EQUIVARIANT RIGIDITY OF RICHARDSON VARIETIES

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Abstract. We prove that Schubert and Richardson varieties in flag varieties are uniquely determined by their equivariant cohomology classes, as well as a stronger result that replaces Schubert varieties with closures of Bialynicki-Birula cells under suitable conditions. This is used to prove a conjecture from [BCP23], stating that any two-pointed curve neighborhood representing a quantum cohomology product with a Seidel class is a Schubert variety.

1. Introduction

A Schubert variety Ω in a flag manifold $X = G/P$ is called *rigid* if it is uniquely determined by its class $[\Omega]$ in the cohomology ring $H^*(X)$. More precisely, if $Z \subset X$ is any irreducible closed subvariety such that $[Z]$ is a multiple of $[\Omega]$ in $H^*(X)$, then Z is a G-translate of Ω. This problem has been studied in numerous papers, see e.g. [Hon05, Hon07, Cos11, RT12, CR13, Cos14, Cos18, HM20] and the references therein.

In this paper we show that all Schubert varieties are *equivariantly rigid*. In other words, if $T \subset G$ is a maximal torus, $\Omega \subset X$ is a T-stable Schubert variety, and $Z \subset X$ is a (non-empty) T-stable closed subvariety such that the T-equivariant class $[Z] \in H^*_T(X)$ is a multiple of $[\Omega]$, then $Z = \Omega$. We use this result to prove a conjecture from [BCP23], stating that a two-pointed curve neighborhood corresponding to a quantum cohomology product with a Seidel class, is an explicitly determined Schubert variety. This conjecture was known in some cases when X is cominuscule, in all cases when X is a flag variety of type A [LLSY22, Tar23], and for $X = \mathrm{SG}(2, 2n)$ [BPX]

More generally, let T be an algebraic torus over an algebraically closed field, let X be a non-singular projective T-variety with finite fixed point set X^T, and assume that all fixed points $p \in X^T$ are *fully definite*, in the sense that all T-weights of the Zariski tangent space T_pX belong to a strict half-space of the character lattice of T. Assume also that $X^T = X^{\mathbb{G}_m}$ holds for some 1-parameter subgroup $\mathbb{G}_m \subset T$, such that the associated Bialynicki-Birula decomposition $X = \bigcup X_p^+$ is a *stratification*, in the sense that each cell closure $\overline{X_p^+}$ is a union of cells. In this situation we prove the following result.

Theorem. Let $Z \subset X$ be a T-stable closed subvariety such that the T-equivariant Chow class of Z is a multiple of the class of a cell closure $\overline{X_p^+}$. Then $Z = \overline{X_p^+}$.

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In addition to flag varieties, this result applies to a class of horospherical varieties, which includes all non-singular horospherical varieties of Picard rank 1 [Pas09]. If X is defined over the field of complex numbers, the Chow class of Z may be replaced with its class in the T-equivariant singular cohomology ring $H_T^*(X)$. In fact, we only use the restrictions $[Z]_p \in H_T^*(\text{point})$ of this class to T-fixed points $p \in X^T$, which do not depend on the chosen cohomology theory.

To prove the theorem, we first show that the fixed point set of Z is given by $Z^T = \{p \in X^T : [Z]_p \neq 0\}$. Under the assumptions of the theorem, this implies that Z and $\overline{X_p^+}$ have the same T-fixed points. We then observe that $Z^T \subset \overline{X_p^+}$ implies $Z \subset \overline{X_p^+}$ when the Bialynicki-Birula decomposition of X is a stratification.

Our paper is organized as follows. In Section 2 we recall some basic facts and notation related to torus actions. In Section 3 we prove that the restricted class $[Z]_p$ is non-zero for each fixed point $p \in Z^T$, and more generally that the equivariant local class $\eta_p Z$ is non-zero when p is a fully definite T-fixed point of Z. FIXME: This is used to prove the above theorem in Section 4 and Section 5. Section 6 interprets the theorem for flag varieties, which is used in Section 7 to prove the conjecture about curve neighborhoods from [BCP23]. Finally, Section 9 interprets our theorem for certain horospherical varieties.

2. Torus actions

We work with varieties over a fixed algebraically closed field $\mathbb{K}$. Varieties are reduced but not necessarily irreducible. A point will always mean a closed point. The multiplicative group of $\mathbb{K}$ is denoted $\mathbb{G}_m = \mathbb{K} \smallsetminus \{0\}$. An (algebraic) torus is a group variety isomorphic to $(\mathbb{G}_m)^r$ for some $r \in \mathbb{N}$.

