding triple intersections in K-theory. Section 9 finally contains a discussion of the overall structure of the bialgebra Γ. We show that if the inverse of the element $t = 1 - G_1$ is joined to Γ then the result Γ_t is a Hopf algebra. We furthermore pose a conjecture which implies that the Abelian group scheme $\operatorname{Spec}\Gamma_t$ looks like an infinite affine space minus a hyperplane. We conclude by raising some additional questions. We hope that the statements in the last two sections will be comprehensible after reading Section 2 and the first seven lines of Section 6.

This paper came out of a project aimed at finding a formula for the structure sheaf of a quiver variety. We will present such a formula in [2], thus generalizing our earlier results with W. Fulton regarding the cohomology class of a quiver variety [3]. The proof of the cohomology formula was relatively simple, because some powerful cohomological tools related to the ring of symmetric functions were already available. In order to obtain the K-theory formula, we have found replacements for these tools which work in K-theory. The first part of this is the construction of the bialgebra Γ which is carried out in the present paper. Especially the coproduct on Γ is important for the applications to quiver varieties. The remaining tools consist of a K-theory parallel of a Gysin formula of Pragacz [21], in addition to some extra combinatorics used to put everything together. This will be described in [2].

We thank Fulton for numerous helpful discussions and suggestions during the project. For carrying out the work in this paper, it has been invaluable to speak to S. Fomin, from whom we have learned very much about Grothendieck polynomials during several fruitful discussions. In particular we thank Fomin for supplying the proof of Lemma 3.4 and for suggestions that led to simplifications of many proofs. We also thank F. Sottile for useful discussions about stable Grothendieck polynomials. We are grateful to A. Knutson for sharing his ideas about K-theoretic triple intersections and for allowing us to report about them here. Finally, we thank A. Lascoux for informing us about a remarkable recursive formula for stable Grothendieck polynomials [15], which has turned a conjecture from the first preprint of this paper into Theorem 6.14.

2. Grothendieck polynomials

In this section we fix the notation regarding Grothendieck polynomials and stable Grothendieck polynomials. Grothendieck polynomials were introduced by Lascoux and Schützenberger as representatives for the structure sheaves of the Schubert varieties in a Flag variety [16, 14]. For any permutation $w \in S_n$ we define the *double Grothendieck polynomial* $\mathfrak{G}_w = \mathfrak{G}_w(x;y)$ as follows. If w is the longest permutation $w_0 = n\,(n-1)\cdots 2\,1$ we set

$$\mathfrak{G}_{w_0} = \prod_{i+j\leq n} (x_i + y_j - x_i\, y_j)\,.$$

Otherwise we can find a simple reflection $s_i = (i, i+1) \in S_n$ such that $\ell(w\, s_i) = \ell(w) + 1$. Here $\ell(w)$ denotes the length of w, which is the smallest number ℓ for which w can be written as a product of ℓ simple reflections. We then define

$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{w s_i})$$

where π_i is the *isobaric divided difference operator* given by

$$\pi_i(f) = \frac{(1-x_{i+1})f(x_1,x_2,\dots) - (1-x_i)f(\dots,x_{i+1},x_i,\dots)}{x_i - x_{i+1}}\,.$$

This definition is independent of our choice of the simple reflection s_i since the operators π_i satisfy the Coxeter relations.

Notice that the longest element in S_{n+1} is $w_0^{(n+1)} = w_0 \cdot s_n \cdot s_{n-1} \cdots s_1$. Since $\pi_n \cdot \pi_{n-1} \cdots \pi_1$ applied to the Grothendieck polynomial for $w_0^{(n+1)}$ is equal to $\mathfrak{G}_{w_0}$, it follows that $\mathfrak{G}_w$ does not depend on which symmetric group w is considered an element of.

Let $1^m \times w \in S_{m+n}$ denote the permutation which is the identity on $\{1,2,\dots,m\}$ and which maps j to $w(j-m)+m$ for $j > m$. Fomin and Kirillov have shown that when m grows to infinity, the coefficient of each fixed monomial in $\mathfrak{G}_{1^m\times w}$ eventually becomes stable [9]. The *stable Grothendieck polynomial* $G_w \in \mathbb{Z}[\![x_i, y_i]\!]_{i\geq 1}$ is defined as the resulting power series:

$$G_w = G_w(x;y) = \lim_{m\to\infty} \mathfrak{G}_{1^m\times w}\,.$$

Fomin and Kirillov also proved that this power series is symmetric in the variables $\{x_i\}$ and $\{y_i\}$ separately, and that

$$G_w(1-e^{-x}; 1-e^{y}) = G_w(1-e^{-x_1}, 1-e^{-x_2}, \dots; 1-e^{y_1}, 1-e^{y_2}, \dots)$$

is super symmetric, i.e. if one sets $x_1 = y_1$ in this expression then the result is independent of x_1 and y_1.

If we put all the variables y_i equal to zero in $\mathfrak{G}_w(x;y)$, we obtain the single Grothendieck polynomial $\mathfrak{G}_w(x) = \mathfrak{G}_w(x;0)$. Similarly the single stable Grothendieck polynomial for w is defined as $G_w(x) = G_w(x;0)$. Notice that the super symmetry of $G_w(1-e^{-x}; 1-e^{y})$ implies that the double stable Grothendieck polynomial $G_w(x;y)$ is uniquely determined by the single polynomial $G_w(x)$ [24, 20]. We will use the notation $G_w(x_1,x_2,\dots,x_p; y_1,y_2,\dots,y_q)$ for the polynomial obtained by setting $x_i = 0$ for $i > p$ and $y_j = 0$ for $j > q$ in the stable polynomial $G_w(x;y)$.

If $\lambda \subset \nu$ are two partitions, let ν/λ denote the skew diagram of boxes in ν which are not in λ, and let $|\nu/\lambda|$ be the number of boxes in this diagram. Now choose a numbering of the north-west to south-east diagonals in the diagram with positive integers, which increase consecutively from south-west to north-east. For example, if $\nu = (4,3,2)$ and $\lambda = (1)$, and if the bottom-left box in ν/λ is in diagonal number 3, then the numbering is given by the picture:

	6	7	8
4	5	6	
3	4		

Let $(i_1, i_2, \dots, i_m)$ be the sequence of diagonal numbers of the boxes in ν/λ when these boxes are read from right to left and then from bottom to top. We then let $w_{\nu/\lambda} = s_{i_1} s_{i_2} \cdots s_{i_m}$ be the product of the corresponding simple reflections. For the skew shape above this gives $w_{\nu/\lambda} = s_4\, s_3\, s_6\, s_5\, s_4\, s_8\, s_7\, s_6 = 1\,2\,5\,7\,3\,9\,4\,6\,8$. Notice that $w_{\nu/\lambda}$ depends on the numbering of the diagonals as well as the diagram ν/λ. A theorem of Billey, Jockusch, and Stanley [1] says that a permutation w can be obtained from a skew diagram in this way, if and only if it is *321-avoiding*, i.e. there are no integers $i < j < k$ for which $w(i) > w(j) > w(k)$.

Permutations obtained from different numberings of the diagonals in the same diagram ν/λ differ only by a shift. In other words, if $w_{\nu/\lambda}$ is the permutation corresponding to the numbering which puts the bottom-left box in diagonal number one, then any other numbering will give a permutation of the form $1^m \times w_{\nu/\lambda}$. We can therefore define the stable Grothendieck polynomial for any skew diagram by $G_{\nu/\lambda}(x;y) = G_{w_{\nu/\lambda}}(x;y)$.

If the skew diagram is a partition λ, and if p is the number of the diagonal containing the box in its upper-left corner, then w_λ is called the *Grassmannian permutation for* λ *with descent in position* p. This permutation is given by $w_\lambda(i) = i + \lambda_{p+1-i}$ for $1 \leq i \leq p$ and $w_\lambda(i) < w_\lambda(i+1)$ for $i \neq p$. Notice that p must be greater than or equal to the length of λ, which is the largest number $\ell = \ell(\lambda)$ for which λ_ℓ is non-zero.

It follows from the definitions that the term of lowest total degree in a Grothendieck polynomial $\mathfrak{G}_w(x;y)$ is the Schubert polynomial $\mathfrak{S}_w(x;-y)$ for the same permutation [16]. This implies that the lowest term of $G_\lambda(x)$ is the Schur function $s_\lambda(x)$ [20, 19]. In particular the polynomials G_λ for all partitions λ are linearly independent. We define Γ to be the linear span of all stable Grothendieck polynomials for Grassmannian permutations:

$$\Gamma = \bigoplus_\lambda \mathbb{Z} \cdot G_\lambda \subset \mathbb{Z}[\![x_1, x_2, \dots, y_1, y_2, \dots]\!] \,.$$

This group is the main object of study in this paper. For example, Corollary 5.5 and Theorem 6.13 below will show that Γ is closed under multiplication and that it contains all stable Grothendieck polynomials.

3. Set-valued tableaux

In this section we will introduce set-valued tableaux and use them to give a formula for stable Grothendieck polynomials indexed by 321-avoiding permutations.

If a and b are two non-empty subsets of the positive integers $\mathbb{N}$, we will write $a < b$ if $\max(a) < \min(b)$, and $a \leq b$ if $\max(a) \leq \min(b)$. We define a *set-valued tableau* to be a labeling of the boxes in a Young diagram or skew diagram with finite non-empty subsets of $\mathbb{N}$, such that the rows are weakly increasing from left to right and the columns strictly increasing from top to bottom. When we speak about a tableau we shall always mean a set-valued tableau unless explicitly stated otherwise. The shape $\mathrm{sh}(T)$ of a tableau T is the partition or skew diagram it is a labeling of. For example

		1	23
	12	234	
2	35	7	

is a tableau of shape $(4,3,3)/(2,1)$ containing the sets $\{2\}$, $\{3,5\}$, $\{1,2\}$, $\{7\}$, $\{2,3,4\}$, $\{1\}$, and $\{2,3\}$ when the boxes are read bottom to top and then left to right. A *semistandard tableau* is a set-valued tableau in which each box contains a single integer.

Given a tableau T, let x^T be the monomial in which the exponent of x_i is the number of boxes in T which contain the integer i. If T is the tableau displayed above we get $x^T = x_1^2\, x_2^4\, x_3^3\, x_4\, x_5\, x_7$. We let $|T|$ denote the total degree of this monomial, i.e. the sum of the cardinalities of the sets in the boxes of T.

Theorem 3.1. *The single stable Grothendieck polynomial $G_{\nu/\lambda}(x)$ is given by the formula*

$$G_{\nu/\lambda}(x) = \sum_T (-1)^{|T|-|\nu/\lambda|}\, x^T$$

where the sum is over all set-valued tableaux T of shape ν/λ.

To prove this proposition we need a result of Fomin and Kirillov. Let $H_n(0)$ be the degenerate Hecke algebra over the polynomial ring $R = \mathbb{Z}[x_1,\dots,x_m]$. This is the free associative R-algebra generated by symbols $u_1,\dots,u_n$, modulo relations

$$\begin{aligned} u_i\, u_j &= u_j\, u_i && \text{if } |i-j| \geq 2, \\ u_i\, u_{i+1}\, u_i &= u_{i+1}\, u_i\, u_{i+1} && \text{, and} \\ u_i^2 &= -u_i\,. \end{aligned}$$

If $w \in S_{n+1}$ is a permutation with reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ we set $u_w = u_{i_1} \cdots u_{i_\ell} \in H_n(0)$. This is independent of the choice of a reduced expression. Furthermore these elements u_w for $w \in S_{n+1}$ form a basis for $H_n(0)$. Now set $A(x) = (1+x\,u_n)\cdots(1+x\,u_2)\cdot(1+x\,u_1)$ and $B(x) = (1+x\,u_1)\cdot(1+x\,u_2)\cdots(1+x\,u_n)$. Then a special case of the theory developed by Fomin and Kirillov [9, 8] is the following.

Theorem 3.2. *The coefficient of u_w in $A(x_m)\cdots A(x_2)\cdot A(x_1)$ is the stable Grothendieck polynomial $G_w(x_1,\dots,x_m)$. The coefficient of u_w in $B(x_m)\cdots B(x_2)\cdot B(x_1)$ is $G_w(0;x_1,\dots,x_m)$.*

Define an *inner corner* of a partition λ to be any box of λ such that the two boxes under and to the right of it are not in λ. By an *outer corner* we will mean a box outside λ such that the two boxes above and to the left of it are in λ.

Proof of Theorem 3.1. It is enough to show that $G_{\nu/\lambda}(x_1,\dots,x_m)$ is the sum of the signed monomials $(-1)^{|T|-|\nu/\lambda|}\, x^T$ for tableaux T with no integers larger than m.

Now number the diagonals of ν/λ from south-west to north-east as described in the previous section and let $V = \bigoplus_{\mu\supset\lambda} \mathbb{Z}\cdot[\mu]$ be the free Abelian group with one basis element for each partition containing λ. Imitating the methods of Fomin and Greene [7], we define a linear action of $H_n(0)$ on V as follows. If a partition μ has an outer corner in the i'th diagonal, then we set $u_i \cdot [\mu] = [\tilde\mu]$ where $\tilde\mu$ is obtained by adding a box to μ in this corner. If μ has an inner corner in the i'th diagonal, and if the box in this corner is not contained in λ, then we set $u_i \cdot [\mu] = -[\mu]$. In all other cases we set $u_i \cdot [\mu] = 0$.

We claim that if $w \in S_{n+1}$ is any permutation such that $u_w \cdot [\lambda] \neq 0$ then $u_w \cdot [\lambda] = [\mu]$ for some partition $\mu \supset \lambda$ and $w = w_{\mu/\lambda}$. This claim is clear if $\ell(w) \leq 1$. If $\ell(w) \geq 2$, write $w = s_i\, w'$ where $\ell(w') = \ell(w) - 1$. Since $u_{w'} \cdot [\lambda] \neq 0$ we can assume by induction that $u_{w'} \cdot [\lambda] = [\mu_0]$ for some partition μ_0, and $w' = w_{\mu_0/\lambda}$. It is enough to show that μ_0 has no inner corner in the i'th diagonal. If it had, then the box in this corner would be outside λ since $u_i \cdot [\mu_0] \neq 0$. But then we could write $w' = w_1'\, s_i\, w_2'$ such that $\ell(w') = \ell(w_1') + \ell(w_2') + 1$ and no reduced expression for w_1' contains s_{i-1}, s_i, or s_{i+1}. This would mean that $w = s_i\, w' = w_1'\, w_2'$, contradicting that $\ell(w) > \ell(w')$.

