THE FROBENIUS MORPHISM ON A TORIC VARIETY

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Let X be a projective toric variety over a field k. In [3] Danilov states the Bott vanishing theorem

$$\mathrm{H}^i(X, \tilde{\Omega}^j_{X/k} \otimes L) = 0$$

where $\tilde{\Omega}^j_{X/k}$ denotes the Zariski differentials, L is an ample line bundle on X and $i > 0$. Batyrev and Cox proves this theorem in the simplicial case in [1]. The purpose of this paper is to show that Bott vanishing is a simple consequence of a very specific condition on the Frobenius morphism in prime characteristic p.

Assume now that $k = \mathbb{Z}/p$, where $p > 0$ and let X be any smooth variety over k. Recall that the absolute Frobenius morphism $F : X \to X$ on X is the identity on point spaces and the p-th power map locally on functions. Assume that there is a flat scheme $X^{(2)}$ over $\mathbb{Z}/p^2$, such that $X \cong X^{(2)} \times_{\mathbb{Z}/p^2} \mathbb{Z}/p$. The condition on F is that there should be a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$ which gives F by reduction mod p. In this case we will say that the Frobenius morphism lifts to $\mathbb{Z}/p^2$. It is known that a lift of the Frobenius morphism to $\mathbb{Z}/p^2$ leads to a quasi-isomorphism

$$\sigma : \bigoplus_{0 \leq i} \Omega^i_X[-i] \to F_* \Omega^\bullet_X$$

where the complex on the left has zero differentials and $\Omega^\bullet_X$ denotes the de Rham complex of X ([4], Remarques 2.2(ii)). Using duality we prove that σ is in fact a split quasi-isomorphism.

In general it is very difficult to decide when Frobenius lifts to $\mathbb{Z}/p^2$. However for varieties which are glued together by monomial automorphisms it is easy. This is the case for toric varieties, where we show that the Frobenius morphism lifts to $\mathbb{Z}/p^2$. This places the Bott vanishing theorem for (singular and smooth) toric varieties and the degeneration of the Danilov spectral sequence ([3], Theorem 7.5.2, Theorem 12.5) in a natural characteristic p framework.

In the second half of this paper we study the Frobenius morphism on flag varieties. This is related to work of Paranjape and Srinivas [12]. They have proved using complex algebraic geometry that if Frobenius for a flag variety X over k lifts to the p-adic numbers $\hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n$, then X is a product of projective spaces. We generalize this result by showing that Frobenius for a large class [1] of generalized flag varieties admits no lift to $\mathbb{Z}/p^2$. This is done using a lemma on fibrations linking non-lifting of Frobenius to Bott non-vanishing cohomology groups for flag

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[1]Our proof is related to the failure of Bott vanishing over $\mathbb{C}$ for flag varieties. It seems likely that if X is a flag variety over $\mathbb{C}$, then Bott vanishing for X implies that X is a product of projective spaces.

varieties of Hermitian symmetric type over the complex numbers. These cohomology groups have been studied thoroughly by M.-H. Sato and D. Snow.

Part of these results have been announced in [2].

We are grateful to D. Cox for his interest in this work and for pointing out the paper [1].

1. Preliminaries

Let k be a perfect field of characteristic $p > 0$ and X a smooth k-variety of dimension n. By Ω_X we denote the sheaf of k-differentials on X and $\Omega^j_X = \wedge^j \Omega_X$. The (absolute) Frobenius morphism $F : X \to X$ is the morphism on X, which is the identity on the level of points and given by $F^\#(f) = f^p : \mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$ on the level of functions. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $F_*\mathcal{F} = \mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure is changed according to the homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$.