Let $T = (\mathbb{G}_m)^r$ be an algebraic torus. Any rational representation V of T is a direct sum $V = \bigoplus_\lambda V_\lambda$ of weight spaces $V_\lambda = \{v \in V \mid t.v = \lambda(t)v \ \forall t \in T\}$ defined by characters $\lambda : T \to \mathbb{G}_m$. The *weights* of V are the characters λ for which $V_\lambda \neq 0$. The group of all characters of T is called the *character lattice* and is isomorphic to $\mathbb{Z}^r$. Given a T-variety X, we let $X^T \subset X$ denote the closed subvariety of T-fixed points. A subvariety $Z \subset X$ is called *T-stable* if $t.z \in X$ for all $t \in T$ and $z \in Z$. In this case Z is itself a T-variety.

Definition 2.1. The T-fixed point $p \in X$ is *non-degenerate* in X if T acts with non-zero weights on the Zariski tangent space T_pX. The point p is *fully definite* if all T-weights of T_pX belong to a strict half-space of the character lattice of T.

Equivalently, $p \in X^T$ is fully definite in X if and only if there exists a cocharacter $\rho : \mathbb{G}_m \to T$ such that $\mathbb{G}_m$ acts with strictly positive weights on T_pX though ρ. For example, if $X = G/P$ is a flag variety and $T \subset G$ is a maximal torus, then all points of X^T are fully definite in X (see Section 6). Any non-degenerate T-fixed point must be isolated in X^T. Fully definite T-fixed points are called *attractive* in many sources, see e.g. [Bri97]; here we follow the terminology from [BB73].

Remark 2.2. If X is a normal quasi-projective T-variety, then $X^{\mathbb{G}_m} = X^T$ holds for all general cocharacters $\rho : \mathbb{G}_m \to T$. Here a cocharacter is called *general* if it avoids finitely many hyperplanes in the lattice of all cocharacters. This follows because X admits an equivariant embedding $X \subset \mathbb{P}(V)$, where V is a rational representation of T [Kam66, Mum65, Sum74].

In the rest of this paper we let X be a non-singular T-variety. The T-equivariant Chow cohomology ring of X will be denoted $H_T^*(X)$, see [Ful98, AF24]. This is an algebra over the ring $H_T^*(\text{point})$, which may be identified with the symmetric algebra of the character lattice of T. Given a class $\sigma \in H_T^*(X)$ and a T-fixed point $p \in X^T$, we let $\sigma_p \in H_T^*(\text{point})$ denote the pullback of σ along the inclusion $\{p\} \to X$. When X is defined over $\mathbb{K} = \mathbb{C}$, Chow cohomology can be replaced with singular cohomology. In fact, our arguments will only depend on equivariant classes $[Z]_p \in H_T^*(\text{point})$ obtained by restricting the class of a T-stable closed subvariety $Z \subset X$ to a fixed point, and these restrictions are independent of the chosen cohomology theory. Similarly, we can use cohomology with coefficients in either $\mathbb{Z}$ or $\mathbb{Q}$.

3. Equivariant local classes

Let Z be a T-variety, fix $p \in Z^T$, and let $\mathfrak{m} \subset \mathcal{O}_{Z,p}$ be the maximal ideal in the local ring of p. Then the tangent cone $C_pZ = \operatorname{Spec}(\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1})$ is a T-stable closed subscheme of the Zariski tangent space $T_pZ = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \operatorname{Spec}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2))$. The *local class* of Z at p is defined by (see [AF24, §17.4])

$$\eta_p Z = [C_pZ] \ \in H_T^*(T_pZ) = H_T^*(\text{point}) \,. \tag{1}$$

When p is a non-singular point of Z, we have $\eta_p Z = 1$.