Using Theorem 3.2 it follows from the claim that the coefficient of the basis element $[\nu]$ in $A(x_m) \cdots A(x_1) \cdot [\lambda]$ is $G_{\nu/\lambda}(x_1, \dots, x_m)$. Finally, it is easy to identify the terms of $A(x_m) \cdots A(x_1)$ which take $[\lambda]$ to $[\nu]$ with set-valued tableaux on ν/λ. In fact, this product expands as a sum of terms of the form

$$(x_{i_1}\, u_{j_1}) \cdot (x_{i_2}\, u_{j_2}) \cdots (x_{i_k}\, u_{j_k})\,.$$

From any such term which contributes to the coefficient of $[\nu]$ we obtain a tableau on ν/λ by joining the integer i_r to the set in the inner corner in diagonal number j_r of the partition $u_{j_r} \cdots u_{j_k} \cdot [\lambda]$ for each $1 \leq r \leq k$. □

Remark 3.3. It is easy to extend the notion of set-valued tableau to obtain formulas for double stable Grothendieck polynomials $G_{\nu/\lambda}(x;y)$. The main point is that integers corresponding to the x-variables should occupy horizontal strips in a set-valued tableau while integers corresponding to y-variables should appear in vertical strips.

While we have the notation of Theorem 3.2 fresh in mind, we shall also establish the following lemma for use later. At an early point in this project we asked Sergey Fomin if the statement of this lemma could possibly be true, after which he proved it immediately.

Lemma 3.4 (Fomin)**.** *Let $w \in S_{n+1}$ be a permutation and let $w_0 \in S_{n+1}$ be the longest permutation. Then $G_{w_0 w w_0}(x;y) = G_w(y;x)$.*

Proof. Since $G_w(1 - e^{-x}; 1 - e^{y})$ is super-symmetric, it is enough to prove that $G_{w_0 w w_0}(x_1, \dots, x_m) = G_w(0; x_1, \dots, x_m)$ for any number of variables m.

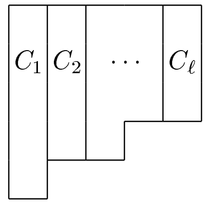
Now $H_n(0)$ has an R-linear automorphism which sends u_i to u_{n+1-i} for each $1 \leq i \leq n$. Since this automorphism takes $A(x_j)$ to $B(x_j)$ and u_w to $u_{w_0 w w_0}$, the lemma follows from Theorem 3.2. □

As a special case we obtain $G_{\lambda'/\mu'}(x;y) = G_{w_0\, w_{\lambda/\mu}\, w_0}(x;y) = G_{\lambda/\mu}(y;x)$.

4. A column bumping algorithm

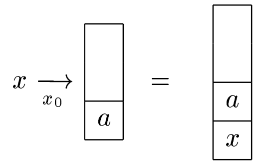
In this section we will present a column bumping algorithm for set-valued tableaux and derive an important bijective correspondence from it. This correspondence will be the main ingredient in the proof of the Littlewood-Richardson rule for stable Grothendieck polynomials.

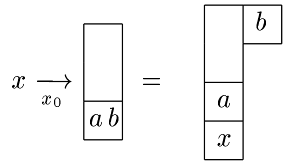
We will use the following notation. If a and b are disjoint sets of integers we will let $\boxed{a\,b}$ denote a single box containing the union of a and b. If T is a tableau with ℓ columns, and C_i denotes its i'th column for each $1 \leq i \leq \ell$, then we will write

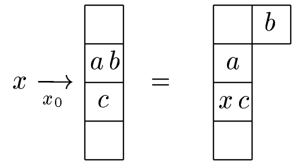
$T =$  $= (C_1, C_2, \ldots, C_\ell)\,.$

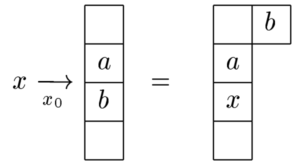
We start with the following definition.

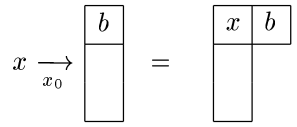
Definition 4.1. Let $x \in \mathbb{N}$, $x_0 \in \mathbb{N} \cup \{\infty\}$, and let C be a tableau with only one column. We then define a new tableau $x \underset{x_0}{\longrightarrow} C$ by the following rules:

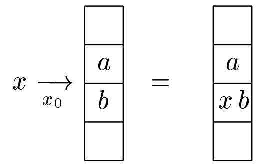
(B1)  if $a < x$

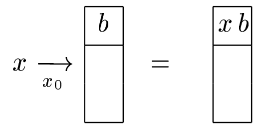
(B2)  if $a < x \leq b$

(B3)  if $a < x \leq b$

(B4)  if $a < x \leq b$ and $x_0 \notin b$

(B5)  if $x \leq b$ and $x_0 \notin b$

(B6)  if $a < x \leq b$ and $x_0 \in b$

(B7)  if $x \leq b$ and $x_0 \in b$

The white areas in these tableaux indicate boxes which are left unchanged by the operation. It is easy to see that exactly one of the cases (B1)–(B7) will apply to define $x \underset{x_0}{\longrightarrow} C$. In the rules (B2) to (B5) we say that the set b is "bumped out".

If $x \subset \mathbb{N}$ is any non-empty set, we extend this definition as follows. Let $x_1 < \cdots < x_k$ be the elements of x in increasing order and let (C_k, y_k) be the tableau $x_k \underset{x_0}{\longrightarrow} C$:

$$x_k \underset{x_0}{\longrightarrow} C = \begin{array}{|c|c|}\hline & y_k \\ \cline{2-2} C_k & \multicolumn{1}{c}{} \\ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Here y_k is the set in the single box in the second column. If $x_k \underset{x_0}{\longrightarrow} C$ has only one column, we let y_k be the empty set. Continue by setting $(C_i, y_i) = (x_i \underset{x_{i+1}}{\longrightarrow} C_{i+1})$ for each $i = k-1, \dots, 1$. We finally define $x \underset{x_0}{\longrightarrow} C$ to be the tableau (C_1, y) where $y = y_1 \cup \cdots \cup y_k$:

$$x \underset{x_0}{\longrightarrow} C = \begin{array}{|c|c|}\hline & y \\ \cline{2-2} C_1 & \multicolumn{1}{c}{} \\ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

If $x_0 \subset \mathbb{N}$ is a set, we will write $x \underset{x_0}{\longrightarrow} C = x \underset{\min(x_0)}{\longrightarrow} C$. We furthermore set $x \cdot C = x \underset{\infty}{\longrightarrow} C$. This defines the product of a one-box tableau with a one-column tableau. Notice that if C has ℓ boxes, then the shape of $x \cdot C$ is one of $(1^{\ell+1})$, $(2, 1^{\ell-1})$, or $(2, 1^\ell)$, where 1^ℓ means a sequence of ℓ ones. Notice also that $x \cdot C = x \underset{x_0}{\longrightarrow} C$ unless one of the rules (B6) or (B7) are used to define $\max(x) \underset{x_0}{\longrightarrow} C$. For example:

$$\begin{array}{|c|}\hline 235 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 12 \\ \hline 45 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 245 \\ \hline 23 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

We will continue by defining the product of a non-empty set x with any tableau $T = (C_1, C_2, \dots, C_\ell)$. Namely, set $(C'_1, y_1) = x \cdot C_1$ and $(C'_i, y_i) = y_{i-1} \cdot C_i$ for $2 \le i \le \ell$. If a product $y_{i-1} \cdot C_i$ has only one column for some i, we let C'_i be this product and set $C'_j = C_j$ for $j > i$ and $y_\ell = \emptyset$. We then define $x \cdot T$ to be the tableau whose j'th column is C'_j for $1 \le j \le \ell$. If $y_\ell \ne \emptyset$ we furthermore add an $\ell + 1$'st column with one box containing this set:

$$x \cdot \begin{array}{|c|c|c|c|}\hline C_1 & C_2 & \cdots & C_\ell \\ \hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline C'_1 & C'_2 & \cdots & C'_\ell & y_\ell \\ \hline\end{array}$$

To see that this is in fact a tableau, we need the following lemma.

Lemma 4.2. *Let $x \in \mathbb{N}$, $y_0 \in \mathbb{N} \cup \{\infty\}$, and let C_1 and C_2 be one-column tableaux which fit together to form a tableau (C_1, C_2). Let $(C'_1, y) = x \cdot C_1$ and $(C'_2, z) = y \underset{y_0}{\longrightarrow} C_2$. Then (C'_1, C'_2) is a tableau.*

Proof. Suppose y was bumped out of box number i in C_1, counted from the top. Then since (C_1, C_2) is a tableau, y is less than or equal to the set in box number i in C_2. This implies that the j'th box of C_2' is equal to the j'th box of C_2 for all $j > i$. For $j \leq i$, the j'th box of C_2' can contain elements from the j'th box of C_2 and from y. Since both of these sets are larger than or equal to the j'th box in C_1', this shows that (C_1', C_2') is a tableau. $\square$

To check that the product $x \cdot T$ defined above is a tableau, it is enough to assume that $T = (C_1, C_2)$ has only two columns. Let $x' = \min(x)$ and $x'' = x \smallsetminus \{x'\}$, and set $(C_1'', y'') = x'' \cdot C_1$ and $(C_2'', z'') = y'' \cdot C_2$. Then set $(C_1', y') = x' \xrightarrow[x'']{} C_1''$ and $(C_2', z') = y' \xrightarrow[y'']{} C_2''$. Then $x \cdot C_1 = (C_1', y)$ where $y = y'' \cup y'$, and $y \cdot C_2 = (C_2', z'' \cup z')$. We must show that (C_1', C_2') is a tableau.

By induction we can assume that (C_1'', C_2'') is a tableau. If $x' \xrightarrow[x'']{} C_1''$ is defined by (B6) or (B7) then the maximal elements in the boxes of C_1' and C_1'' are equal, and since y' is empty we have $C_2' = C_2''$. This makes it clear that (C_1', C_2') is a tableau. Otherwise we have $x' \xrightarrow[x'']{} C_1'' = x' \cdot C_1''$ in which case (C_1', C_2') is a tableau by Lemma 4.2.

Define a *rook strip* to be a skew shape (between two partitions) which has at most one box in any row or column. It then follows from our earlier observations regarding the shape of a product of a set with a one-column tableau, that the shape of $x \cdot T$ differs from that of T by a rook strip.

Now let C be a tableau with one column and let T be any tableau. Suppose the boxes of C contain the sets $x_1, x_2, \ldots, x_\ell$, read from top to bottom. We then define the product of C and T to be the tableau $C \cdot T = x_\ell \cdot (x_{\ell-1} \cdot (\cdots x_2 \cdot (x_1 \cdot T) \cdots))$.

Example 4.3. It is not possible to extend this product to an associative product on all set-valued tableau. In fact, if this was possible we would have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2\,3 \\ \cline{1-1} \end{array} = \begin{array}{|c|} \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 1\,2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} = \begin{array}{|c|} \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 123 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 2\,3 \\ \hline 2 \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

$$= \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 2\,3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 2 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

which is of course wrong.

In the following lemmas we shall study the shape of a product $C \cdot T$.

Lemma 4.4. *Let $x_1 < x_2$ be non-empty sets of integers and let C be a one-column tableau. Let $x_1 \cdot C = (C_1, y_1)$ and $x_2 \cdot C_1 = (C_2, y_2)$. Then $y_1 < y_2$.*

Proof. Notice at first that $\min(x_2) \leq y_2$, which follows directly from Definition 4.1. Since all of the integers in y_2 come from C_1, it suffices to show that all integers from C_1 which are $\geq \min(x_2)$ are also strictly greater than y_1.

Let k be the number of the box in C_1 (counted from the top) which contains $\max(x_1)$. There are then two possibilities. Either $\max(x_1)$ is the largest element in this box of C_1, in which case all integers in y_1 must come from the boxes 1 through k of C. Furthermore, the boxes in C_1 strictly below the k'th box contain the same sets as the corresponding boxes of C, so they are strictly greater than y_1. Since these are also the only boxes that can be greater than or equal to $\min(x_2)$, the statement follows in this case.

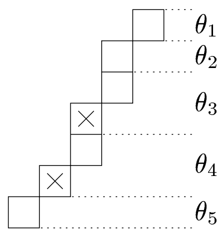
Otherwise $\max(x_1)$ is not the largest element in the k'th box of C_1, which implies that all integers in y_1 come from the boxes 1 through $k-1$ in C. Since the integers in C_1 which are strictly greater than $\max(x_1)$ all come from the k'th box in C or from boxes below this box, the statement is also true in this case. □

Lemma 4.5. *Let T be a tableau and let $x_1 < x_2$ be non-empty sets of integers. Set $T_1 = x_1 \cdot T$ and $T_2 = x_2 \cdot T_1$, and let $\theta_1 = \operatorname{sh}(T_1)/\operatorname{sh}(T)$ and $\theta_2 = \operatorname{sh}(T_2)/\operatorname{sh}(T_1)$ be the rook strips giving the differences between the shapes of these tableaux. Then all boxes of θ_2 are strictly south of the boxes in θ_1.*

Proof. Suppose the southernmost box of θ_1 occurs in column j. Then let U be the tableau consisting of the leftmost $j-1$ columns of T and let V be the rest of T. We will write $T = (U, V)$ to indicate this. Similarly we let $(U_1, y_1) = x_1 \cdot U$ and $(U_2, y_2) = x_2 \cdot U_1$, i.e. U_1 and U_2 are the leftmost $j-1$ columns of these products. Finally set $V_1 = y_1 \cdot V$ and $V_2 = y_2 \cdot V_1$. We then have $T_1 = (U_1, V_1)$ and $T_2 = (U_2, V_2)$.

Since V_1 has one more box in the first column than V, this box of V_1 must contain a subset of y_1. Since $y_2 > y_1$ by Lemma 4.4, this means that $V_2 = y_2 \cdot V_1$ consists of V_1 with y_2 attached below the first column (or $V_2 = V_1$ if y_2 is empty). The lemma follows from this. □

It follows immediately from this lemma that the shape of a product $C \cdot T$ adds a vertical strip to the shape of T, but more detailed information can be obtained. As above, let $x_1, x_2, \dots, x_\ell$ be the sets contained in the boxes of C from top to bottom. Set $T_0 = T$ and $T_i = x_i \cdot T_{i-1}$ for $1 \le i \le \ell$. Let $\theta_i = \operatorname{sh}(T_i)/\operatorname{sh}(T_{i-1})$ be the rook strip between the shapes of T_i and T_{i-1}, and let $\theta = \operatorname{sh}(C \cdot T)/\operatorname{sh}(T)$ be the union of these rook strips. Then Lemma 4.5 says that the θ_i split θ up into disjoint segments running from north to south.

$\theta =$ 

Define the *extra boxes* of θ to be the boxes that are not the upper-right box of any rook strip θ_i. These boxes are marked with a cross in the above picture. There are exactly $|\theta| - \ell$ extra boxes in θ, and at most one in each column of θ. Furthermore, if a column of θ has an extra box then it is the northernmost one. Define $\operatorname{col}(C, T)$ to be the set of columns of θ which contain an extra box. The rook strips θ_i are then uniquely determined by θ and this set. In what follows we shall see that if x is a set and T is a tableau, then x and T can be recovered from their product $x \cdot T$ if one knows the shape of T. As a consequence we see that C and T are uniquely determined by their product $C \cdot T$, the shape of T, and the set $\operatorname{col}(C, T)$.