1.1. The Cartier operator. The universal derivation $d : \mathcal{O}_X \to \Omega_X$ gives rise to a family of k-homomorphisms $d^j : \Omega^j_X \to \Omega^{j+1}_X$ making $\Omega^\bullet_X$ into a complex of k-modules which is called the de Rham complex of X. By applying F_* to the de Rham complex, we obtain a complex $F_*\Omega^\bullet_X$ of $\mathcal{O}_X$-modules. Let $B^i_X \subseteq Z^i_X \subseteq F_*\Omega^i_X$ denote the coboundaries and cocycles in degree i. There is the following very nice description of the cohomology of $F_*\Omega^\bullet_X$ due to Cartier.

Theorem 1. There is a uniquely determined graded $\mathcal{O}_X$-algebra isomorphism

$$C^{-1} : \Omega^\bullet_X \to \mathcal{H}^\bullet(F_*\Omega^\bullet_X)$$

which in degree 1 is given locally as

$$C^{-1}(da) = a^{p-1} da$$

Proof. [8], Theorem 7.2. □

With some abuse of notation, we let C denote the natural homomorphism $Z^i_X \to \Omega^i_X$, which after reduction modulo B^i_X gives the inverse isomorphism to C^{-1}. The isomorphism $\bar{C} : Z^i_X/B^i_X \to \Omega^i_X$ is called the Cartier operator.

2. Liftings of Frobenius to $W_2(k)$

There is a very interesting connection ([11], §5.3), between the Cartier operator and liftings of the Frobenius morphism to flat schemes of characteristic p^2 due to Mazur. We go on to explore this next.

2.1. Witt vectors of length two. The Witt vectors $W_2(k)$ ([10], Lecture 26) of length 2 over k can be interpreted as the set $k \times k$, where multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ in $W_2(k)$ are defined by

$$a\, b = (a_0\, b_0, a_0^p b_1 + b_0^p a_1)$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 + \sum_{j=1}^{p-1} p^{-1} \binom{p}{j} a_0^j\, b_0^{p-j})$$

In the case $k = \mathbb{Z}/p$, one can prove that $W_2(k) \cong \mathbb{Z}/p^2$. The projection on the first coordinate $W_2(k) \to k$ corresponds to the reduction $W_2(k) \to W_2(k)/p \cong k$ modulo p. The ring homomorphism $F^{(2)} : W_2(k) \to W_2(k)$ given by $F^{(2)}(a_0, a_1) = (a_0^p, a_1^p)$ reduces to the Frobenius homomorphism F on k modulo p.

2.2. Splittings of the de Rham complex. The previous section shows that there is a canonical morphism $\operatorname{Spec} k \to \operatorname{Spec} W_2(k)$. Assume that there is a flat scheme $X^{(2)}$ over $\operatorname{Spec} W_2(k)$ such that

$$X \cong X^{(2)} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k \tag{1}$$

We shall say that the Frobenius morphism F lifts to $W_2(k)$ if there exists a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$ covering the Frobenius homomorphism $F^{(2)}$ on $W_2(k)$, which reduces to F via the isomorphism (1). When we use the statement that Frobenius lifts to $W_2(k)$ we will always implicitly assume the existence of the flat lift $X^{(2)}$.

Theorem 2. If the Frobenius morphism on X lifts to $W_2(k)$ then there is a split quasi-isomorphism

$$0 \to \bigoplus_{0 \le i} \Omega^i_X[-i] \xrightarrow{\sigma} F_* \Omega^\bullet_X$$

Proof. For an affine open subset $\operatorname{Spec} A^{(2)} \subseteq X^{(2)}$ there is a ring homomorphism $F^{(2)} : A^{(2)} \to A^{(2)}$ such that

$$F^{(2)}(b) = b^p + p \cdot \varphi(b)$$

where $\varphi : A^{(2)} \to A = A^{(2)}/pA^{(2)}$ is some function and $p\cdot : A \to A^{(2)}$ is the $A^{(2)}$-homomorphism derived from tensoring the short exact sequence of $W_2(k)$-modules

$$0 \longrightarrow p\,W_2(k) \longrightarrow W_2(k) \xrightarrow{p\,\cdot} p\,W_2(k) \longrightarrow 0$$