Proposition 3.1. *Let Z be a T-variety and let $p \in Z^T$ be fully definite in Z. Then $\eta_p Z \neq 0$ in $H_T^*(\text{point})$.*

Proof. We may assume that p is a singular point of Z, so that C_pZ has positive dimension. Choose $\mathbb{G}_m \subset T$ such that $\mathbb{G}_m$ acts with positive weights on T_pZ. It suffices to show that the class of C_pZ is non-zero in $H_{\mathbb{G}_m}^*(T_pZ)$. Let $\{v_1, \dots, v_n\}$ be a basis of T_pZ consisting of eigenvectors of $\mathbb{G}_m$. Then the action of $\mathbb{G}_m$ is given by $t.v_i = t^{a_i} v_i$ for positive integers $a_1, \dots, a_n > 0$. Set $A = \prod_{i=1}^n a_i$, and let $\mathbb{G}_m$ act on $U = \mathbb{K}^n$ by $t.u = t^A u$. Then the map $\phi : T_pZ \to U$ defined by

$$\phi(c_1 v_1 + \cdots + c_n v_n) = (c_1^{A/a_1}, \dots, c_n^{A/a_n})$$

is a finite $\mathbb{G}_m$-equivariant morphism. By [EG98, Thm. 4] we obtain

$$H_{\mathbb{G}_m}^*(U \smallsetminus \{0\}) \otimes \mathbb{Q} = H^*(\mathbb{P}U) \otimes \mathbb{Q} \,,$$

where $\mathbb{P}U = (U \smallsetminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^{n-1}$ is the projective space of lines in U, and

$$\phi_*[C_pZ]\,|_{U \smallsetminus \{0\}} = \deg(\phi)\,[\phi(C_pZ \smallsetminus \{0\})/\mathbb{G}_m] \ \in H^*(\mathbb{P}U) \otimes \mathbb{Q} \,.$$

The result now follows from the fact that every non-empty closed subvariety of projective space defines a non-zero Chow class. □

Corollary 3.2. *Let X be a non-singular T-variety, $Z \subset X$ a T-stable closed subvariety, and $p \in Z^T$ a T-fixed point of Z. If p is non-degenerate in X and fully definite in Z, then $[Z]_p \neq 0 \in H_T^*(\text{point})$.*

Proof. By [AF24, Prop. 17.4.1] we have $[Z]_p = c_m(T_pX/T_pZ) \cdot \eta_p Z$, where $m = \dim T_pX - \dim T_pZ$. The result therefore follows from Proposition 3.1, noting that T acts with non-zero weights on T_pX/T_pZ. □

The following example rules out some potential generalizations of Corollary 3.2.

Example 3.3. Let $\mathbb{G}_m$ act on $\mathbb{A}^4$ by

$$t.(a,b,c,d) = (ta,\, tb,\, t^{-1}c,\, t^{-1}d)\,.$$

Set $Z = V(ad-bc) \subset \mathbb{A}^4$, and let $p = (0,0,0,0)$ be the origin in $\mathbb{A}^4$. Then $T_pZ = T_p\mathbb{A}^4 = \mathbb{A}^4$ and $C_pZ = Z$. Since $\mathbb{G}_m$ acts trivially on the equation $ad-bc$, we have $\eta_p Z = [Z] = 0$ in $H^*_{\mathbb{G}_m}(\mathbb{A}^4)$ (see [AF24, §2.3]).

4. Rigidity of fixed point inclusive subvarieties

Let T be an algebraic torus and let X be a non-singular T-variety. We will show in Section 6 that Schubert varieties and Richardson varieties in a flag variety X satisfy the following two definitions.

Definition 4.1. A T-stable closed subvariety $\Omega \subset X$ is *T-equivariantly rigid* if it is uniquely determined by its T-equivariant cohomology class up to a constant. More precisely, if $Z \subset X$ is any T-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H^*_T(X)$ for some $0 \neq c \in \mathbb{Q}$, then $Z = \Omega$.

Definition 4.2. A T-stable closed subvariety $\Omega \subset X$ is *T-fixed point inclusive* if, for any T-stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$.

When the action of T is clear from the context, we frequently drop T from the notation and write simply *equivariantly rigid* and *fixed point inclusive*. Both notions are properties of the T-equivariant embedding $\Omega \subset X$; for example, any T-variety is fixed point inclusive as a subvariety of itself. Intersections of T-fixed point inclusive subvarieties are again T-fixed point inclusive (with the reduced scheme structure). Most of this paper concerns applications of the following observation.