We will state this a bit sharper. Given a vertical strip θ and a non-negative integer $d \ge 0$, let $\mathcal{C}_d(\theta)$ be the set of all sets of the non-empty columns of θ which have cardinality d and avoids the last column of θ. If $c(\theta)$ is the number of non-empty columns of θ then $\mathcal{C}_d(\theta)$ has cardinality $\binom{c(\theta)-1}{d}$. Let $\mathcal{T}_\lambda$ be the set of all set-valued tableaux of shape λ. We then have a map

$$\mathcal{T}_{(1^\ell)} \times \mathcal{T}_\lambda \longrightarrow \coprod_\nu \mathcal{T}_\nu \times \mathcal{C}_{|\nu/\lambda|-\ell}(\nu/\lambda) \tag{4.1}$$

which takes (C,T) to the pair $(C\cdot T, \operatorname{col}(C,T))$ in the set $\mathcal{T}_\nu \times \mathcal{C}_{|\nu/\lambda|-\ell}(\nu/\lambda)$ where $\nu = \operatorname{sh}(C\cdot T)$. The disjoint union is over all partitions ν containing λ such that ν/λ is a vertical strip. In the remaining part of this section we will construct an inverse to this map, thus proving the following.

Theorem 4.6. *The map of (4.1) is bijective.*

As a first consequence, we obtain a bijective proof of Lenart's Pieri formula [18].

Corollary 4.7 (Lenart)**.** *For any partition λ and $\ell \geq 1$ we have*

$$G_{(1^\ell)} \cdot G_\lambda = \sum_\nu (-1)^{|\nu/\lambda|-\ell} \binom{c(\nu/\lambda)-1}{|\nu/\lambda|-\ell} G_\nu$$

where the sum is over all partitions $\nu \supset \lambda$ such that ν/λ is a vertical strip, and $c(\nu/\lambda)$ is the number of non-empty columns in this diagram.

In order to construct the inverse map of (4.1) we will define a reverse column bumping algorithm for set-valued tableaux. We start with the following definition.

Definition 4.8. Let $T = (C,y)$ be a tableau whose second column has one box containing a single integer $y \in \mathbb{N}$. For any $y_0 \in \mathbb{N}\cup\{0\}$ we define the pair $\mathcal{R}_{y_0}(C,y)$ by the following rules:

$$\text{(R1)}\qquad \mathcal{R}_{y_0}\left(\begin{array}{|c|c|}\hline & y \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1} b\,c & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \left(b\ ,\ \begin{array}{|c|}\hline \\ \hline a\,y \\ \hline c \\ \hline \\ \hline \end{array}\right) \qquad \text{if } b \leq y < c$$

$$\text{(R2)}\qquad \mathcal{R}_{y_0}\left(\begin{array}{|c|c|}\hline & y \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \left(a\ ,\ \begin{array}{|c|}\hline \\ \hline y \\ \hline b \\ \hline \\ \hline \end{array}\right) \qquad \text{if } a \leq y < b \text{ and } y_0 \notin a$$

$$\text{(R3)}\qquad \mathcal{R}_{y_0}\left(\begin{array}{|c|c|}\hline & y \\ \cline{2-2} & \multicolumn{1}{c}{} \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \left(a\ ,\ \begin{array}{|c|}\hline \\ \\ \hline y \\ \hline \end{array}\right) \qquad \text{if } a \leq y \text{ and } y_0 \notin a$$

$$\text{(R4)}\qquad \mathcal{R}_{y_0}\left(\begin{array}{|c|c|}\hline & y \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \left(\emptyset\ ,\ \begin{array}{|c|}\hline \\ \hline a\,y \\ \hline b \\ \hline \\ \hline \end{array}\right) \qquad \text{if } a \leq y < b \text{ and } y_0 \in a$$

$$\text{(R5)}\qquad \mathcal{R}_{y_0}\left(\begin{array}{|c|c|}\hline & y \\ \cline{2-2} & \multicolumn{1}{c}{} \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right) = \left(\emptyset\ ,\ \begin{array}{|c|}\hline \\ \\ \hline a\,y \\ \hline \end{array}\right) \qquad \text{if } a \leq y \text{ and } y_0 \in a$$

We extend this definition to tableaux (C, y) where $y \subset \mathbb{N}$ is any non-empty set of integers as follows: Let $y_1 < \cdots < y_k$ be the elements of y in increasing order. Let $(x_1, C_1) = \mathcal{R}_{y_0}(C, y_1)$ and $(x_i, C_i) = \mathcal{R}_{y_{i-1}}(C_{i-1}, y_i)$ for $2 \leq i \leq k$. Then we set $\mathcal{R}_{y_0}(C, y) = (x, C_k)$ where $x = x_1 \cup \cdots \cup x_k$. If $y_0 \subset \mathbb{N}$ is a set, we will write $\mathcal{R}_{y_0}(C, y) = \mathcal{R}_{\max(y_0)}(C, y)$. Finally we set $\mathcal{R}(C, y) = \mathcal{R}_0(C, y)$.

Lemma 4.9. *Let C be a one-column tableau with ℓ boxes and let $x \subset \mathbb{N}$ be a set such that $x \cdot C$ has shape $(2, 1^{\ell-1})$. Then $\mathcal{R}(x \cdot C) = (x, C)$. Similarly, if T is a tableau of shape $(2, 1^{\ell-1})$ and $\mathcal{R}(T) = (x, C)$, then C has ℓ boxes and $x \cdot C = T$.*

Proof. Suppose at first $x \in \mathbb{N}$ is a single integer, and let $x \cdot C = (C', y)$. Then by the definition of $\mathcal{R}(C', y)$, the minimal element of y will bump out x from C', after which the remaining elements of y will be added to the same box as the minimal element went into. This recovers the tableau C.

If x has more than one element, let $x_1 = \min(x)$ and $x_2 = x \smallsetminus \{x_1\}$, and write $x_2 \cdot C = (C_2, y_2)$ and $x_1 \underset{x_2}{\longrightarrow} C_2 = (C_1, y_1)$. Then $x \cdot C = (C_1, y)$ where $y = y_1 \cup y_2$. By induction we can assume $\mathcal{R}(C_2, y_2) = (x_2, C)$. There are two cases to consider.

Either $y_1 = \emptyset$. In this case C_1 is obtained from C_2 by joining x_1 to the box containing $\min(x_2)$. This means that if $\mathcal{R}(C_2, \min(y_2)) = (x', C')$ then we have $\mathcal{R}(C_1, \min(y_2)) = (\{x_1\} \cup x', C')$. This proves that $\mathcal{R}(C_1, y) = (x, C)$ in this case.

Otherwise we have $y_1 \neq \emptyset$, in which case $(C_1, y_1) = x_1 \cdot C_2$. We then know that $\mathcal{R}(C_1, y_1) = (x_1, C_2)$. Since $\mathcal{R}(C_2, y_2) = (x_2, C)$, it is enough to show that $\mathcal{R}(C_2, y_2) = \mathcal{R}_{y_1}(C_2, y_2)$. Here it is enough to check that none of the rules (R4) or (R5) are used to define $\mathcal{R}_{\max(y_1)}(C_2, \min(y_2))$. But since $\min(x_2)$ is bumped out when $\mathcal{R}(C_2, \min(y_2))$ is formed, this would imply that $\min(x_2)$ and $\max(y_1)$ are in the same box of C_2. This contradicts the fact that x_1 is bumping y_1 but not $\min(x_2)$ out when multiplied to C_2.

The proof of the second statement is similar and left to the reader. $\square$

To recover the factors of a product $x \cdot C$ which adds two boxes to the shape of C, we continue with the following definition.

Definition 4.10. Let $T = (C, y)$ be a tableau with at least two boxes in the first column and one box in second column. Suppose $\mathcal{R}(C, y)$ has the form

$$\mathcal{R}(C, y) = \left(x \ , \ \begin{array}{|c|}\hline \\ \\ \hline a \\ \hline b \\ \hline \end{array} \right) . \tag{4.2}$$

Then we define $\mathcal{R}^*(C, y)$ as follows:

$$\mathcal{R}^*(C, y) = \left(x \cup b \ , \ \begin{array}{|c|}\hline \\ \\ \hline a \\ \hline \end{array} \right) \qquad \text{if } b \not\subset y \tag{D1}$$

$$\mathcal{R}^*(C, y) = \left(x \ , \ \begin{array}{|c|}\hline \\ \\ \hline a\,b \\ \hline \end{array} \right) \qquad \text{if } b \subset y \tag{D2}$$

Lemma 4.11. *Let C be a one-column tableau with ℓ boxes and let $x \subset \mathbb{N}$ be a set such that $x \cdot C$ has shape $(2, 1^\ell)$. Then $\mathcal{R}^*(x \cdot C) = (x, C)$. Similarly, if T is a tableau of shape $(2, 1^\ell)$ and $\mathcal{R}^*(T) = (x, C)$, then C has ℓ boxes and $x \cdot C = T$.*

Proof. Notice at first that if $x \cdot C$ has shape $(2, 1^\ell)$ then $\max(x) \cdot C$ must be defined by one of the rules (B1) or (B2). Let $x_2 \subset x$ be the largest subset of the form $x \cap [k, \infty[$ such that none of the rules (B3)–(B5) are used in the definition of $x_2 \cdot C$, and let $x_1 = x \smallsetminus x_2$. The lemma is easy to prove if x_1 is empty, so we will assume $x_1 \neq \emptyset$. Let $x_2 \cdot C = (C_2, y_2)$ and $x_1 \cdot C_2 = (C_1, y_1)$. Then $x \cdot C = (C_1, y)$ where $y = y_1 \cup y_2$. Furthermore we have $\mathcal{R}(C_1, y_1) = (x_1, C_2)$ by Lemma 4.9. There are two cases to consider.

First suppose $\max(x_2) \cdot C$ is defined by (B1). Then C_2 is equal to C with x_2 attached in a new box at the bottom, and y_2 is empty. This means that $x \cdot C = (C_1, y_1)$, so $\mathcal{R}(x \cdot C) = (x_1, C_2)$. By rule (D1) we therefore have $\mathcal{R}^*(x \cdot C) = (x_1 \cup x_2, C)$ as claimed.

Otherwise $\max(x_2) \cdot C$ is defined by (B2). If the bottom box of C contains $a \cup b$ with $a < \max(x_2) \leq b$, then $y_2 = b$ and C_2 is obtained by removing b from the bottom box of C and attaching x_2 in a new box below it. It follows that $\mathcal{R}_{y_1}(C_2, y_2) = (x_2, C')$ where C' is obtained from C by moving the elements of b from the bottom box to a new box below it. We conclude that $\mathcal{R}(x \cdot C) = (x, C')$, so $\mathcal{R}^*(x \cdot C) = (x, C)$ by (D2).

The proof of the second statement is similar and left to the reader. □

Now let T be a tableau of shape ν and let $\lambda \subsetneq \nu$ be a proper subpartition such that ν/λ is a rook strip. We will then produce a tableau T' of shape λ and a set $x \subset \mathbb{N}$ such that $x \cdot T' = T$.

Write $T = (C_1, C_2, \ldots, C_k)$ where C_i is the i'th column, and suppose the upper-right box of ν/λ is in column j. Let C'_j be the result of removing the bottom box from C_j, and let x_j be the set from this removed box. Now for each $i = j-1, \ldots, 1$, we define

$$(x_i, C'_i) = \begin{cases} \mathcal{R}(C_i, x_{i+1}) & \text{if } \nu/\lambda \text{ does not have a box in column } i; \\ \mathcal{R}^*(C_i, x_{i+1}) & \text{if } \nu/\lambda \text{ has a box in column } i. \end{cases}$$

We then set $x = x_1$ and $T' = (C'_1, \ldots, C'_j, C_{j+1}, \ldots, C_k)$. A argument similar to the proof of Lemma 4.2 shows that T' is a tableau, and by definition the shape of T' is λ. Furthermore, it follows from Lemma 4.9 and Lemma 4.11 that $x \cdot T' = T$. We let $\mathcal{R}_{\nu/\lambda} : \mathcal{T}_\nu \longrightarrow \mathcal{T}_1 \times \mathcal{T}_\lambda$ be the map defined by $\mathcal{R}_{\nu/\lambda}(T) = (x, T')$.

Proof of Theorem 4.6. It follows from Lemma 4.9 and Lemma 4.11 that the maps $\mathcal{R}_{\nu/\lambda}$ define an inverse to the map of (4.1) when $\ell = 1$. If $\ell \geq 2$ and $(T, S) \in \mathcal{T}_\nu \times \mathcal{C}_{|\nu/\lambda|-\ell}(\nu/\lambda)$ is any element, there are unique rook strips $\theta_1, \ldots, \theta_\ell$ which split the vertical strip $\theta = \nu/\lambda$ up into disjoint intervals from north to south, such that S contains the columns of the extra boxes in θ. Then set $(x_\ell, T_\ell) = \mathcal{R}_{\theta_\ell}(T)$ and $(x_i, T_i) = \mathcal{R}_{\theta_i}(T_{i+1})$ for $i = \ell - 1, \ldots, 1$, and let $C \in \mathcal{T}_{(1^\ell)}$ be the column whose i'th box contains x_i. An argument similar to the proof of Lemma 4.4 shows that $x_1 < \cdots < x_\ell$ which implies that C is a tableau. Finally Lemma 4.9 and Lemma 4.11 show that the map $(T, S) \mapsto (C, T_1)$ gives an inverse to (4.1). □

Remark 4.12. Although we have skipped some details of the proof of Theorem 4.6, the arguments given here do suffice to establish that the map of (4.1) is injective. In stead of writing down proofs of the remaining statements, one can also use Lenart's proof of Corollary 4.7 [18] to deduce that the two sets in (4.1) have the same number of elements (which is finite if we only consider tableaux containing integers between 1 and m for any $m \geq 1$).

5. Stable Grothendieck polynomials in non-commutative variables

In this section we define stable Grothendieck polynomials in non-commutative variables and show that they span a commutative ring. As a consequence we obtain an explicit Littlewood-Richardson rule for multiplying Grothendieck polynomials of Grassmannian permutations.

Following [6, 7], we define the *local plactic algebra* to be the free associative $\mathbb{Z}$-algebra $\mathcal{L}$ in variables $u_1, u_2, \ldots$, modulo the relations

$$\begin{aligned} u_i\, u_j &= u_j\, u_i && \text{if } |i-j| \geq 2, \\ u_i\, u_{i+1}\, u_i &= u_{i+1}\, u_i\, u_i && \text{, and} \\ u_{i+1}\, u_i\, u_{i+1} &= u_{i+1}\, u_{i+1}\, u_i\,. \end{aligned}$$

We shall work in the completion $\hat{\mathcal{L}}$ of $\mathcal{L}$ which consists of formal power series in these variables.