with the flat $W_2(k)$ module $A^{(2)}$ identifying $A \cong A^{(2)}/pA^{(2)}$ with $p\,A^{(2)}$. We get the following properties of φ:

$$\varphi(a+b) = \varphi(a) + \varphi(b) - \sum_{j=1}^{p-1} p^{-1} \binom{p}{j} \bar{a}^j \bar{b}^{p-j}$$
$$\varphi(a\,b) = \bar{a}^p \varphi(b) + \bar{b}^p \varphi(a)$$

where $\bar{\cdot}$ means reduction $\mod p$. Now it follows that

$$a \mapsto a^{p-1} da + d\varphi(\tilde{a})$$

where $\tilde{a}$ is any lift of a, is a well defined derivation $\delta : A \to Z^1_{\operatorname{Spec} A} \subset F_* \Omega^1_{\operatorname{Spec} A}$, which gives a homomorphism $\varphi : \Omega^1_{\operatorname{Spec} A} \to Z^1_{\operatorname{Spec} A} \subset F_* \Omega^1_{\operatorname{Spec} A}$. This homomorphism can be extended via the algebra structure to give an A-algebra homomorphism $\sigma : \oplus_i \Omega^i_{\operatorname{Spec} A} \to Z^\bullet_{\operatorname{Spec} A} \subseteq F_* \Omega^\bullet_{\operatorname{Spec} A}$, which composed with the canonical homomorphism $Z^\bullet_{\operatorname{Spec} A} \to \mathcal{H}^\bullet(F_* \Omega^\bullet_{\operatorname{Spec} A})$ gives the inverse Cartier operator. Since an affine open covering $\{\operatorname{Spec} A^{(2)}\}$ of $X^{(2)}$ gives rise to an affine open covering $\{\operatorname{Spec} A^{(2)}/pA^{(2)}\}$ of X, we have proved that σ is a quasi-isomorphism of complexes inducing the inverse Cartier operator on cohomology.

Now we give a splitting homomorphism of $\sigma_i : \Omega^i_X \to F_*\Omega^i_X$. Notice that $\sigma_0 : \mathcal{O}_X \to F_*\mathcal{O}_X$ is the Frobenius homomorphism and that σ_i $(i > 0)$ splits C in the exact sequence

$$0 \longrightarrow B^i_X \longrightarrow Z^i_X \xrightarrow{C} \Omega^i_X \longrightarrow 0$$

The natural perfect pairing $\Omega^i_X \otimes \Omega^{n-i}_X \to \Omega^n_X$ gives an isomorphism between $\mathcal{H}om_X(\Omega^{n-i}_X, \Omega^n_X)$ and Ω^i_X. It is easy to check that the homomorphism

$$F_*\Omega^i_X \to \mathcal{H}om_X(\Omega^{n-i}_X, \Omega^n_X) \cong \Omega^i_X$$

given by $\omega \mapsto \varphi(\omega)$, where $\varphi(\omega)(z) = C(\sigma_{n-i}(z) \wedge \omega)$, splits σ_i. $\square$

2.3. Bott vanishing. Let X be a normal variety and let j denote the inclusion of the smooth locus $U \subseteq X$. If the Frobenius morphism lifts to $W_2(k)$ on X, then the Frobenius morphism on U also lifts to $W_2(k)$. Define the Zariski sheaf $\tilde{\Omega}^i_X$ of i-forms on X as $j_*\Omega^i_U$. Since $\operatorname{codim}(X-U) \geq 2$ it follows ([6], Proposition 5.10) that $\tilde{\Omega}^i_X$ is a coherent sheaf on X.

Theorem 3. Let X be a projective normal variety such that F lifts to $W_2(k)$. Then

$$\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) = 0$$

for $s > 0$ and L an ample line bundle.