Theorem 4.3. *Let X be a non-singular projective T-variety such that all fixed points $p \in X^T$ are fully definite in X. Then any irreducible T-fixed point inclusive subvariety of X is T-equivariantly rigid.*

Proof. Let $\Omega \subset X$ be irreducible and fixed point inclusive, and let $Z \subset X$ be any T-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H^*_T(X)$, with $0 \neq c \in \mathbb{Q}$. Then Corollary 3.2 shows that $Z^T = \Omega^T = \{p \in X^T : [Z]_p \neq 0\}$. Since Ω is fixed point inclusive, we obtain $Z \subset \Omega$. Finally, the assumption $[Z] = c\,[\Omega]$ implies that Z and Ω have the same dimension, so we must have $Z = \Omega$. □

5. Rigidity of Bialynicki-Birula cells

The multiplicative group $\mathbb{G}_m$ is identified with the complement of the origin in $\mathbb{A}^1$. Given a morphism of varieties $f : \mathbb{G}_m \to X$, we write $\lim_{t\to 0} f(t) = p$ if f can be extended to a morphism $\bar{f} : \mathbb{A}^1 \to X$ such that $\bar{f}(0) = p$. This limit is unique when it exists, and it always exists when X is complete.

Let X be a non-singular projective $\mathbb{G}_m$-variety such that $X^{\mathbb{G}_m}$ is finite. Then each fixed point $p \in X^{\mathbb{G}_m}$ defines the (positive) Bialynicki-Birula cell

$$X_p^+ = \{x \in X \mid \lim_{t\to 0} t.x = p\}\,.$$

A negative cell is similarly defined by $X_p^- = \{x \in X \mid \lim_{t\to 0} t^{-1}.x = p\}$. By [BB73, Thm. 4.4], these cells form a locally closed decomposition of X,

$$X = \bigcup_{p \in X^{\mathbb{G}_m}} X_p^+\,, \tag{2}$$

that is, a disjoint union of locally closed subsets. In addition, each cell X_p^+ is isomorphic to an affine space.

Lemma 5.1. *For any $\mathbb{G}_m$-stable closed subset $Z \subset X$, we have $Z \subset \bigcup_{p \in Z^{\mathbb{G}_m}} X_p^+$.*

Proof. For any point $x \in Z$, we have $x \in X_p^+$, where $p = \lim_{t \to 0} t.x \in Z^{\mathbb{G}_m}$. □

Definition 5.2. A locally closed decomposition $X = \bigcup X_i$ is called a *stratification* if each subset X_i is non-singular and its closure $\overline{X_i}$ is a union of subsets X_j of the decomposition.

The Bialynicki-Birula decomposition (2) typically fails to be a stratification, for example when X is the blow-up of $\mathbb{P}^2$ at the point $[0,1,0]$, where $\mathbb{G}_m$ acts on $\mathbb{P}^2$ by $t.[x,y,z] = [x, ty, t^2 z]$, see [BB76, Ex. 1]. Lemma 5.1 shows that the Bialynicki-Birula decomposition is a stratification if and only if $X_q^+ \subset \overline{X_p^+}$ holds for each fixed point $q \in (\overline{X_p^+})^{\mathbb{G}_m}$. It was proved in [BB76, Thm. 5] that the decomposition is a stratification when each positive cell X_p^+ meets each negative cell X_q^- transversally. In particular, this holds when $X = G/P$ is a flag variety and $\mathbb{G}_m \subset G$ is a general 1-parameter subgroup, see [McG02, Ex. 4.2] or Lemma 6.1. When both the positive and negative Bialynicki-Birula decomposition are stratifications, all cells X_p^+ and X_q^- of complementary dimensions meet transversally, hence the positive and negative cell closures form a pair of Poincare dual bases of the cohomology ring $H^*(X)$, see [BP, Lemma 3.11]. In this paper we utilize the following application, which follows from Lemma 5.1.

Proposition 5.3. *Assume that the Bialynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+} \subset X$ is $\mathbb{G}_m$-fixed point inclusive.*

Corollary 5.4. *Let T be an algebraic torus and X a non-singular projective T-variety such that all fixed points $p \in X^T$ are fully definite in X. Assume that $X^T = X^{\mathbb{G}_m}$ for some 1-parameter subgroup $\mathbb{G}_m \subset T$, such that the associated Bialynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+}$ is T-fixed point inclusive and T-equivariantly rigid.*

Proof. The cell X_p^+ is T-stable because T is commutative and $p \in X^T$. The result now follows from Theorem 4.3 and Proposition 5.3. □

Question 5.5. We do not know whether Proposition 5.3 and Corollary 5.4 are true without the assumption that the Bialynicki-Birula decomposition of X is a stratification. It would be very interesting to settle this question.