Let T be a set-valued tableau. We define the (column) *word* of T to be the sequence $w(T)$ of the integers contained in its boxes when these are read from bottom to top and then from left to right. The integers within a single box are arranged in increasing order. The word of the tableau displayed in the start of Section 3 is $(2,3,5,1,2,7,2,3,4,1,2,3)$.

If T has word $w(T) = (i_1, i_2, \ldots, i_\ell)$, then we let u^T be the non-commutative monomial $u^T = u_{i_1}\, u_{i_2} \cdots u_{i_\ell} \in \mathcal{L}$. It is not hard to see that one gets the same monomial if the boxes of T are read from left to right first and then from bottom to top, but we shall not need this fact. If ν/λ is any skew diagram we then define a stable Grothendieck polynomial in the variables u_i by

$$G_{\nu/\lambda}(u) = \sum_T (-1)^{|T|-|\nu/\lambda|}\, u^T \in \hat{\mathcal{L}}$$

where this sum is over all tableaux T of shape ν/λ.

If C is a one-column tableau and T is any tableau, one may easily verify from Definition 4.1 that $u^C \cdot u^T = u^{C \cdot T}$ in $\mathcal{L}$. From Theorem 4.6 we therefore obtain a non-commutative version of Lenart's Pieri rule.

Lemma 5.1. *If λ is any partition and $\ell \geq 1$ an integer, then*

$$G_{(1^\ell)}(u) \cdot G_\lambda(u) = \sum_\nu (-1)^{|\nu/\lambda|-\ell} \binom{c(\nu/\lambda)-1}{|\nu/\lambda|-\ell} G_\nu(u)$$

holds in $\hat{\mathcal{L}}$, the sum is over all partitions $\nu \supset \lambda$ such that ν/λ is a vertical strip.

A first consequence of this is that the elements $G_{(1^\ell)}(u)$ for $\ell \geq 1$ commute in $\hat{\mathcal{L}}$. Now let $\mathcal{G} = \bigoplus_\lambda \mathbb{Z} \cdot G_\lambda(u)$ be the span of the polynomials $G_\lambda(u)$ for all partitions λ, and let $\hat{\mathcal{G}} \subset \hat{\mathcal{L}}$ be its completion, consisting of all infinite linear combinations of the $G_\lambda(u)$.

Lemma 5.2. *The group $\hat{\mathcal{G}}$ consists of all formal power series in the polynomials $G_{(1^\ell)}(u)$ for $\ell \geq 1$. In particular $\hat{\mathcal{G}}$ is a commutative subring of $\hat{\mathcal{L}}$.*

Proof. Let $n \in \mathbb{N}$ be any positive integer. It is enough to show that for any partition λ there exists a polynomial $P_\lambda(u) \in \mathbb{Z}\left[G_{(1^\ell)}(u)\right]_{\ell \geq 1}$ such that $G_\lambda(u) - P_\lambda(u)$ is a linear combination of Grothendieck polynomials $G_\mu(u)$ for partitions μ of length $\ell(\mu) \geq n$.

Define a partial order on partitions by writing $\mu < \lambda$ if $\mu_1 < \lambda_1$, or $\mu_1 = \lambda_1$ and $\ell(\mu) > \ell(\lambda)$. Notice that given a partition λ there are only finitely many partitions μ such that $\mu < \lambda$ and $\ell(\mu) < n$.

We shall prove that the polynomials $P_\lambda(u)$ exist by induction on this order. Since the smallest partitions λ have only one column, the existence of $P_\lambda(u)$ is clear for these partitions. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of length $\ell < n$ and assume that $P_\mu(u)$ exists for all $\mu < \lambda$. Let $\sigma = (\lambda_1 - 1, \dots, \lambda_\ell - 1)$ be the partition obtained by removing the first column of λ. Then $\sigma < \lambda$. By Lemma 5.1 we furthermore have

$$G_\lambda(u) = G_{(1^\ell)}(u) \cdot G_\sigma(u) - \sum_{\nu \supsetneq \lambda} (-1)^{|\nu/\lambda| - \ell} \binom{c(\nu/\lambda) - 1}{|\nu/\lambda| - \ell} G_\nu(u)$$

where the sum is over all partitions ν properly containing λ such that ν/λ is a vertical strip. Notice that any such partition ν for which $|\nu/\lambda| \geq \ell$ must satisfy $\nu < \lambda$. We can therefore define

$$P_\lambda(u) = G_{(1^\ell)}(u) \cdot P_\sigma(u) - \sum_{\nu \supsetneq \lambda} (-1)^{|\nu/\lambda| - \ell} \binom{c(\nu/\lambda) - 1}{|\nu/\lambda| - \ell} P_\nu(u)\,.$$

This finishes the proof. □

The lemma shows that any product $G_\lambda(u) \cdot G_\mu(u)$ is an element of $\hat{\mathcal{G}}$, so we may write

$$G_\lambda(u) \cdot G_\mu(u) = \sum_\nu c_{\lambda\mu}^\nu \, G_\nu(u) \tag{5.1}$$

where the coefficients $c_{\lambda\mu}^\nu$ are integers. Notice that this linear combination could be infinite.

We say that a sequence of positive integers $w = (i_1, i_2, \dots, i_\ell)$ has *content* $(c_1, c_2, \dots, c_r)$ if w consists of c_1 1's, c_2 2's, and so on up to c_r r's. If the content of each subsequence $(i_k, \dots, i_\ell)$ of w is a partition, then w is called a *reverse lattice word*. Now if $\nu = (\nu_1, \dots, \nu_p)$ is a partition, we define $u^\nu = u_p^{\nu_p} \cdots u_2^{\nu_2} u_1^{\nu_1} \in \mathcal{L}$.

Lemma 5.3. *A sequence $w = (i_1, i_2, \dots, i_\ell)$ is a reverse lattice word with content ν if and only if $u_{i_1} u_{i_2} \cdots u_{i_\ell} = u^\nu$ in $\mathcal{L}$.*

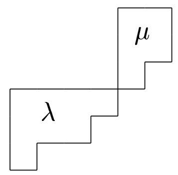
Proof. If w is a reverse lattice word, then the rectification of w in the plactic monoid is the semistandard Young tableau $U(\nu)$ of shape ν in which all boxes in row i contain the integer i [10, Lemma 5.1]. This implies that the identity $u_{i_1} u_{i_2} \cdots u_{i_\ell} = u^\nu$ holds even with the weaker relations of the plactic algebra.

On the other hand, if $u_{i_1} u_{i_2} \cdots u_{i_\ell} = u^\nu$ then one can obtain the sequence $(p^{\nu_p}, \dots, 2^{\nu_2}, 1^{\nu_1})$ from w by replacing subsequences in the following ways:

$$\begin{aligned} (i,j) &\leftrightarrow (j,i) \qquad \text{if } |i-j| \geq 2 \\ (i,i+1,i) &\leftrightarrow (i+1,i,i) \\ (i+1,i,i+1) &\leftrightarrow (i+1,i+1,i) \end{aligned}$$

Since all of these moves preserve reverse lattice words, w must be a reverse lattice word with content ν. □

If λ and μ are partitions, we let $\lambda * \mu$ be the skew diagram obtained by putting λ and μ corner to corner as shown:

$\lambda * \mu =$ 

Theorem 5.4. *The coefficient $c^{\nu}_{\lambda\mu}$ is equal to $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of set-valued tableaux T of shape $\lambda * \mu$ such that $w(T)$ is a reverse lattice word with content ν.*

Proof. Start by noticing that the only tableau of shape ν whose word is a reverse lattice word is the tableau $U(\nu)$. It follows from this that the coefficient of u^{ν} in the right hand side of (5.1) is $c^{\nu}_{\lambda\mu}$.

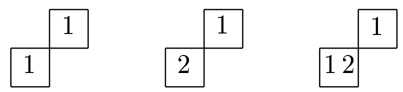
On the other hand, the left hand side is equal to $G_{\lambda*\mu}(u)$. If T is a tableau on $\lambda * \mu$, then u^T is equal to u^{ν} exactly when $w(T)$ is a reverse lattice word with content ν by the lemma. The theorem follows from this. □

Since there are only finitely many tableaux T of a given shape such that the word of T is a reverse lattice word, Theorem 5.4 implies that the linear combination in (5.1) is finite. In other words $\mathcal{G}$ is a commutative subring of $\hat{\mathcal{L}}$.

Corollary 5.5. $\Gamma = \bigoplus \mathbb{Z} \cdot G_\lambda$ *is closed under multiplication. The structure constants $c^{\nu}_{\lambda\mu}$ such that $G_\lambda \cdot G_\mu = \sum_\nu c^{\nu}_{\lambda\mu} G_\nu$ are given by Theorem 5.4.*

Proof. By replacing each non-commutative variable u_i with x_i in (5.1), we obtain the identity $G_\lambda(x) \cdot G_\mu(x) = \sum_\nu c^{\nu}_{\lambda\mu} G_\nu(x)$ for single stable Grothendieck polynomials. Since $G_w(1-e^{-x}; 1-e^{y})$ is super symmetric, the same equation must hold for the double polynomials as well. □

Example 5.6. For the shape $(1) * (1)$ we can find the following three set-valued tableaux whose contents are reverse lattice words:



It follows that $G_1 \cdot G_1 = G_2 + G_{(1,1)} - G_{(2,1)}$.

Our methods seem insufficient to prove that a stable Grothendieck polynomial $G_{\nu/\lambda}(u)$ in non-commutative variables is in $\hat{\mathcal{G}}$, except when the skew shape ν/λ is a product of partitions like in Theorem 5.4. If this could be established, then the proof of Theorem 5.4 would also prove a rule for writing $G_{\nu/\lambda}$ as a linear combination of the stable polynomials G_μ for Grassmannian permutations. We shall instead derive such a formula from the statement of Theorem 5.4 in the next section.

6. A coproduct on stable Grothendieck polynomials

Our main task in this section is to show that the ring Γ has a natural coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ which makes it a bialgebra. We will show that for certain integers

$d^{\nu}_{\lambda\mu}$ given by an explicit Littlewood-Richardson rule similar to that of Theorem 5.4, we have

$$G_\nu(x,z;w,y) = \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu}\, G_\lambda(x;w)\cdot G_\mu(z;y) \tag{6.1}$$

whenever x, y, z, and w are different sets of variables. The coproduct can then be defined by $\Delta\, G_\nu = \sum d^{\nu}_{\lambda\mu}\, G_\lambda \otimes G_\mu$.

Lemma 6.1. *Let $\lambda \subset \nu$ be partitions and let $p, q \geq 1$ be integers. Then*

$$G_{\nu/\lambda}(x_1,\dots,x_{p+q}) = \sum_{\tau/\sigma \text{ rook strip}} (-1)^{|\tau/\sigma|}\, G_{\tau/\lambda}(x_1,\dots,x_p)\cdot G_{\nu/\sigma}(x_{p+1},\dots,x_{p+q})$$

where the sum is over all partitions σ and τ such that $\lambda \subset \sigma \subset \tau \subset \nu$ and τ/σ is a rook strip.

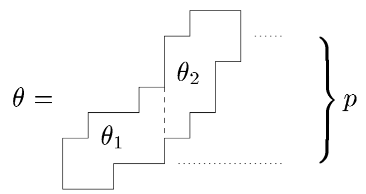
Proof. By Theorem 3.1 the left hand side is the signed sum of monomials x^T for all tableaux T of shape ν/λ such that all contained integers are between 1 and $p+q$. If T is such a tableaux, let T_1 be the subtableau obtained by removing all integers strictly larger than p (as well as all boxes that become empty), and let T_2 be obtained by removing integers less than or equal to p. Then the shape of T_1 is τ/λ for some partition τ, while the shape of T_2 is of the form ν/σ, and we have $\lambda \subset \sigma \subset \tau \subset \nu$. Since τ/σ is the shape where T_1 and T_2 overlap, this skew shape must be a rook strip. Finally, since $x^T = x^{T_1}\cdot x^{T_2}$ this gives a bijection between the terms of the two sides of the claimed identity. □

Notice that since the polynomials $G_{\nu/\lambda}(1-e^{-x}; 1-e^{y})$ are super-symmetric, this lemma implies that if x, y, z, and w are different set of variables, then we have

$$G_{\nu/\lambda}(x,z;w,y) = \sum_{\tau/\sigma \text{ rook strip}} (-1)^{|\tau/\sigma|}\, G_{\tau/\lambda}(x;w)\cdot G_{\nu/\sigma}(z;y)\,. \tag{6.2}$$

This can be deduced by writing each polynomial $G_w(1-e^{-x}; 1-e^{y})$ as a linear combination of double Schur functions [20].

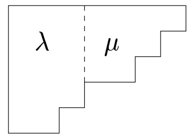
Lemma 6.2. *Let θ be a skew shape which is broken up into two smaller skew shapes θ_1 and θ_2 by a vertical line as shown.*



Let p be the number of boxes between the top edge of the leftmost column of θ_2 and the bottom edge of the rightmost column of θ_1. Then we have $G_\theta(x_1,\dots,x_p) = G_{\theta_1}(x_1,\dots,x_p)\cdot G_{\theta_2}(x_1,\dots,x_p)$.