Proof. Let U be the smooth locus of X and let j denote the inclusion of U into X. On U we have by Theorem 2 a split sequence

$$0 \to \Omega^r_U \to F_*\Omega^r_U$$

which pushes down to the split sequence (F commutes with j)

$$0 \to \tilde{\Omega}^r_X \to F_*\tilde{\Omega}^r_X$$

Now tensoring with L and using the projection formula we get injections for $s > 0$

$$\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) \hookrightarrow \mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L^p)$$

Iterating these injections and using that the Zariski sheaves are coherent one gets the desired vanishing theorem by Serre's theorem. $\square$

2.4. Degeneration of the Hodge to de Rham spectral sequence. Let X be a projective normal variety with smooth locus U. Associated with the complex $\tilde{\Omega}^\bullet_X$ there is a spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}^p_X) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}^\bullet_X)$$

where $\mathrm{H}^\bullet(X, \tilde{\Omega}^\bullet_X)$ denotes the hypercohomology of the complex $\tilde{\Omega}^\bullet_X$. This is the Hodge to de Rham spectral sequence for Zariski sheaves.

Theorem 4. If the Frobenius morphism on X lifts to $W_2(k)$, then the spectral sequence degenerates at the E_1-term.

Proof. As complexes of sheaves of abelian groups $\tilde{\Omega}^\bullet$ and $F_*\tilde{\Omega}^\bullet$ are the same so their hypercohomology agree. Applying hypercohomology to the split injection (Theorem 2)

$$\sigma : \bigoplus_{0\leq i} \tilde{\Omega}^i_{X/k}[-i] \to F_*\tilde{\Omega}^\bullet_X$$

we get

$$\begin{aligned} \sum_{p+q=n} \dim_k E_\infty^{pq} = \dim_k \mathrm{H}^n(X, \tilde{\Omega}^\bullet_X) &= \dim_k \mathrm{H}^n(X, F_*\tilde{\Omega}^\bullet_X) \geq \\ \sum_{p+q=n} \dim_k \mathrm{H}^q(X, \tilde{\Omega}^p_X) &= \sum_{p+q=n} \dim_k E_1^{pq} \end{aligned}$$

Since E_∞^{pq} is a subquotient of E_1^{pq}, it follows that $E_\infty^{pq} \cong E_1^{pq}$ so that the spectral sequence degenerates at E_1. □

3. Toric varieties

In this section we briefly sketch the definition of toric varieties following Fulton [5] and demonstrate how the results of Section 2 may be applied.

3.1. Convex geometry. Let N be a lattice, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual lattice, and let V be the real vector space $V = N \otimes_{\mathbb{Z}} \mathbb{R}$. It is natural to identify the dual space of V with $M \otimes_{\mathbb{Z}} \mathbb{R}$, and we think of $N \subset V$ and $M \subset V^*$ as the subsets of integer points.

By a cone in N we will mean a subset $\sigma \subset V$ taking the form $\sigma = \{r_1v_1 + \cdots + r_sv_s \mid r_i \geq 0\}$ for some $v_i \in N$. The vectors $v_1, \ldots, v_s$ are called generators of σ. We define the dual cone to be $\sigma^\vee = \{u \in V^* | \forall v \in \sigma : \langle u, v\rangle \geq 0\}$. One may show that $\sigma^\vee$ is a cone in M. A face of σ is any set $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$. Any face of σ is clearly a cone in N, generated by the v_i for which $\langle u, v_i\rangle = 0$.

Now let σ be a strongly convex cone in N, this means that $\{0\}$ is a face of σ or equivalently that no nontrivial subspace of V is contained in σ. We define S_σ to be the semi group $\sigma^\vee \cap M$. Since $\sigma^\vee$ is a cone in M, S_σ is finitely generated.

3.2. Affine toric varieties. If k is any commutative ring the semigroup ring $k[S_\sigma]$ is a finitely generated commutative k-algebra, and $U_\sigma = \operatorname{Spec} k[S_\sigma]$ is an affine scheme of finite type over k. Schemes of this form are called affine toric schemes.