Example 5.6. Let X be a non-singular projective toric variety, with torus $T \subset X$, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$. We show that the conclusion of Corollary 5.4 holds, even though the Bialynicki-Birula decomposition is rarely a stratification. All fixed points $p \in X^T$ are fully definite in X, as the weights of T_pX form a basis of the character lattice of T. The T-orbits $O_\tau \subset X$ correspond to the cones τ of the fan defining X, and we have $O_\sigma \subset \overline{O_\tau}$ if and only if τ is a face of σ, see [Ful93, §3.1]. In particular, the T-fixed points in X correspond to the maximal cones σ. Since X is complete, each cone τ is the intersection of the maximal cones σ corresponding to the T-fixed points in $\overline{O_\tau}$. Since all cell closures $\overline{X_p^+}$ are T-orbit closures, it suffices to show that each orbit closure $\overline{O_\tau}$ is T-fixed point inclusive. Let $Z \subset X$ be a T-stable closed subvariety such that $Z^T \subset \overline{O_\tau}$. We may assume that Z is irreducible, in which case $Z = \overline{O_\kappa}$ is also a T-orbit closure. Since κ is the intersection of the maximal cones given by the fixed points in Z^T, we obtain $\tau \subset \kappa$ and $\overline{O_\kappa} \subset \overline{O_\tau}$, as required. Now assume that X has dimension two. By [BB73, Cor. 1 of Thm. 4.5], there is a unique repulsive fixed point $b \in X^{\mathbb{G}_m}$ with $X_b^+ = \{b\}$, and a unique attractive fixed point $a \in X^{\mathbb{G}_m}$ such that X_a^+ is a dense open subset of X. For all other fixed points $p \in X^{\mathbb{G}_m} \smallsetminus \{a, b\}$, the cell $X_p^+ \cong \mathbb{A}^1$ is a line. If the Bialynicki-Birula decomposition of X is a stratification, then $b \in \overline{X_p^+}$ for all $p \in X^{\mathbb{G}_m}$. The T-fixed point b corresponds to a maximal cone σ, and b is connected to exactly two T-stable lines corresponding to the rays forming the boundary of this cone. We deduce that X contains at most four T-fixed points. Higher dimensional toric varieties for which the Bialynicki-Birula decomposition is not a stratification can be constructed by taking products. We do not know if the cell closures $\overline{X_p^+}$ are $\mathbb{G}_m$-fixed point inclusive when X is a toric variety.

6. Rigidity of Richardson varieties

Let $X = G/P = \{g.P \mid g \in G\}$ be a flag variety defined by a connected reductive linear algebraic group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B^- \cap B = T$. Let Φ be the root system of non-zero weights of $T_1 G$, the tangent space of G at the identity element. The positive roots Φ^+ are the non-zero weights of $T_1 B$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives of the cosets in W/W_P. The set of T-fixed points in X is given by $X^T = \{w.P \mid w \in W\}$, where each point $w.P$ depends only on the coset wW_P in W/W_P. Each fixed point $w.P$ defines the *Schubert varieties* $X_w = \overline{Bw.P}$ and $X^w = \overline{B^- w.P}$. For $w \in W^P$ we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$. The Bruhat order $\leq$ on W^P is defined by

$$u \leq w \quad \Leftrightarrow \quad X_u \subset X_w \quad \Leftrightarrow \quad X^u \supset X^w \quad \Leftrightarrow \quad X^u \cap X_w \neq \emptyset \,.$$

A *Richardson variety* is any non-empty intersection $X_w^u = X_w \cap X^u$ of opposite Schubert varieties in X. More generally, any G-translate of X_w^u will be called a Richardson variety. Any Richardson variety is reduced, irreducible, and rational, see [Deo77] and [BK05, §2].

Recall that a cocharacter $\rho : \mathbb{G}_m \to T$ is *strongly dominant* if $\langle \alpha, \rho \rangle > 0$ for all positive roots $\alpha \in \Phi^+$, where $\langle \alpha, \rho \rangle \in \mathbb{Z}$ is defined by $\alpha(\rho(t)) = t^{\langle \alpha, \rho \rangle}$ for $t \in \mathbb{G}_m$. The following lemma is well known, see e.g. [McG02, Ex. 4.2] or [BP, Cor. 3.14].