Proof. Number the rows of θ such that the top box in the leftmost column of θ_2 is in row number one, and so that the numbers increase from top to bottom. Then suppose T_1 and T_2 are tableaux of shapes θ_1 and θ_2 for which all contained integers are less than or equal to p. Then all integers in row i of T_2 will be greater than or equal to i because they have at least $i-1$ boxes above them. Similarly the integers in row i of T_1 will be smaller than or equal to i because they have $p-i$ boxes below them. Therefore T_1 and T_2 fit together to form a tableau T of shape θ. This shows that the terms of G_θ and $G_{\theta_1} \cdot G_{\theta_2}$ are in bijective correspondence. ☐

Proposition 6.3. *Let ν be a Young diagram which is broken up into two smaller Young diagrams λ and μ by a vertical line after column q.*

$\nu =$ 

Then if p is the length of the last column of λ we have

$$G_\nu(x_1,\dots,x_p;y_1,\dots,y_q) = G_\lambda(x_1,\dots,x_p;y_1,\dots,y_q)\cdot G_\mu(x_1,\dots,x_p)\,.$$

Proof. In this proof we will write x for the variables $x_1,\dots,x_p$ and y for $y_1,\dots,y_q$. Then by Lemma 6.1 we have

$$G_\nu(x;y) = \sum G_\tau(0;y)\cdot G_{\nu/\sigma}(x)$$

where the sum is over all partitions $\sigma \subset \tau \subset \nu$ such that τ/σ is a rook strip. Notice that when τ has more than q columns, then $G_\tau(0;y) = G_{\tau'}(y) = 0$ by Lemma 3.4 and Theorem 3.1. Therefore we only need to include terms for which $\tau \subset \lambda$ in the sum. For such terms Lemma 6.2 implies that $G_{\nu/\sigma}(x) = G_{\lambda/\sigma}(x)\cdot G_\mu(x)$. The lemma follows from this by applying Lemma 6.1 to $G_\lambda(x;y)$. ☐

Corollary 6.4. *If $\nu_{p+1} \geq q+1$ then $G_\nu(x_1,\dots,x_p;y_1,\dots,y_q) = 0$.*

Proof. Let λ be the first q columns of ν and let μ be rest like in Proposition 6.3. Then since $\ell(\mu) > p$ we get $G_\mu(x_1,\dots,x_p) = 0$. The statement therefore follows from the proposition. ☐

Corollary 6.5 (Factorization formula)**.** *Let $R = (q)^p$ be a rectangle with p rows and q columns, and let σ and τ be partitions such that $\ell(\sigma) \leq p$ and $\tau_1 \leq q$. Let $(R+\sigma,\tau)$ denote the partition $(q+\sigma_1,\dots,q+\sigma_p,\tau_1,\tau_2,\dots)$ obtained by attaching σ and τ to the right and bottom sides of R. Then*

$$\begin{aligned}&G_{R+\sigma,\tau}(x_1,\dots,x_p;y_1,\dots,y_q) = \\ &\qquad G_\tau(0;y_1,\dots,y_q)\cdot G_R(x_1,\dots,x_p;y_1,\dots,y_q)\cdot G_\sigma(x_1,\dots,x_p)\,.\end{aligned}$$

Proof. Let x denote $x_1,\dots,x_p$ and let y denote $y_1,\dots,y_q$. Then Proposition 6.3 implies that $G_{R+\sigma,\tau}(x;y) = G_{R,\tau}(x;y)\cdot G_\sigma(x)$. Now using Lemma 3.4 we obtain $G_{R,\tau}(x;y) = G_{R'+\tau'}(y;x) = G_{R'}(y;x)\cdot G_{\tau'}(y) = G_\tau(0;y)\cdot G_R(x;y)$. ☐

We are now ready to define the structure constants for the coproduct in Γ. We will say that a sequence of integers $w = (i_1,\dots,i_\ell)$ is a *partial reverse lattice word* with respect to an integer interval $[a,b]$, if for all $1 \leq k \leq \ell$ and $p \in [a,b-1]$, the subsequence $(i_k,\dots,i_\ell)$ has more occurrences of p than of $p+1$.

Given three partitions $\lambda = (\lambda_1,\dots,\lambda_p)$, $\mu = (\mu_1,\dots,\mu_q)$, and ν, we then define $d^\nu_{\lambda\mu}$ to be $(-1)^{|\lambda|+|\mu|-|\nu|}$ times the number of set-valued tableaux T of shape ν such that $w(T)$ is a partial reverse lattice word with respect to both of the intervals $[1,p]$ and $[p+1,p+q]$, and with content $(\lambda,\mu) = (\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q)$. Notice that if R is a rectangle which is taller than λ and wider than μ then $d^\nu_{\lambda\mu} = c^{R+\lambda,\mu}_{\nu,R}$ by Theorem 5.4.

Theorem 6.6. *For any partition ν we have*

$$G_\nu(x;y) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, G_\lambda(x) \cdot G_\mu(0;y)\,.$$

Proof. It is enough to show this for finitely many variables $x_1, \dots, x_p$ and $y_1, \dots, y_q$, as long as p and q can be arbitrarily large. Let $R = (q)^p$ be a rectangle with p rows and q columns. If ρ is a partition such that G_ρ occurs with non-zero coefficient in $G_R \cdot G_\nu$ then first of all $R \subset \rho$. Furthermore, Corollary 6.4 shows that $G_\rho(x;y)$ is non-zero only if ρ has the form $\rho = (R + \lambda, \mu)$ for partitions λ and μ. By these observations we get

$$G_R(x;y) \cdot G_\nu(x;y) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, G_{R+\lambda,\mu}(x;y) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, G_R(x;y) \cdot G_\lambda(x) \cdot G_\mu(0;y)\,.$$

Since Theorem 3.1 shows that $G_R(x;y) \neq 0$, this proves the theorem. □

Again using the fact that $G_\lambda(1 - e^{-x}; 1 - e^{y})$ is super-symmetric, this theorem implies that $G_\nu(x, z; w, y) = \sum d^\nu_{\lambda\mu}\, G_\lambda(x;w) \cdot G_\mu(z;y)$ whenever x, y, z, and w are different sets of variables.

Corollary 6.7. *The ring $\Gamma = \bigoplus \mathbb{Z} \cdot G_\lambda$ is a commutative and cocommutative bialgebra with product $\Gamma \otimes \Gamma \to \Gamma$ given by*

$$G_\lambda \cdot G_\mu = \sum_\nu c^\nu_{\lambda\mu}\, G_\nu$$

and coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ given by

$$\Delta\, G_\nu = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, G_\lambda \otimes G_\mu\,.$$

The linear map $\Gamma \to \mathbb{Z}$ sending $G_{(0)} = 1$ to one and G_λ to zero for $\lambda \neq \emptyset$ is a counit. Furthermore, conjugation of partitions defines an involution of this bialgebra

$$G_\lambda(x;y) \mapsto G_{\lambda'}(x;y) = G_\lambda(y;x)\,.$$

Proof. These statements are clear from Theorems 5.4 and 6.6, and Lemma 3.4. □

Example 6.8. Using the definition of the coefficients $d^\nu_{\lambda\mu}$ we may compute

$$\Delta\, G_1 = G_1 \otimes 1 + 1 \otimes G_1 - G_1 \otimes G_1\,.$$

We will finish this section by showing that Γ indeed contains all stable Grothendieck polynomials. We start by generalizing the Littlewood-Richardson rule of Theorem 5.4 to a rule for the stable Grothendieck polynomial of any 321-avoiding permutation.

Given partitions $\lambda \subset \nu$ we define $G_{\nu /\!\!/ \lambda} = \sum_\mu d^\nu_{\lambda\mu}\, G_\mu$. It then follows from Corollary 6.7 that

$$\Delta\, G_\nu = \sum_{\lambda \subset \nu} G_\lambda \otimes G_{\nu /\!\!/ \lambda}\,, \tag{6.3}$$

and $G_{\nu /\!\!/ \lambda}$ is furthermore uniquely defined by this identity. Therefore we deduce from Lemma 6.1 that

$$G_{\nu /\!\!/ \lambda} = \sum_{\lambda/\sigma \text{ rook strip}} (-1)^{|\lambda/\sigma|}\, G_{\nu/\sigma} \tag{6.4}$$

where the sum is over all partitions $\sigma \subset \lambda$ such that λ/σ is a rook strip.

Now given a skew shape $\theta = \nu/\lambda$ and a partition μ, let $\alpha_{\theta,\mu} = \alpha_{\nu/\lambda,\mu}$ be $(-1)^{|\mu|-|\theta|}$ times the number of set-valued tableaux T of shape θ such that $w(T)$ is a reverse lattice word with content μ.

Theorem 6.9. *For any skew shape $\theta = \nu/\lambda$ we have*

$$G_{\nu/\lambda} = \sum_{\mu} \alpha_{\theta,\mu}\, G_\mu \,.$$

Proof. We will start by comparing the coefficients $d^\nu_{\lambda\mu}$ and $\alpha_{\theta,\mu}$. Suppose T is a tableau of shape ν such that $w(T)$ is a partial reverse lattice word with respect to the intervals $[1, \ell(\lambda)]$ and $[\ell(\lambda)+1, \ell(\lambda)+\ell(\mu)]$ and with content (λ, μ). Then all integers in T which come from the interval $[1, \ell(\lambda)]$ must be located in the upper-left corner in T of shape λ. Furthermore, any such integer i can occur only in row i. Now let the skew shape ν/σ be the region in which the integers larger than $\ell(\lambda)$ are located in T. Since this region can only overlap λ in a rook strip, λ/σ must be a rook strip. If you remove all integers smaller than or equal to $\ell(\lambda)$ from T and subtract $\ell(\lambda)$ from the rest, then the result is a tableau of shape ν/σ whose word is a reverse lattice word with content μ. Since T is uniquely determined by λ and this skew tableau, we obtain

$$d^\nu_{\lambda\mu} = \sum_{\lambda/\sigma \text{ rook strip}} (-1)^{|\lambda/\sigma|}\, \alpha_{\nu/\sigma,\mu}$$

where this sum is over all partitions $\sigma \subset \lambda$ such that λ/σ is a rook strip.

To finish the proof we set $\tilde{G}_{\nu/\sigma} = \sum_\mu \alpha_{\nu/\sigma,\mu}\, G_\mu$. Then we obtain

$$G_{\nu /\!\!/ \lambda} = \sum_\mu \sum_{\lambda/\sigma \text{ rook strip}} (-1)^{|\lambda/\sigma|}\, \alpha_{\nu/\sigma,\mu}\, G_\mu = \sum_{\lambda/\sigma \text{ rook strip}} (-1)^{|\lambda/\sigma|}\, \tilde{G}_{\nu/\sigma} \,.$$

By comparing with (6.4) and noting that the transition matrix between the $G_{\nu/\lambda}$ and the $G_{\nu/\!\!/\lambda}$ for fixed ν is invertible, we conclude that $G_{\nu/\lambda} = \tilde{G}_{\nu/\lambda}$. ☐

Let us remark that all of the theorems 3.1, 5.4, 6.6, and 6.9 can be summarized in the following statement. We leave the details to the reader.

Corollary 6.10. *The $n-1$ fold coproduct applied to $G_{\nu/\lambda}$ is given by*

$$\Delta^{n-1}\, G_{\nu/\lambda} = \sum_{\mu(1),\dots,\mu(n)} d^{\nu/\lambda}_{\mu(1),\dots,\mu(n)}\, G_{\mu(1)} \otimes \cdots \otimes G_{\mu(n)}$$

where $d^{\nu/\lambda}_{\mu(1),\dots,\mu(n)}$ is $(-1)^{|\nu/\lambda|+\sum|\mu(i)|}$ times the number of set-valued tableaux T of shape ν/λ such that $w(T)$ is a partial reverse lattice word with respect to each of the intervals $\left[1+\sum_{i=1}^{k-1}\ell(\mu(i))\ ,\ \sum_{i=1}^{k}\ell(\mu(i))\right]$ for $1 \le k \le n$, and has content $(\mu(1),\dots,\mu(n))$.

Finally, we will give an independent argument showing that the stable Grothendieck polynomial $G_w = G_w(x; y)$ for any permutation w is contained in Γ. Recall that the single Schur function $s_\lambda(x)$ for a partition λ is defined by

$$s_\lambda(x) = \sum_T x^T$$

where the sum is over all semistandard tableaux T of shape λ [20, 10]. This is the term of lowest degree in the single stable polynomial $G_\lambda(x)$.

If μ is a partition containing λ, let $g_{\lambda\mu}$ be the number of row and column strict tableaux of shape μ/λ such that all entries in the i'th row are between 1 and $i-1$. The boxes of these tableaux should contain single integers. Then Lenart has proved the following result [18].

Theorem 6.11 (Lenart)**.** *Let w_λ be the Grassmannian permutation for λ with descent in position $p \geq \ell(\lambda)$. Then the single Grothendieck polynomial $\mathfrak{G}_{w_\lambda}(x)$ is given by*

$$\mathfrak{G}_{w_\lambda}(x) = \sum_{\mu \supset \lambda} (-1)^{|\mu/\lambda|} g_{\lambda\mu}\, s_\mu(x_1, \dots, x_p)$$

where the sum is over all partitions μ containing λ.

As a consequence we obtain the formula

$$G_\lambda(x) = \sum_{\mu \supset \lambda} (-1)^{|\mu/\lambda|} g_{\lambda\mu}\, s_\mu(x)\,. \tag{6.5}$$

This formula can also be derived from Theorem 3.1. We will briefly sketch the argument. It is sufficient to construct a bijection between set-valued tableaux T of shape λ and pairs (U,T) where U is a semistandard tableau of some shape μ containing λ and S is one of the row and column strict tableaux contributing to $g_{\lambda\mu}$. We shall do this by induction on $\ell = \ell(\lambda)$.

Given a set-valued tableau T of shape λ, let R be the top row of T and let T' be the rest. By induction we can assume that T' corresponds to a pair (U', S'). Now let $\tilde{R}$ be the unique semistandard tableau of shape $(\lambda_1, 1^m)$ where $m = |R| - \lambda_1$, such that $x^{\tilde{R}} = x^R$ and each box in the top row of $\tilde{R}$ contains the smallest integer in the corresponding box of R. For example:

$$R = \begin{array}{|c|c|c|c|}\hline 12 & 2 & 23 & 4 \\ \hline \end{array} \quad \text{gives} \quad \tilde{R} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

Now let $U = U' \cdot \tilde{R}$ be the product of U' and $\tilde{R}$ in the sense of [10, §1]. Then U has shape $\mu = (\lambda_1, \sigma)$ where the partition σ is obtained by adding a vertical strip to the shape of U'. Finally, let S be the skew tableau of shape μ/λ obtained by copying the i'th row of S' to the $i+1$'st row of S; if S needs an additional box in this row, we put i in this box. One may then check that the map $T \mapsto (U,S)$ gives the desired bijection. For example:

$$T = \begin{array}{|c|c|c|}\hline 1 & 12 & 34 \\ \hline 23 & 45 \\ \cline{1-2} \end{array} \quad \text{gives} \quad (U,S) = \left(\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \;,\; \begin{array}{|c|c|c|}\hline & & \\ \cline{3-3} & & 1 \\ \hline 1 & 2 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \right)$$

Equation (6.5) shows that a stable polynomial $G_\lambda(x)$ is an infinite linear combination of Schur functions $s_\mu(x)$ for partitions $\mu \supset \lambda$ such that μ and λ have the same number of columns. Since the coefficient of $s_\lambda(x)$ is $g_{\lambda\lambda} = 1$, we can also go in the opposite direction and write each Schur function $s_\mu(x)$ as an infinite linear combination of single stable Grothendieck polynomials for partitions with the same number of columns as μ. In fact Lenart also gives an explicit formula for doing this [18].

We also need the following result of Fomin and Greene [7]. If $w \in S_n$ is a permutation and λ is a partition, let $g_{w\lambda}$ be the number of semistandard tableaux T of shape λ' such that if the column word of T is $w(T) = (i_1, \dots, i_r)$ then $u_{i_1} \cdots u_{i_r} = \pm u_w$ in the degenerate Hecke algebra $H_n(0)$. Notice that this implies that $g_{w\lambda}$ is zero if λ has n or more columns.