3.3. Glueing affine toric varieties. Let $\tau = \sigma \cap u^\perp$ be a face of σ. One may assume that $u \in S_\sigma$. Then it follows that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$, so that $k[S_\tau] = k[S_\sigma]_u$. In this way U_τ becomes a principal open subscheme of U_σ. This may be used to glue affine toric schemes together. We define a fan in N to be a nonempty set Δ of strongly convex cones in N satisfying that the faces of any cone in Δ are also in Δ and the intersection of two cones in Δ is a face of each. The affine varieties arising from cones in Δ may be glued together to form a scheme $X_k(\Delta)$ as follows. If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau \in \Delta$ is a face of both τ and σ, so $U_{\sigma\cap\tau}$ is isomorphic to open subsets $U_{\sigma\tau}$ in U_σ and $U_{\tau\sigma}$ in U_τ. Take the transition morphism $\phi_{\sigma\tau} : U_{\sigma\tau} \to U_{\tau\sigma}$ to be the one going through $U_{\sigma\cap\tau}$. A scheme $X_k(\Delta)$ arising from a fan Δ in some lattice is called a toric scheme.

3.4. Liftings of the Frobenius morphism on toric varieties. Let $X = X_k(\Delta)$ be a toric scheme over the commutative ring k of characteristic $p > 0$. We are going to construct explicitly a lifting of the absolute Frobenius morphism on X to $W = W_2(k)$. Define $X^{(2)}$ to be $X_W(\Delta)$. Since all the rings $W[S_\sigma]$ are free W-modules, this is clearly a flat scheme over $W_2(k)$. Moreover, the identities $W[S_\sigma] \otimes_W k \cong k[S_\sigma]$ immediately give an isomorphism $X^{(2)} \times_{\operatorname{Spec} W} \operatorname{Spec} k \cong X$.

For $\sigma \in \Delta$, let $F^{(2)}_\sigma : W[S_\sigma] \to W[S_\sigma]$ be the ring homomorphism extending $F^{(2)} : W \to W$ and mapping $u \in S_\sigma$ to u^p. It is easy to see that these maps are compatible with the transition morphisms, so we may take $F^{(2)} : X^{(2)} \to X^{(2)}$ to be the morphism which is defined by $F^{(2)}_\sigma$ locally on $\operatorname{Spec} W[S_\sigma]$. This gives the lift of F to $W_2(k)$ and completes the construction.

3.5. Bott vanishing and the Danilov spectral sequence. Since toric varieties are normal we get the following corollary of Section 2:

Theorem 5. Let X be a projective toric variety over k. Then

$$\mathrm{H}^q(X, \tilde{\Omega}^p_X \otimes L) = 0$$

where $q > 0$ and L is an ample line bundle. Furthermore the Danilov spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}^p_X) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}^\bullet_X)$$

degenerates at the E_1-term.

Remark 1. One may use the above to prove similar results in characteristic zero. The key issue is that we have proved that Bott vanishing and degeneration of the Danilov spectral sequence holds in any prime characteristic.

4. Flag varieties

In this section we generalize Paranjape and Srinivas result on non-lifting of Frobenius on flag varieties not isomorphic to $\mathbb{P}^n$. The key issue is that one can reduce to flag varieties with rank 1 Picard group. In many of these cases one can exhibit ample line bundles with Bott non-vanishing.

We now set up notation. Let G be a semisimple algebraic group over k and fix a Borel subgroup B in G. Recall that (reduced) parabolic subgroups $P \supseteq B$ are given by subsets of the simple root subgroups of B. These correspond bijectively to subsets of nodes in the Dynkin diagram associated with G. A parabolic subgroup Q is contained in P if and only if the simple root subgroups in Q is a subset of the simple root subgroups in P. A maximal parabolic subgroup is the maximal parabolic subgroup not containing a specific simple root subgroup.