Lemma 6.1. *Let $\rho : \mathbb{G}_m \to T$ be a strongly dominant cocharacter. Then the associated Bialynicki-Birula cells of X are given by $X_p^+ = B.p$, for $p \in X^T$.*

Proof. Let $\mathbb{G}_m$ act on G by conjugation through ρ. The fixed point set for this action is [Spr98, (7.1.2), (7.6.4)]

$$T = \{g \in G \mid tgt^{-1} = g \;\forall\, t \in \mathbb{G}_m\} \,,$$

and the corresponding Bialynicki-Birula cell is [Spr98, (8.2.1)]

$$B = \{g \in G \mid \lim_{t \to 0} tgt^{-1} \in T\} \,.$$

This implies $B.p \subset X_p^+$ for any fixed point $p \in X^{\mathbb{G}_m}$. We deduce from (2) that the positive Bialynicki-Birula cells in X are the B-orbits. □

Lemma 6.2. *Let $\Omega \subset X$ be a T-stable closed subvariety. Any T-stable G-translate of Ω has the form $w.\Omega$, with $w \in N_G(T)$.*

Proof. Let $\Omega' = g.\Omega$ be a T-stable translate, and let $H \subset G$ be the stabilizer of Ω'. Since T and gTg^{-1} are maximal tori in H, we can choose $h \in H$ such that $T = hgTg^{-1}h^{-1}$. We obtain $hg \in N_G(T)$ and $\Omega' = h.\Omega' = hg.\Omega$, as required. □

Theorem 6.3. *Any T-stable Richardson variety in the flag variety $X = G/P$ is T-fixed point inclusive and T-equivariantly rigid.*

Proof. It follows from Proposition 5.3 and Lemma 6.1 that all Schubert varieties X_w and X^u are fixed point inclusive. This implies that every Richardson variety $X_w^u = X_w \cap X^u$ is fixed point inclusive, hence all T-stable Richardson varieties in X are fixed point inclusive by Lemma 6.2. The B-fixed point $p = 1.P$ is fully definite in X because the weights of T_pX are a subset of the negative roots of G. Since W acts transitively on X^T, this implies that all T-fixed points in X are fully definite. The result therefore follows from Theorem 4.3. □

Let $E = G/B$ denote the variety of complete flags, and let $\pi : E \to X$ be the natural projection. A *projected Richardson variety* in X is the image $\Pi_w^u(X) = \pi(E_w^u)$ of a Richardson variety in E. Projected Richardson varieties in the Grassmannian $X = \mathrm{Gr}(m,n)$ of type A, obtained as images of Richardson varieties in $\mathrm{Fl}(n)$, are also called *positroid varieties*.

Corollary 6.4. *Let $X = \mathrm{Gr}(m,n)$ be a Grassmannian of type A, and let $T = (\mathbb{G}_m)^n$ act on X through the diagonal action on $\mathbb{K}^n$. Then all positroid varieties in X are T-fixed point inclusive and T-equivariantly rigid.*

Proof. It was proved in [KLS13] that any positroid variety Ω is defined by Plucker equations. Equivalently, Ω is an intersection of T-stable Schubert divisors, so Ω is fixed point inclusive by Theorem 6.3 and equivariantly rigid by Theorem 4.3. □

Remark 6.5. Corollary 6.4 does not hold for projected Richardson varieties in arbitrary flag varieties $X = G/P$. Each simple root β defines a projected Richardson divisor $D_\beta = \Pi_{w_0^P}^{s_\beta}(X)$, where w_0^P denotes the longest element in W^P. It frequently happens that two distinct divisors $D_{\beta'}$ and $D_{\beta''}$ have the same T-equivariant cohomology and K-theory classes, which implies that these divisors are not equivariantly rigid. For example, this is the case for the quadric hypersurfaces of dimensions 7 and 8, of Lie types B_4 and D_5, and the two-step flag variety $\mathrm{Fl}(1,4;5)$ of type A_4. For other flag varieties X, all projected Richardson varieties have distinct equivariant classes, but some projected Richardson divisor D_β contains all T-fixed points in X, which rules out that D_β is fixed point inclusive. For example, this is the case for the Lagrangian Grassmannian $\mathrm{LG}(2,4)$ of type C_2 and the maximal orthogonal Grassmannian $\mathrm{OG}(4,8)$ of type D_4. This is a special case of [BP, Lemma 3.1], which can be used to produce many more examples.

Any element $u \in W$ has a unique factorization $u = u^P u_P$ for which $u^P \in W^P$ and $u_P \in W_P$, called the *parabolic factorization* with respect to P. This factorization is *reduced* in the sense that $\ell(u) = \ell(u^P) + \ell(u_P)$. The parabolic factorization of the longest element $w_0 \in W$ is $w_0 = w_0^P w_{0,P}$, where w_0^P and $w_{0,P}$ are the longest elements in W^P and W_P, respectively. Since w_0 and $w_{0,P}$ are self-inverse, we have $w_{0,P} = w_0 w_0^P$. As preparation for the next section, we prove the following identity of Schubert varieties.