Theorem 6.12 (Fomin and Greene)**.** *The single stable Grothendieck polynomial $G_w(x)$ is given by*

$$G_w(x) = \sum_\lambda (-1)^{|\lambda|-\ell(w)} g_{w\lambda}\, s_\lambda(x)\,.$$

We can finally prove:

Theorem 6.13. *Let $w \in S_n$ be a permutation. Then the double stable Grothendieck polynomial $G_w(x;y)$ can be written as a finite linear combination*

$$G_w(x;y) = \sum_\lambda \alpha_{w\lambda}\, G_\lambda(x;y) \tag{6.6}$$

of the polynomials $G_\lambda(x;y)$ for $|\lambda| \geq \ell(w)$. In particular G_w is an element of Γ.

Proof. It follows from (6.5) and Theorem 6.12 that the single polynomial $G_w(x)$ is a possibly infinite linear combination of polynomials $G_\lambda(x)$ for partitions λ with at most $n-1$ columns:

$$G_w(x) = \sum_\lambda \alpha_{w\lambda} G_\lambda(x)\,.$$

Using that $G_w(1-e^{-x}; 1-e^{y})$ is super symmetric, this implies that (6.6) is true with the same coefficients.

Let $w_0 = n \cdots 2\,1$ be the longest permutation of S_n. Using Lemma 3.4 we then deduce that $\alpha_{w\lambda} = \alpha_{w_0ww_0,\lambda'}$ is zero unless $\ell(\lambda) = \lambda'_1 < n$. We conclude that $G_w(x;y)$ is a finite linear combination of the polynomials $G_\lambda(x;y)$. $\square$

Since the term of lowest degree in $G_w(x)$ is the stable Schubert polynomial or Stanley symmetric function $F_w(x)$, it follows that when $|\lambda| = \ell(w)$ the coefficient $\alpha_{w\lambda}$ is the one defined by Stanley [23]. The coefficients $\alpha_{w\lambda}$ also generalize the structure constants $c^\nu_{\lambda\mu}$ and $d^\nu_{\lambda\mu}$ of Γ. On the other hand the coefficients $\alpha_{w\lambda}$ are special cases of quiver coefficients $c_r(\mu)$ which will be introduced in [2]. We believe that these quiver coefficients have alternating signs. A particular case of this has been confirmed by A. Lascoux [15].

Theorem 6.14 (Lascoux)**.** *The coefficients $\alpha_{w\lambda}$ have alternating signs, in other words $(-1)^{|\lambda|-\ell(w)}\alpha_{w\lambda}$ is non-negative.*

Lascoux's proof is based on a beautiful recursive formula for writing a stable Grothendieck polynomial G_w as a linear combination of the basis elements G_λ.

It would be interesting to define a non-commutative polynomial $G_w(u)$ in the local plactic algebra for any permutation w. Such a definition might lead to a Littlewood-Richardson rule for the coefficients $\alpha_{w\lambda}$.

7. Consequences of the Littlewood-Richardson rule

In this section we will derive some consequences of the formulas proved in the past two sections. We will start with a Pieri rule for the coproduct in Γ which is analogous to Lenart's Pieri rule of Corollary 4.7. If $\lambda = (\lambda_1, \dots, \lambda_p)$ is a partition, we let $\bar\lambda = (\lambda_2, \dots, \lambda_p)$ be the partition obtained by removing the top row of λ.

Corollary 7.1. *If λ/μ is a horizontal strip and $k \geq 0$ an integer, then*

$$d^{\lambda}_{\mu,(k)} = (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar{\lambda})}{k-|\lambda/\mu|}$$

where $r(\mu/\bar{\lambda})$ is the number of rows in $\mu/\bar{\lambda}$. If λ/μ is not a horizontal strip then $d^{\lambda}_{\mu,(k)} = 0$.

This is an immediate consequence of Theorem 6.6. Notice that this implies that

$$G_\lambda(x_1,\dots,x_p) = \sum_{\mu,k} (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar{\lambda})}{k-|\lambda/\mu|} G_\mu(x_1,\dots,x_{p-1})\, x_p^k$$

where the sum is over all integers $k \geq 0$ and all partitions $\mu \subset \lambda$ such that λ/μ is a horizontal strip. This gives a practical way to calculate stable Grothendieck polynomials $G_\lambda(x)$, which can easily be extended to double stable Grothendieck polynomials and 321-avoiding permutations. For example, if $R = (q)^p$ is a rectangle with p rows and q columns, then

$$G_R(x_1,\dots,x_p;y_1,\dots,y_q) = \prod_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} (x_i + y_j - x_i\, y_j)\,. \tag{7.1}$$

To see this, use the Pieri rule and Corollary 6.5 to write

$$\begin{aligned} G_R(x_1,\dots,x_p;y) &= \sum_{q\leq j+k\leq q+1} (-1)^{j+k-q}\, G_{(q^{p-1},j)}(x_1,\dots,x_{p-1};y)\cdot x_p^k \\ &= \sum_{q\leq j+k\leq q+1} (-1)^{j+k-q}\, G_{(q)^{p-1}}(x_1,\dots,x_{p-1};y)\cdot G_j(0;y)\cdot x_p^k \\ &= G_{(q)^{p-1}}(x_1,\dots,x_{p-1};y)\cdot G_q(x_p;y) \end{aligned}$$

where y denotes the variables $y_1,\dots,y_q$. Since $G_R(x;y) = G_{R'}(y;x)$, equation (7.1) follows from this by induction on p and q. Notice that $G_1(x_i;y_j) = x_i + y_j - x_i\, y_j$ by Example 6.8, Theorem 3.1, and Lemma 3.4.

As another application of Corollary 7.1, notice that if λ/μ is a horizontal strip then

$$\sum_{k\geq 0} d^{\lambda}_{\mu,(k)} = \sum_{k\geq 0} (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar{\lambda})}{k-|\lambda/\mu|} = \begin{cases} 1 & \text{if } \mu = \bar{\lambda}; \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$G_\lambda(1,x_1,x_2,\dots) = \sum_{\mu,k} d^{\lambda}_{\mu,(k)}\, G_\mu(x)\cdot 1^k = G_{\bar{\lambda}}(x)\,.$$

In particular, if we let $\phi_p : \Gamma \to \Gamma$ be the linear map which maps G_λ to G_μ where $\mu = (\lambda_{p+1},\lambda_{p+2},\dots)$ is obtained by removing the top p rows of λ, then ϕ_p is a ring homomorphism. Being used to calculating in the ring of symmetric functions, we find this somewhat surprising. Similarly we have $\Delta\,\phi_p = (\phi_i \otimes \phi_j)\,\Delta$ whenever $i+j=p$. This implies lots of identities among the structure constants $c^{\nu}_{\lambda\mu}$ and $d^{\nu}_{\lambda\mu}$ of Γ. For example, if $\emptyset \neq \mu \subset \lambda$ are partitions and $p \geq \ell(\lambda)$, then by comparing the coefficients of $G_\mu \otimes 1$ in $\Delta\,\phi_p\, G_\lambda = (1\otimes\phi_p)\,\Delta\, G_\lambda$ we obtain the identity $\sum_\tau d^{\lambda}_{\mu\tau} = 0$. The map ϕ_p is related to the proper pushforward map on Grothendieck groups from a Grassmann bundle to its base variety. See [2] for details about this.

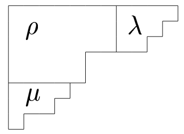
We will next describe some identities involving rectangles. Given a rectangle $R = (q)^p$ with p rows and q columns and a Young diagram μ contained in R, we let $\hat{\mu}$ denote μ rotated 180 degrees and put in the lower-right corner of R. Then Theorem 6.6 implies the following multiplicity free formula for the coefficients of $\Delta\, G_R$:

$$d^R_{\lambda\mu} = \begin{cases} (-1)^{|\lambda|+|\mu|-|R|} & \text{if } \lambda \cup \hat{\mu} = R \text{ and } \lambda \cap \hat{\mu} \text{ is a rook strip.} \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

There is a similar formula for multiplying the stable Grothendieck polynomials of two rectangles R_1 and R_2. Let $R = R_1 \cap R_2$ be their intersection and $\rho = R_1 \cup R_2$ their union. Then we have

$$G_{R_1} \cdot G_{R_2} = \sum_{\lambda,\mu} d^R_{\lambda\mu}\, G_{\rho+\lambda,\mu} \tag{7.3}$$

where $(\rho + \lambda, \mu)$ is the partition

$(\rho + \lambda, \mu) =$ 

Assuming that R_2 is the tallest of the two rectangles, this formula is easy to prove from Theorem 5.4 by counting set-valued tableaux of shape $R_1 * R_2$.

J. Stembridge has recently classified all multiplicity free products of Schur functions [25]. We will say that a product $G_\lambda \cdot G_\mu$ is multiplicity free if all constants $c^\nu_{\lambda\mu}$ are ± 1 or zero. Stembridge's result then has the following analogue for Grothendieck polynomials.

Proposition 7.2. *The product $G_\lambda \cdot G_\mu$ is multiplicity free if and only if both partitions λ and μ are rectangles or one of them is a single box or empty.*

Proof. It is enough to show that if λ is not a rectangle and $\ell = \ell(\mu) \geq 2$ then the absolute value of some coefficient $c^\nu_{\lambda\mu}$ is at least two. The assumptions on λ and ℓ imply that we can find a partition ρ containing λ such that ρ/λ is a vertical strip with $\ell + 1$ boxes and at least three columns. Then it follows from Lenart's Pieri formula that $c^\rho_{\lambda,(1^\ell)} \leq -2$, which means that there exist two different tableaux T_1 and T_2 of shape λ such that the concatenations $w(T_1) \circ (\ell, \ell-1, \dots, 1)$ and $w(T_2) \circ (\ell, \ell-1, \dots, 1)$ are reverse lattice words with the same content ρ. But then $w(T_1) \circ w(U(\mu))$ and $w(T_2) \circ w(U(\mu))$ must also be reverse lattice words with the same content, so $c^\nu_{\lambda,\mu} \leq -2$ if ν is the content of $w(T_i) \circ w(U(\mu))$. □

Similarly, a coproduct $\Delta\, G_\lambda$ is multiplicity free if and only if λ is a rectangle.

A related consequence of Theorem 5.4, which we will need in Section 8, is that if a basis element G_R for a rectangular partition R occurs with non-zero coefficient in a product $G_\lambda \cdot G_\mu$, then R is the disjoint union of λ and $\hat{\mu}$ (which in particular means that $c^R_{\lambda\mu} = 1$). This can be proved as follows. Start with R and the partition μ, and look for a setvalued tableau of shape $\lambda * \mu$ for some partition λ, such that this tableau has content R. The filling of μ then has to be $U(\mu)$. Now construct λ and the tableau on λ by first filling 1 in some boxes, then 2, etc. It is then easy to see that at each step there is only one choice, i.e. both λ and the tableau on λ are uniquely determined by the requirement that the word of the tableau on $\lambda * \mu$ is a reverse lattice word with content R.

As noted earlier, the polynomials $G_{\nu/\lambda}$ are uniquely determined by (6.4). It is not hard to see that the formula in the opposite direction is

$$G_{\nu/\lambda} = \sum_{\sigma \subset \lambda} G_{\nu /\!\!/ \sigma} \tag{7.4}$$

where this sum is over all partitions σ contained in λ. From this we obtain the following inverse of the relation $d^{\nu}_{\lambda\mu} = c^{R+\lambda,\mu}_{\nu,R}$ between the structure constants of Γ. Namely, if R is a rectangle which contains λ and μ then

$$c^{\nu}_{\lambda\mu} = \alpha_{\mu * \lambda, \nu} = \alpha_{(R+\lambda,\mu)/R,\nu} = \sum_{\sigma \subset R} d^{R+\lambda,\mu}_{\nu,\sigma} \,.$$

We will finish this section by proving some results concerning the shapes of partitions which give non-zero constants $c^{\nu}_{\lambda\mu}$, $d^{\nu}_{\lambda\mu}$, or $\alpha_{\nu/\lambda,\mu}$. We will need a few lemmas which allow us to make small changes to set-valued tableaux. Let the *shared boxes* of a tableau be the boxes that contain two or more integers.

Lemma 7.3. *Let T be a set-valued tableau of shape ν/λ with at least one shared box, and let y be the largest integer contained in a shared box of T. Suppose v is a sequence of integers such that the concatenation $w(T) \circ v$ of the word of T with v is a reverse lattice word. Then there exists a set-valued tableau $\tilde{T}$ of shape ν/λ such that $w(\tilde{T}) \circ v$ is a reverse lattice word, and so that the integers contained in $\tilde{T}$ are the same as those in T, except one integer $x \geq y$ is left out. (In other words, the content of $(x) \circ w(\tilde{T})$ is equal to the content of $w(T)$.)*

Proof. Start by locating the leftmost shared box A of T which contains y. To construct $\tilde{T}$ we start by removing y from this box. Then look for the nearest box B below or to the left of A which contains $y+1$, such that the box above B does not contain y and the box to the left of B does not contain $y+1$. If no such box exists, then $w(T) \circ v$ stays a reverse lattice word even if y is removed from A. If we can locate a box B satisfying these requirements, we replace $y+1$ with y in this box. Notice that B can't be a shared box by the assumptions. We then continue in the same way, with B in the role of A and $y+1$ in the role of y. $\tilde{T}$ is the tableau resulting when no new box B can be obtained. $\square$

The following picture shows an example of the transformation described in the proof. The initial box A is the one in the upper-right corner and $y = 6$.

$$\begin{array}{|c|c|c|c|c|} \cline{4-5} \multicolumn{3}{c|}{} & 35 & 56 \\ \cline{3-5} \multicolumn{2}{c|}{} & 2 & 6 & 7 \\ \hline 23 & 3 & 7 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & 45 & 8 & \multicolumn{2}{c}{} \\ \cline{1-3} 8 & 8 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|c|} \cline{4-5} \multicolumn{3}{c|}{} & 35 & 5 \\ \cline{3-5} \multicolumn{2}{c|}{} & 2 & 6 & 6 \\ \hline 23 & 3 & 7 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & 45 & 8 & \multicolumn{2}{c}{} \\ \cline{1-3} 7 & 8 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

Corollary 7.4 (of proof)**.** *With the assumptions of Lemma 7.3, there exists a partition ρ obtained by adding a single box to ν and a tableau $\tilde{T}_1$ of shape ρ/λ such that $w(\tilde{T}_1) \circ v$ is a reverse lattice word and the content of $w(\tilde{T}_1)$ is equal to the content of $w(T)$.*

Proof. If λ is empty so the shape of T is the partition ν, then we obtain $\tilde{T}_1$ as the product $x \cdot \tilde{T}$ where x is the integer which $\tilde{T}$ lacks compared to T. When this product is formed, the only boxes that can be effected are those containing integers strictly larger than y or those containing y which are to the left of the original box A in the construction of $\tilde{T}$. Since this implies that no shared boxes are modified, the shape of $\tilde{T}_1$ is only one box larger than ν. When ν/λ is not a partition, the same trick will work if we pretend that the boxes of λ are actually filled with small integers when the product $x \cdot \tilde{T}$ is formed. □

Notice that a sequence w of integers between 1 and b is a partial reverse lattice word with respect to two integer intervals $[1, a]$ and $[a+1, b]$ if and only if $w \circ (a^N, \dots, 2^N, 1^N)$ is a reverse lattice word for large N. Therefore Lemma 7.3 and Corollary 7.4 are still true if one replaces "reverse lattice word" with "partial reverse lattice word" for given integer intervals.