We shall need the following result from the appendix to [9]

Proposition 1. If the sequence

$$0 \to B^1_X \to Z^1_X \xrightarrow{C} \Omega^1_X \to 0$$

splits, then the Frobenius morphism on X lifts to $W_2(k)$.

We also need the following fact derived from ([7], Proposition II.8.12 and Exercise II.5.16(d))

Proposition 2. Let $f : X \to Y$ be a smooth morphism between smooth varieties X and Y. Then for every $n \in \mathbb{N}$ there is a filtration $F^0 \supseteq F^1 \supseteq \dots$ of Ω^n_X such that

$$F^i/F^{i+1} \cong f^*\Omega^i_Y \otimes \Omega^{n-i}_{X/Y}$$

Lemma 1. Let $f : X \to Y$ be a surjective, smooth and projective morphism between smooth varieties X and Y such that the fibers have no non-zero global n-forms, where $n > 0$. Then there is a canonical isomorphism

$$\Omega^\bullet_Y \to f_*\Omega^\bullet_X$$

and a splitting $\sigma : \Omega^1_X \to Z^1_X$ of the Cartier operator $C : Z^1_X \to \Omega^1_X$ induces a splitting $f_*\sigma : \Omega^1_Y \to Z^1_Y$ of $C : Z^1_Y \to \Omega^1_Y$.

Proof. Notice first that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism of rings as f is projective and smooth. The assumption on the fibers translates into $f_*\Omega^n_{X/Y} \otimes k(y) \cong \mathrm{H}^0(X_y, \Omega^n_{X_y}) = 0$ for geometric points $y \in Y$, when $n > 0$. So we get $f_*\Omega^n_{X/Y} = 0$ for $n > 0$. By Proposition 2 this means that all of the natural homomorphisms $\Omega^n_Y \to f_*\Omega^n_X$ induced by $\mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*\Omega^1_X$ are isomorphisms giving an isomorphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \Omega^1_Y & \longrightarrow & \Omega^2_Y & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*\mathcal{O}_X & \longrightarrow & f_*\Omega^1_X & \longrightarrow & f_*\Omega^2_X & \longrightarrow & \dots \end{array}$$

This means that the middle arrow in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^1_Y & \longrightarrow & Z^1_Y & \xrightarrow{C} & \Omega_Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*B^1_X & \longrightarrow & f_*Z^1_X & \xrightarrow{f_*C} & f_*\Omega_X & \longrightarrow & 0 \end{array}$$

is an isomorphism and the result follows. □

Corollary 1. Let $Q \subseteq P$ be two parabolic subgroups of G. If the Frobenius morphism on G/Q lifts to $W_2(k)$, then the Frobenius morphism on G/P lifts to $W_2(k)$.

Proof. It is well known that $G/Q \to G/P$ is a smooth projective fibration, where the fibers are isomorphic to $Z = P/Q$. Since Z is a rational projective smooth variety it follows from ([7], Exercise II.8.8) that $\mathrm{H}^0(Z, \Omega^n_Z) = 0$ for $n > 0$. Now the result follows from Lemma 1 and Proposition 1. □

In specific cases one can prove using the "standard" exact sequences that certain flag varieties do not have Bott vanishing. We go on to do this next.

Let Y be a smooth divisor in a smooth variety X. Suppose that Y is defined by the sheaf of ideals $I \subseteq \mathcal{O}_X$. Then ([7], Proposition II.8.17(2) and Exercise II.5.16(d)) gives for $n \in \mathbb{N}$ an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \Omega^{n-1}_Y \otimes I/I^2 \to \Omega^n_X \otimes \mathcal{O}_Y \to \Omega^n_Y \to 0$$

From this exact sequence and induction on n it follows that $\mathrm{H}^0(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n} \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.1. Quadric hypersurfaces in $\mathbb{P}^n$. Let Y be a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$. There is an exact sequence

$$0 \to \mathcal{O}_Y(1-n) \to \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}(3-n) \otimes \mathcal{O}_Y \to \Omega^1_Y \otimes \mathcal{O}_Y(3-n) \to 0$$