Lemma 6.6. *Let $Q \subset G$ be a parabolic subgroup containing B and set $w = w_0^Q$. Then $w^{-1}.X^w = X_{w_0 w}$.*

Proof. Since $X_{w_{0,Q}}$ is a Q-stable Schubert variety, we have $X_{w_{0,Q}} = w_{0,Q}.X_{w_{0,Q}}$. By translating both sides by $w = w_0^Q$, we obtain $w.X_{w_0 w} = w_0.X_{w_0 w} = X^w$. □

7. Seidel Neighborhoods

In this section we prove a conjecture about curve neighborhoods from [BCP23]. Since this conjecture and its proof relies on the moduli space of stable maps, we will restrict our attention to varieties defined over the field $\mathbb{K} = \mathbb{C}$ of complex numbers. As in Section 6, we let $X = G/P$ denote a flag variety.

For any effective degree $d \in H_2(X, \mathbb{Z})$, we let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of degree d and genus zero, see [FP97]. The evaluation map $\mathrm{ev}_i : M_d \to X$, defined for $1 \leq i \leq 3$, sends a stable map to the image of the i-th marked point in its domain. Given two opposite Schubert varieties X_v and X^u, the *Gromov-Witten variety* $M_d(X_v, X^u)$ is the variety of stable maps that send the first two marked points to X_v and X^u:

$$M_d(X_v, X^u) \,=\, \mathrm{ev}_1^{-1}(X_v) \cap X_2^{-1}(X^u) \,\subset M_d\,.$$

The *curve neighborhood* $\Gamma_d(X_v, X^u)$ is the union of all stable curves of degree d in X connecting X_v and X^u:

$$\Gamma_d(X_v, X^u) \,=\, \mathrm{ev}_3(M_d(X_v, X^u)) \,\subset X\,.$$

Let $\mathbb{Z}[q] = \mathrm{Span}_{\mathbb{Z}}\{q^d : d \in H_2(X, \mathbb{Z}) \text{ effective}\}$ be the semigroup ring defined by the effective curve classes on X. The equivariant quantum cohomology ring of X is an algebra over $H_T^*(\mathrm{point}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$, which is defined by $\mathrm{QH}_T(X) = H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module. The *quantum product* of two opposite Schubert classes is given by

$$[X_v] \star [X^u] \,=\, \sum_{d \geq 0} q^d \, \mathrm{ev}_{3,*}[M_d(X_v, X^u)]\,,$$

where the sum is over all effective degrees $d \in H_2(X; \mathbb{Z})$.

A simple root $\gamma \in \Phi^+$ is called *cominuscule* if, when the highest root is written in the basis of simple roots, the coefficient of γ is one. The flag variety G/Q is cominuscule if Q is a maximal parabolic subgroup corresponding to a cominuscule simple root γ, that is, s_γ is the unique simple reflection in W^Q. Let $W^{\mathrm{comin}} \subset W$ be the subset of point representatives of cominuscule flag varieties of G, together with the identity element:

$$W^{\mathrm{comin}} \,=\, \{w_0^Q \mid G/Q \text{ is cominuscule}\} \cup \{1\}\,.$$

This is a subgroup of W, which is isomorphic to the quotient of the coweight lattice of Φ modulo the coroot lattice [Bou81, Prop. VI.2.6]. The isomorphism sends w_0^Q to the class of the fundamental coweight $\omega_\gamma^\vee$ corresponding to Q. Notice that γ is the unique simple root for which $w_0^Q.\gamma < 0$. In the following we set $d(w_0^Q, u) = \omega_\gamma^\vee - u^{-1}.\omega_\gamma^\vee \in H_2(X; \mathbb{Z})$ for any $u \in W$. Here we identify the group $H_2(X, \mathbb{Z})$ with a quotient of the coroot lattice, by mapping each simple coroot $\beta^\vee$ to the curve class $[X_{s_\beta}]$ if $s_\beta \in W^P$, and to zero otherwise.