The following result says that the non-zero structure constants of Γ come in paths which start in the usual Littlewood-Richardson coefficients.

Proposition 7.5. *Let λ, μ, and ν be partitions.*

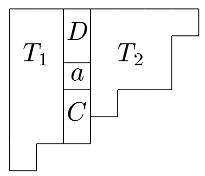
(i) *If $c^{\nu}_{\lambda\mu} \neq 0$ and $|\nu| > |\lambda| + |\mu|$ then there exists a partition $\tilde{\nu} \subset \nu$ of weight $|\tilde{\nu}| = |\nu| - 1$ such that $c^{\tilde{\nu}}_{\lambda\mu} \neq 0$.*
(ii) *If $c^{\nu}_{\lambda\mu} \neq 0$ and $|\nu| > |\lambda| + |\mu|$ then there exists a partition $\tilde{\mu} \supset \mu$ of weight $|\tilde{\mu}| = |\mu| + 1$ such that $c^{\nu}_{\lambda\tilde{\mu}} \neq 0$.*
(iii) *If $d^{\nu}_{\lambda\mu} \neq 0$ and $|\nu| < |\lambda| + |\mu|$ then there exists a partition $\tilde{\nu} \supset \nu$ of weight $|\tilde{\nu}| = |\nu| + 1$ such that $d^{\tilde{\nu}}_{\lambda\mu} \neq 0$.*
(iv) *If $d^{\nu}_{\lambda\mu} \neq 0$ and $|\nu| < |\lambda| + |\mu|$ then there exists a partition $\tilde{\mu} \subset \mu$ of weight $|\tilde{\mu}| = |\mu| - 1$ such that $d^{\nu}_{\lambda\tilde{\mu}} \neq 0$.*
(v) *If $\alpha_{\nu/\lambda,\mu} \neq 0$ and $|\mu| > |\nu/\lambda|$ then there exists a partition $\tilde{\mu} \subset \mu$ of weight $|\tilde{\mu}| = |\mu| - 1$ such that $\alpha_{\nu/\lambda,\tilde{\mu}} \neq 0$.*
(vi) *If $\alpha_{\nu/\lambda,\mu} \neq 0$ and $|\mu| > |\nu/\lambda|$ then there exists a partition $\tilde{\nu} \supset \nu$ of weight $|\tilde{\nu}| = |\nu| + 1$ such that $\alpha_{\tilde{\nu}/\lambda,\mu} \neq 0$.*
(vii) *If $\alpha_{\nu/\lambda,\mu} \neq 0$ and $|\mu| > |\nu/\lambda|$ then there exists a partition $\tilde{\lambda} \subset \lambda$ of weight $|\tilde{\lambda}| = |\lambda| - 1$ such that $\alpha_{\nu/\tilde{\lambda},\mu} \neq 0$.*

Proof. Lemma 7.3 implies (i), (iv), and (v), while Corollary 7.4 implies (ii), (iii), and (vi). For example, to prove (ii) from the corollary, recall that if $c^{\nu}_{\lambda\mu} \neq 0$ then there exists a tableau T of shape μ such that $w(T) \circ w(U(\lambda))$ is a reverse lattice word with content ν. Since $|\nu| > |\lambda| + |\mu|$, T must contain a shared box. We can therefore let $\tilde{\mu}$ be the shape of the tableau $\tilde{T}_1$ of Corollary 7.4.

Finally, to prove (vii) we need to show that if T is a tableau of shape ν/λ with at least one shared box such that $w(T)$ is a reverse lattice word with content ν, then there exists a tableau $\tilde{T}$ of a shape $\nu/\tilde{\lambda}$ such that $w(\tilde{T})$ is a reverse lattice word with the same content ν. Let y be the smallest integer contained in a shared box of T, and let A be the northernmost shared box containing y. Then start by removing y from this box. If all integers in the row above A are larger than or equal to y, then we can add a new box containing y at the left end of this row. Otherwise let B be the rightmost box in the row above A which contains an integer strictly less than y. Now replace the integer in B with y and continue in the same way with this integer in the role of y and B in the role of A. Using the induction hypothesis that some box strictly north of A contains y, it is not hard to check that this process stops before we reach the top row of T, and that the result is a tableau $\tilde{T}$ with the desired properties. □

Lemma 7.6. *Let T be a set-valued tableau whose shape is a partition λ, such that T has at least one shared box. Let y be the smallest integer contained in a shared box of T. Suppose v is a sequence of integers such that $w(T) \circ v$ is a reverse lattice word. Then there exists a tableau $\tilde{T}$ of shape λ and an integer $x \leq y$ such that $w(\tilde{T}) \circ v$ is a reverse lattice word and the content of $(x) \circ w(\tilde{T})$ is equal to the content of $w(T)$.*

Proof. Let a be the set in the leftmost shared box which contains y and let T_1, C, D, and T_2 be as in the picture.

$T =$ 

Let (T_1, y) be the tableau obtained by attaching a box containing y to the right side of T_1 and let $\theta = \mathrm{sh}(T_1, y)/\mathrm{sh}(T_1)$ be the skew diagram of this box. Then set $(x, \tilde{T}_1) = \mathcal{R}_\theta(T_1, y)$, and let $\tilde{T}$ be the tableau obtained from T by replacing T_1 with $\tilde{T}_1$ and a with $\tilde{a} = a \smallsetminus \{y\}$. Notice that x must be a single integer, since only integers less than or equal to y in T_1 are affected when forming $\mathcal{R}_\theta(T_1, y)$, and none of these are in shared boxes.

Since $w(T) \circ v = w(T_1) \circ w(C) \circ w(a) \circ w(D) \circ w(T_2) \circ v$ is a reverse lattice word, so is $w(T_1, y) \circ w(C) \circ w(\tilde{a}) \circ w(D) \circ w(T_2) \circ v$. But then Lemma 5.3 implies that $(x) \circ w(\tilde{T}_1) \circ w(C) \circ w(\tilde{a}) \circ w(D) \circ w(T_2) \circ v = (x) \circ w(\tilde{T}) \circ v$ is a reverse lattice word. The lemma follows from this. $\square$

Proposition 7.7. *If $c^\nu_{\lambda\mu} \neq 0$ then ν is contained in the union of all partitions ρ of weight $|\rho| = |\lambda| + |\mu|$, such that the Littlewood-Richardson coefficient $c^\rho_{\lambda\mu}$ is non-zero.*

Proof. It is enough to show that for each $1 \leq i \leq \ell(\nu)$ there is a partition ρ of weight $|\rho| = |\lambda| + |\mu|$ such that $c^\rho_{\lambda\mu} \neq 0$ and $\rho_i = \nu_i$. We will do this by induction on $|\nu|$, the case $|\nu| = |\lambda| + |\mu|$ being trivial.

By Theorem 5.4 there exists a tableau T of shape λ such that $w(T) \circ w(U(\mu))$ is a reverse lattice word with content ν. Since $|\nu| > |\lambda| + |\mu|$, this tableau must have at least one shared box. If this box contains an integer which is larger than i, then Lemma 7.3 gives us a tableau $\tilde{T}$ in which the number of i's is the same as in T. Otherwise some shared box contains an integer smaller than i, in which case we use Lemma 7.6 to produce $\tilde{T}$. Now if $\tilde{\nu} = \mathrm{sh}(\tilde{T})$, then $c^{\tilde{\nu}}_{\lambda\mu} \neq 0$ and $\tilde{\nu}_i = \nu_i$. Since $|\tilde{\nu}| = |\nu| - 1$, the required partition ρ exists by induction. $\square$

Let us finally remark that the triples of partitions (λ, μ, ν) for which $c^\nu_{\lambda\mu} \neq 0$ do not form a semigroup as is the case if one only considers Littlewood-Richardson coefficients [26]. For example, $c^{(2,1)}_{(1),(1)} = -1$ but $c^{(4,2)}_{(2),(2)} = 0$. However, Γ might still have the property that if $c^{N\nu}_{N\lambda,N\mu}$ is not zero for some integer $N > 1$ then $c^\nu_{\lambda\mu} \neq 0$. For Littlewood-Richardson coefficients this has been proved by Knutson and Tao [12]. The same question applies to the coefficients $d^\nu_{\lambda\mu}$ and $\alpha_{\nu/\lambda,\mu}$ as well.

8. K-theory of Grassmannians

This section establishes the link between K-theory of Grassmann varieties and the bialgebra Γ. This is then used to describe some results of A. Knutson regarding K-theoretic triple intersections on Grassmannians.

If E and F are vector bundles over a variety X and w is a permutation, we define an element $G_w(F-E)$ in the Grothendieck group of X as follows. Suppose first that $E = L_1 \oplus \cdots \oplus L_e$ and $F = M_1 \oplus \cdots \oplus M_f$ are direct sums of line bundles. Then we set

$$G_w(F-E) = G_w(1-M_1^{-1},\dots,1-M_f^{-1};1-L_1,\dots,1-L_e)\,.$$

Since the stable Grothendieck polynomial $G_w(x;y)$ is symmetric in both the x_i and the y_i separately, this expression can be written as a polynomial in the exterior powers of $F^\vee$ and E. For this reason the definition makes sense even when E and F do not have decompositions into line bundles. The fact that $G_\lambda(1-e^{-x};1-e^{y})$ is super symmetric translates into the formula $G_\lambda(F\oplus H - E\oplus H) = G_\lambda(F-E)$ where H is an arbitrary vector bundle. Lemma 3.4 says that $G_\lambda(F-E) = G_{\lambda'}(E^\vee - F^\vee)$.

Now let $X = \mathrm{Gr}(d,\mathbb{C}^n)$ be the Grassmann variety of d dimensional subspaces of $\mathbb{C}^n$, and let $S \subset \mathbb{C}^n \times X$ be the tautological subbundle of rank d on X. Let λ be a partition with at most d rows and at most $n-d$ columns. Then the class of the structure sheaf of the Schubert variety $\Omega_\lambda \subset X$ defined by (1.1) is given by

$$[\mathcal{O}_{\Omega_\lambda}] = G_\lambda(S^\vee) = G_\lambda(S^\vee - 0)\,. \tag{8.1}$$

To see this, let $Y = \mathrm{F}\ell(\mathbb{C}^n) = \{V_1 \subset \cdots \subset V_n = \mathbb{C}^n\}$ be the variety of full flags in $\mathbb{C}^n$ with tautological flag $F_1 \subset \cdots \subset F_n = \mathbb{C}^n \times Y$. For any permutation $w \in S_n$ there is a Schubert variety in Y defined by

$$\Omega_w = \{V_\bullet \in Y \mid \dim(V_p \cap \mathbb{C}^k) \geq p - r_w(p,n-k)\ \forall p,k\}$$

where $r_w(p,k) = \#\{i \leq p \mid w(i) \leq k\}$. It follows from [16] or [11, Thm. 3] that $[\mathcal{O}_{\Omega_w}] = \mathfrak{G}_w(1-L_1,\dots,1-L_n)$ in $K^\circ Y$, where $\mathfrak{G}_w(x)$ is the single Grothendieck polynomial associated to w and $L_i = F_i/F_{i-1}$. Now if $\rho : Y \to X$ is the map sending a flag $V_\bullet$ to the subspace V_d of dimension d, one can check [10, Prop. 10.9] that $\rho^{-1}(\Omega_\lambda) = \Omega_{w_\lambda}$, where $w_\lambda \in S_n$ is the Grassmannian permutation for λ with descent in position d. Since the pullback map $\rho^* : K^\circ X \to K^\circ Y$ is injective, it is therefore enough to show that $[\mathcal{O}_{\Omega_{w_\lambda}}] = G_{w_\lambda}(F_d^\vee)$ in $K^\circ(Y)$. This is true because $\mathfrak{G}_{w_\lambda}(x) = G_\lambda(x_1,\dots,x_d)$ by Theorem 6.11.

Now given any partition ν, let $I_\nu \subset \Gamma$ be the ideal spanned by the elements G_λ for all partitions λ which are not contained in ν.

Theorem 8.1. *The map $G_\lambda \mapsto G_\lambda(S^\vee)$ induces an isomorphism of rings $\Gamma/I_R \cong K^\circ\,\mathrm{Gr}(d,\mathbb{C}^n)$ where $R = (n-d)^d$ is a rectangle with d rows and $n-d$ columns.*

Proof. Since the map $G_\lambda \mapsto G_\lambda(S^\vee)$ is surjective by (8.1) and since Γ/I_R and $K^\circ X$ are free Abelian groups of the same rank, it is enough to show that $G_\lambda(S^\vee) = 0$ when $\lambda \not\subset R$. If $\ell(\lambda) > d$ this follows from Theorem 3.1 since S has rank d. Now if $0 \to S \to \mathbb{C}^n \to Q \to 0$ denotes the universal exact sequence on X, we get $G_\lambda(S^\vee) = G_\lambda(S^\vee - \mathbb{C}^n) = G_{\lambda'}(\mathbb{C}^n - S) = G_{\lambda'}(Q \oplus S - S) = G_{\lambda'}(Q)$ which is zero if $\lambda_1 > n-d = \mathrm{rank}(Q)$. □

As mentioned in the introduction, the coproduct on Γ is also closely related to K-theory of Grassmannians. Given positive integers $d_1 < n_1$ and $d_2 < n_2$, set $X_1 = \mathrm{Gr}(d_1, \mathbb{C}^{n_1})$, $X_2 = \mathrm{Gr}(d_2, \mathbb{C}^{n_2})$, and $X = \mathrm{Gr}(d_1 + d_2, \mathbb{C}^{n_1+n_2})$, and let S_1, S_2, and S be the tautological subbundles on these varieties. Let P be the product $P = X_1 \times X_2$ with projections $\pi_i : P \to X_i$, and let $\phi : P \to X$ be the embedding which maps a pair (V_1, V_2) of subspaces $V_1 \subset \mathbb{C}^{n_1}$ and $V_2 \subset \mathbb{C}^{n_2}$ to the subspace $V_1 \oplus V_2 \subset \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$. Then since $\phi^* S = \pi_1^* S_1 \oplus \pi_2^* S_2$ it follows that the pullback on Grothendieck rings $\phi^* : K^\circ X \to K^\circ P = K^\circ X_1 \otimes K^\circ X_2$ is given by $\phi^*(G_\nu(S)) = G_\nu(\pi_1^* S_1 \oplus \pi_2^* S_2) = \sum d^\nu_{\lambda\mu} G_\lambda(S_1) \otimes G_\mu(S_2)$.