From this it is easy to deduce that

$$\mathrm{H}^{n-2}(Y, \Omega^1_Y \otimes \mathcal{O}_Y(3-n)) \cong \mathrm{H}^1(Y, \Omega^{n-2}_Y \otimes \mathcal{O}_Y(n-3)) \cong k$$

using that $\mathrm{H}^0(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n} \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.2. The incidence variety in $\mathbb{P}^n \times \mathbb{P}^n$. Let X be the incidence variety of lines and hyperplanes in $\mathbb{P}^n \times \mathbb{P}^n$, where $n \geq 2$. Recall that X is the zero set of $x_0 y_0 + \cdots + x_n y_n$, so that there is an exact sequence

$$0 \to \mathcal{O}(-1) \times \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n} \times \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

Using Künneth it is easy to deduce that

$$\mathrm{H}^{2n-2}(X, \Omega^1_X \otimes \mathcal{O}(1-n) \times \mathcal{O}(1-n)) \cong \mathrm{H}^1(X, \Omega^{2n-2} \otimes \mathcal{O}(n-1) \times \mathcal{O}(n-1)) \cong k$$

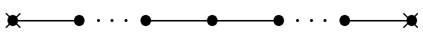
4.3. Bott non-vanishing for flag varieties. In this section we search for specific maximal parabolic subgroups P and ample line bundles L on $Y = G/P$, such that

$$\mathrm{H}^i(Y, \Omega^j_Y \otimes L) \neq 0$$

where $i > 0$. These are instances of Bott non-vanishing. This will be used in Section 4.4 to prove non-lifting of Frobenius for a large class of flag varieties.

Let $\mathcal{O}(1)$ be the ample generator of $\operatorname{Pic} Y$. By flat base change one may produce examples of Bott non-vanishing for Y for fields of arbitrary prime characteristic by restricting to the field of the complex numbers. This has been done in the setting of Hermitian symmetric spaces, where the cohomology groups $\mathrm{H}^p(Y, \Omega^q \otimes \mathcal{O}(n))$ have been thoroughly investigated by Sato [13] and Snow [14][15]. We now show that these examples exist. In each of the following subsections Y will denote G/P, where P is the maximal parabolic subgroup not containing the root subgroup corresponding to the marked simple root in the Dynkin diagram. These flag manifolds are the irreducible Hermitian symmetric spaces.

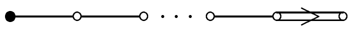
4.3.1. *Type A.*



If Y is a Grassmann variety not isomorphic to projective space ($Y = G/P$, where P corresponds to leaving out a simple root which is not the left or right most one), one may prove ([14], Theorem 3.3) that

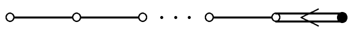
$$\mathrm{H}^1(Y, \Omega^3_Y \otimes \mathcal{O}(2)) \neq 0$$

4.3.2. *Type B.*



Here Y is a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$ and Bott non-vanishing follows from Section 4.1.

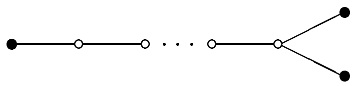
4.3.3. *Type C.*



By ([15], Theorem 2.2) it follows that

$$\mathrm{H}^1(Y, \Omega_Y^2 \otimes \mathcal{O}(1)) \neq 0$$

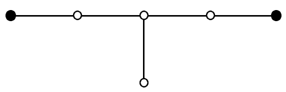
4.3.4. *Type D.*



For the maximal parabolic P corresponding to the leftmost marked simple root, Y=G/P is a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$ and Bott non-vanishing follows from Section 4.1. For the maximal parabolic subgroup corresponding to one of the two rightmost marked simple roots we get by ([15], Theorem 3.2) that