We will next report on some unpublished results of A. Knutson regarding triple intersections of Schubert structure sheaves[1]. Let $\rho : X = \mathrm{Gr}(d, \mathbb{C}^n) \to \{*\}$ be a map to a point and let $\rho_* : K^\circ X \to \mathbb{Z}$ be the induced map on Grothendieck groups. The triple intersection number of the structure sheaves $\mathcal{O}_{\Omega_\lambda}$, $\mathcal{O}_{\Omega_\mu}$, and $\mathcal{O}_{\Omega_\nu}$ is the integer $\rho_*([\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] \cdot [\mathcal{O}_{\Omega_\nu}])$. This is a natural K-theory parallel of the symmetric Littlewood-Richardson coefficients studied in e.g. [12]. Let $\mathcal{I}_\lambda = \mathcal{I}_{\Omega_\lambda \smallsetminus \Omega^\circ_\lambda} \subset \mathcal{O}_{\Omega_\lambda}$ denote the ideal sheaf of the complement of the open Schubert cell Ω°_λ in Ω_λ. To analyze triple intersections, Knutson proved that these ideal sheaves form a dual basis to the basis of Schubert structure sheaves with respect to the pairing $(\alpha, \beta) = \rho_*(\alpha \cdot \beta)$ on $K^\circ X$. Knutson furthermore worked out the change of basis matrices. We will apply the methods developed in the present paper to give simple proofs of these results. In addition we will prove an explicit formula for triple intersections and give an example showing that these numbers can be negative, although the signs of triple intersections do not alternate in a simple way.

Let $t = 1 - G_1 \in \Gamma$. We will abuse notation and write t also for its image $1 - [\mathcal{O}_{\Omega_1}]$ in $K^\circ X$, which by definition of the polynomial G_1 is equal to the class of the line bundle $\bigwedge^d S$. Corollary 4.7 implies that for any partition λ we have $t \cdot G_\lambda = \sum (-1)^{|\sigma/\lambda|} G_\sigma$ where the sum is over all partitions $\sigma \supset \lambda$ such that σ/λ is a rook strip. Since $\rho_*([\mathcal{O}_{\Omega_\sigma}]) = 1$ for each $\sigma \subset R = (n-d)^d$, it follows from this that $\rho_*(t \cdot [\mathcal{O}_{\Omega_\lambda}])$ is equal to one when $\lambda = R$ and zero otherwise. If λ is contained in R, let $\tilde{\lambda}$ be the partition obtained by rotating the skew diagram R/λ 180 degrees, i.e. $\tilde{\lambda} = (n - d - \lambda_d, \dots, n - d - \lambda_1)$. As we noted in Section 7, the coefficient of G_R in a product $G_\lambda \cdot G_\mu$ is non-zero if and only if $\mu = \tilde{\lambda}$, and in this case we have $c^R_{\lambda\mu} = 1$. It follows from this that $\rho_*(t \cdot [\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}])$ is equal to one if $\mu = \tilde{\lambda}$ and zero otherwise. We conclude that the elements $t \cdot [\mathcal{O}_{\Omega_\lambda}]$ form a dual basis to the basis of Schubert structure sheaves, with $[\mathcal{O}_{\Omega_\lambda}]$ and $t \cdot [\mathcal{O}_{\Omega_{\tilde{\lambda}}}]$ dual to each other. Since $\Omega_\lambda \smallsetminus \Omega^\circ_\lambda$ is a zero section of the line bundle $\bigwedge^d S^\vee$ restricted to Ω_λ [22], we finally deduce that $[\mathcal{I}_\lambda] = [\bigwedge^d S \otimes \mathcal{O}_{\Omega_\lambda}] = t \cdot [\mathcal{O}_{\Omega_\lambda}]$.

Now, calculating a triple intersection number $\rho_*([\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] \cdot [\mathcal{O}_{\Omega_\nu}])$ is equivalent to expanding $[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}]$ in terms of the dual basis $\{[\mathcal{I}_\sigma]\}$ and extracting the coefficient of $[\mathcal{I}_{\tilde{\nu}}]$. This is the same as the coefficient of $G_{\tilde{\nu}}$ when the formal power series $t^{-1} \cdot G_\lambda \cdot G_\mu$ is written as an infinite linear combination of the basis elements for Γ. Notice that multiplication by t^{-1} takes any basis element G_λ to the sum of all elements G_σ for partitions σ containing λ; this is the inverse operation to multiplication by t. We therefore obtain the formula

$$\rho_*([\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] \cdot [\mathcal{O}_{\Omega_\nu}]) = \sum_{\sigma \subset \tilde{\nu}} c^\sigma_{\lambda\mu} \,. \tag{8.2}$$

[1] While Knutson's results hold for arbitrary partial flag varieties, we shall only be concerned with Grassmannians here.

Alternatively we have $\rho_*([\mathcal{O}_{\Omega_\lambda}]\cdot[\mathcal{O}_{\Omega_\mu}]\cdot[\mathcal{O}_{\Omega_\nu}]) = \sum_{\sigma\supset\lambda} c^{\tilde\nu}_{\sigma\mu}$. It turns out that many of these intersection numbers are non-negative. For example, when $\tilde\nu$ contains the union of all partitions σ with non-zero coefficient $c^\sigma_{\lambda\mu}$, then the intersection number is equal to one, which follows from the fact that the map ϕ_p of Section 7 is a ring homomorphism. Similarly one can check that all triple intersections on Grassmannians of dimension smaller than 20 are non-negative. However, a direct calculation shows that the coefficient of $G_{5\,4\,3\,1}$ in $t^{-1}(G_{3\,2\,1})^2$ is -1. In other words negative triple intersections can be found on $\mathrm{Gr}(4,\mathbb{C}^9)$.

Let us remark here that the signs showing up in the structure constants of Γ are to some extend a matter of choice. To be precise, all structure constants of Γ with respect to the basis $\{(-1)^{|\lambda|}\,G_\lambda\}$ are non-negative. This viewpoint is equivalent to working with Fomin and Kirillov's β-polynomials, with $\beta = 1$ [8]. We have chosen to keep a notation that leads to signs in order to comply with standard definitions and to honor the fact that the Schubert structure sheaves on a Grassmann variety do multiply with alternating signs.

9. The structure of Γ

In this last section we will give a discussion of the overall structure of Γ, including its relation to the ring of symmetric functions.

Recall that the *ring of symmetric functions* is the span $\Lambda = \bigoplus_\lambda \mathbb{Z}\cdot s_\lambda$ of all Schur functions $s_\lambda = s_\lambda(x)$ [20, 10]. This is in fact a Hopf algebra [4, Ch. 1]. Its structure constants are the Littlewood-Richardson coefficients, i.e.

$$s_\lambda\cdot s_\mu = \sum c^\nu_{\lambda\mu}\, s_\nu \qquad\text{and}\qquad \Delta\, s_\nu = \sum c^\nu_{\lambda\mu}\, s_\lambda\otimes s_\mu$$

where the first sum is over partitions ν and the second over partitions λ and μ, such that $|\nu| = |\lambda| + |\mu|$ in both cases. The antipode is given by $S(s_\lambda) = (-1)^{|\lambda|}\, s_{\lambda'}$.

As noted in the introduction, Λ is the associated graded bialgebra to Γ with respect to the filtration $\Gamma_p = \bigoplus_{|\lambda|\geq p}\mathbb{Z}\cdot G_\lambda$. This is an immediate consequence of the fact that $c^\nu_{\lambda\mu}$ and $d^\nu_{\lambda\mu}$ are both equal to the usual Littlewood-Richardson coefficient when $|\nu| = |\lambda|+|\mu|$. Furthermore, if we let $\hat\Gamma$ and $\hat\Lambda$ be the completions of Γ and Λ, consisting of infinite linear combinations of stable Grothendieck polynomials and Schur functions, respectively, then $\hat\Gamma \cong \hat\Lambda$ as bialgebras. This is true because if we set the variables y_i to zero, then $\hat\Gamma$ and $\hat\Lambda$ both consist of all symmetric power series in $\mathbb{Z}[\![x_1,x_2,\dots]\!]$.

Despite from these facts, Γ and Λ are not isomorphic as bialgebras themselves. In fact, there exists no antipode which makes Γ a Hopf algebra. Recall that an antipode is a linear map $S:\Gamma\to\Gamma$, such that $S(1) = 1$ and for each non-empty partition ν we have $\sum d^\nu_{\lambda\mu}\, S(G_\lambda)\cdot G_\mu = 0$, or equivalently

$$\sum_{\lambda\subset\nu} S(G_\lambda)\, G_{\nu/\!\!/\lambda} = 0 \tag{9.1}$$

where $G_{\nu/\!\!/\lambda}$ is given by (6.3). Taking $\nu = (1)$ we get $S(G_1)\cdot(1-G_1) + 1\cdot G_1 = 0$ which implies that $S(t) = t^{-1}$ where $t = 1 - G_1$. Since $t^{-1}\in\hat\Gamma$ is equal to the sum of the elements G_λ for all partitions λ, this is not an element of Γ.

However, Γ is not far from being a Hopf algebra. In fact, if we let Γ_t be the localization generated by Γ and t^{-1}, then Γ_t is a Hopf algebra. To see this, notice that (6.4) implies that $G_{\nu/\!\!/\nu} = t^m$ where m is the number of inner corners of ν, i.e. the number of indices i such that $\nu_i > \nu_{i+1}$. By (9.1) we therefore see that an antipode $S : \Gamma_t \to \Gamma_t$ must satisfy

$$S(G_\nu) = -t^{-m} \sum_{\lambda \subsetneq \nu} S(G_\lambda)\, G_{\nu /\!\!/ \lambda}\,. \tag{9.2}$$

This equation can be used to define $S : \Gamma_t \to \Gamma_t$. The obtained antipode is furthermore a ring homomorphism, since it must agree with the antipode on the ring of symmetric functions extended to $\hat{\Lambda} = \hat{\Gamma}$.

Regarding the structure of Γ as an abstract ring, we conjecture:

Conjecture 9.1. *(a) Any stable polynomial G_λ can be written as a polynomial in the elements G_R for rectangular partitions R contained in λ.*

(b) The elements $\{\dots, G_3, G_2, G_1, G_{(1,1)}, G_{(1,1,1)}, \dots\}$ corresponding to partitions with only one row or one column are algebraically independent.

Part (a) of this conjecture has been verified for all partitions λ of weight at most 9. For (b), if we define the degree of a monomial in the G_k and $G_{(1^\ell)}$ to be the total number of boxes in the partitions of the factors, then all such monomials of degree up to 9 are linearly independent. Furthermore, it is not hard to prove that for any integer $k \geq 2$, the elements $\{G_k, G_1, G_{(1,1)}, G_{(1,1,1)}, \dots\}$ are algebraically independent. Namely, if one uses the lexicographic order on partitions, then the monomials in these elements all have a different maximal partition λ for which the coefficient of G_λ is non-zero.

The conjecture has some interesting consequences for the structure of Γ. If (a) is true, then Γ_t is generated by the the elements G_k, $G_{(1^\ell)}$ in addition to t^{-1}. To see this, notice that if $R = (q)^p$ is a rectangular partition with at least two rows and two columns, and $\lambda = (q^{p-1}, q-1)$ is the partition obtained by removing the box in the corner of R, then

$$t \cdot G_R = G_1\, G_\lambda - G_{\lambda+(1)} - G_{\lambda,(1)} + G_{\lambda+(1),(1)}\,.$$

Using (a) this shows that $t \cdot G_R$ can be written as a polynomial in G_{q+1}, $G_{(1^{p+1})}$, and the elements $G_{\tilde{R}}$ for rectangles $\tilde{R}$ which are strictly contained in R. This shows that G_R is in the ring generated by the elements G_k, $G_{(1^\ell)}$, and t^{-1} by induction on the size of R.

However, if (b) is true then the elements G_k and $G_{(1^\ell)}$ do not generate Γ as a ring. In fact, the identity

$$t \cdot G_{(2,2)} = G_1\, G_2 + G_1\, G_{(1,1)} - G_2\, G_{(1,1)} - (G_1)^3$$

implies that $G_{(2,2)}$ can't be written as a polynomial in these elements if they are algebraically independent.

Geometrically, the fact that Γ is a commutative and cocommutative bialgebra implies that $\operatorname{Spec}\Gamma$ is an Abelian semigroup scheme. The existence of an antipode on Γ_t means that the dense open subset $\operatorname{Spec}\Gamma_t$ is a group scheme. Furthermore, if Conjecture 9.1 is true then this open subset looks like an infinite dimensional affine space with a hyperplane removed.

We will finish this paper by raising some additional questions. First of all, several people have asked us when a symmetric power series in $\mathbb{Z}[\![x_1, x_2, \dots, y_1, y_2, \dots]\!]$ is an element in Γ. Even when the variables y_i are set to zero, we do not know the answer to this.

In view of Conjecture 9.1 it would be very interesting to know the relations between the elements G_R for rectangular partitions. We have also been wondering if Γ might be a polynomial ring, i.e. are there algebraically independent elements $h_1, h_2, \ldots$ in Γ such that $\Gamma = \mathbb{Z}[h_1, h_2, \ldots]$? We think this is not the case but haven't been able to prove it.

It is not hard to see that the single Grothendieck polynomials $\mathfrak{G}_w(x)$ for all permutations w form a basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. For example, one can check directly that the Grothendieck polynomials for permutations in S_n give a basis for the linear span of all monomials $x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}}$ for which $k_j \leq n - j$ for each j. Alternatively one can use a stronger result of Lenart [17] which expresses any single Grothendieck polynomial $\mathfrak{G}_w(x)$ as an explicit linear combination of Schubert polynomials $\mathfrak{S}_{w'}(x)$ with alternating signs, i.e. the sign of the coefficient of $\mathfrak{S}_{w'}(x)$ is $(-1)^{\ell(w\,w')}$. On the other hand, Lascoux has conjectured that each single Schubert polynomial is a non-negative linear combination of Grothendieck polynomials.

Now define Grothendieck structure constants $c_{u,v}^w \in \mathbb{Z}$ by

$$\mathfrak{G}_u(x) \cdot \mathfrak{G}_v(x) = \sum_w c_{u,v}^w \, \mathfrak{G}_w(x) \, .$$

These constants are generalizations of the structure constants for Schubert polynomials as well as the coefficients $c_{\lambda\mu}^\nu$ discussed in this paper. If w_λ, w_μ, and w_ν are Grassmannian permutations for λ, μ, and ν with descents at the same position, then $c_{\lambda\mu}^\nu = c_{w_\lambda, w_\mu}^{w_\nu}$. Based on our results for Grothendieck polynomials of Grassmannian permutations given in this paper, as well as on some computational evidence, we pose:

Conjecture 9.2. *The structure constants for single Grothendieck polynomials have alternating signs, i.e.* $(-1)^{\ell(uvw)} c_{u,v}^w \geq 0$.

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