$$\mathrm{H}^2(Y, \Omega_Y^4 \otimes \mathcal{O}(2)) \neq 0$$

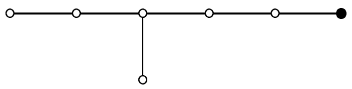
4.3.5. *Type E_6.*



By ([15], Table 4.4) it follows that

$$\mathrm{H}^3(Y, \Omega^5 \otimes \mathcal{O}(2)) \neq 0$$

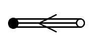
4.3.6. *Type E_7.*



By ([15], Table 4.5) it follows that

$$\mathrm{H}^4(Y, \Omega^6 \otimes \mathcal{O}(2)) \neq 0$$

4.3.7. *Type G_2.*



Here Y is a smooth quadric hypersurface in $\mathbb{P}^6$ and Bott non-vanishing follows from Section 4.1.

4.4. Non-lifting of Frobenius for flag varieties. We now get the following

Theorem 6. Let Q be a parabolic subgroup contained in a maximal parabolic subgroup P in the list 4.3.1 - 4.3.7. Then the Frobenius morphism on G/Q does not lift to $W_2(k)$. Furthermore if G is of type A, then the Frobenius morphism on any flag variety $G/Q \ncong \mathbb{P}^m$ does not lift to $W_2(k)$.

Proof. If P is a maximal parabolic subgroup in the list 4.3.1-4.3.7, then the Frobenius morphism on G/P does not lift to $W_2(k)$. By Corollary 1 we get that the Frobenius morphism on G/Q does not lift to $W_2(k)$. In type A the only flag variety not admitting a fibration to a Grassmann variety $\ncong \mathbb{P}^m$ is the incidence variety. Non-lifting of Frobenius in this case follows from Section 4.2. $\square$

Remark 2. The above case by case proof can be generalized to include projective homogeneous G-spaces with non-reduced stabilizers. It would be nice to prove in general that the only flag variety enjoying the Bott vanishing property is $\mathbb{P}^n$. We know of no other visible obstruction to lifting Frobenius to $W_2(k)$ for flag varieties than the non-vanishing Bott cohomology groups.

References

1. V. Batyrev and D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math J. **75** (1994), 293–338.
2. A. Buch, J. F. Thomsen, N. Lauritzen and V. Mehta, *Frobenius morphisms modulo p^2*, C. R. Acad. Sci. Paris, (to appear).
3. V. I. Danilov, *The geometry of toric varieties*, Russ. Math. Surveys **33:2** (1978), 97–154.
4. P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. math. **89** (1987), 247–270.
5. W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.
6. A. Grothendieck, *Local Cohomology*, vol. Lecture Notes in Mathematics 41, Springer Verlag, 1967.
7. R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
8. N. Katz, *Nilpotent connections and the monodromy theorem*, Publ. Math. IHES **39** (1970), 175–232.
9. V. Mehta and V. Srinivas, *Varieties in positive characteristic with trivial cotangent bundle*, Comp. Math. **64** (1987), 191–212.
10. D. Mumford, *Lectures on Curves on an Algebraic Surface*, Princeton University Press, 1966.
11. J. Oesterlé, *Dégénérescence de la suite spectrale de Hodge vers de Rham*, Séminaire Bourbaki, Asterisque, vol. 153, 1986, pp. 35–57.
12. K. Paranjape and V. Srinivas, *Self maps of homogeneous spaces*, Invent. math. **98** (1989), 425–444.
13. M.-H. Sato, *Generic torelli theorem for hypersurfaces in compact irreducible Hermitian symmetric spaces*, Algebraic Geometry and Commutative Algebra in honor of Masayoshi Nagata (H. Hijikata et al., ed.), vol. 2, Academic Press, 1988.
14. D. Snow, *Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces*, Math. Ann. **276** (1986), 159–176.
15. ———, *Vanishing theorems on compact Hermitian symmetric spaces*, Math. Z. **198** (1988), 1–20.

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