The *Seidel representation* of W^{comin} on $\text{QH}(X)/\langle q-1\rangle$ is defined by $w.[X^u] = [X^w] \star [X^u]$ for $w \in W^{\text{comin}}$ and $u \in W$. In fact, we have [Sei97, Bel04, CMP09]

$$[X^w] \star [X^u] \,=\, q^{d(w,u)}\, [X^{wu}] \tag{3}$$

in the (non-equivariant) quantum ring $\text{QH}(X)$. This implies that $d(w,u)$ is the unique minimal degree d for which $\Gamma_d(X_{w_0 w}, X^u)$ is not empty [FW04, BCLM20]. More generally, it was proved in [CMP09, CP23] that the identity

$$[X^w] \star [w.X^u] \,=\, q^{d(w,u)}\, [X^{wu}] \tag{4}$$

holds in the equivariant quantum cohomology ring $\text{QH}_T(X)$. We will discuss generalizations to quantum K-theory in Section 8.

It follows from (3) and the definition of the quantum product in $\text{QH}(X)$ that $[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)] = [X^{wu}]$ holds in $H^*(X)$. Conjecture 3.11 from [BCP23] asserts that $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is in fact equal to the translated Schubert variety $w^{-1}.X^{wu}$. This is proved below as a consequence of Theorem 6.3 and (4). This result was known when $X = G/P$ is cominuscule and $w = w_0^P$ [BCP23], when X is a Grassmannian of type A and $[X^w]$ is a special Seidel class [LLSY22, Cor. 4.6], when X is any flag variety of type A [Tar23], and when X is the symplectic Grassmannian $\text{SG}(2,2n)$ [BPX, Thm. 8.1].

Theorem 7.1. *Let $X = G/P$ be a complex flag variety. For $w \in W^{\text{comin}}$ and $u \in W$ we have $\Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1}.X^{wu}$.*

Proof. By applying w^{-1} to both sides of (4) and using Lemma 6.6, we obtain

$$[X_{w_0 w}] \star [X^u] \,=\, q^{d(w,u)}\, [w^{-1}.X^{wu}]$$

in $\text{QH}_T(X)$. By definition of the quantum product, this implies that

$$[w^{-1}.X^{wu}] \,=\, \text{ev}_{3,*}[M_{d(w,u)}(X_{w_0 w}, X^u)] \,=\, c\,[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)]$$

holds in $H_T^*(X)$, where c is the degree of the map $\text{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u)$. The result therefore follows from Theorem 6.3. $\square$

8. Seidel products in quantum K-theory

In this section we discuss a generalization of the Seidel multiplication formula to quantum K-theory. We start by briefly recalling the definition of quantum K-theory. A more detailed discussion can be found in [BCMP18a, §2].

Let $X = G/P$ be a flag variety defined over $\mathbb{K} = \mathbb{C}$. The equivariant K-theory ring $K^T(X)$ is an algebra over the representation ring $\Gamma = K^T(\text{point})$. The equivariant quantum K-theory ring $\text{QK}_T(X)$ was originally constructed by Givental and Lee [Giv00, Lee04]. This ring is an algebra over the formal power series ring $\Gamma[\![q]\!] = \Gamma[\![q_\beta : s_\beta \in W^P]\!]$, which has one variable q_β for each simple reflection s_β in W^P. As a module over $\Gamma[\![q]\!]$ we have $\text{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[\![q]\!]$. The *undeformed product* of two opposite Schubert classes in $\text{QK}_T(X)$ is defined by

$$[\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}] \,=\, \sum_{d \geq 0} q^d\, \text{ev}_{3,*}[\mathcal{O}_{M_d(X_v, X^u)}]\,.$$

Let $\Psi : \text{QK}_T(X) \to \text{QK}_T(X)$ be the $\Gamma[\![q]\!]$-linear map defined by

$$\Psi([\mathcal{O}_{X^w}]) \,=\, \sum_{d \geq 0} q^d\, [\mathcal{O}_{\Gamma_d(X^w)}]\,,$$

where the curve neighborhood $\Gamma_d(X^w) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X^w))$ is defined using the evaluation maps from M_d. This curve neighborhood is a Schubert variety in X by [BCMP13, Prop. 3.2(b)], whose Weyl group element was determined in [BM15]. By [BCMP18a, Prop. 2.3], Givental's *quantum K-theory product* $\star$ is given by

$$[\mathcal{O}_{X_v}] \star [\mathcal{O}_{X^u}] \;=\; \Psi^{-1}([\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}])\,. \tag{5}$$

The following conjecture is the K-theoretic analogue of the Seidel multiplication formula (4) in $\mathrm{QH}_T(X)$ proved in [CMP09, CP23].

Conjecture 8.1. *Let $X = G/P$ be a flag variety. For $w \in W^{\mathrm{comin}}$ and $u \in W$ we have*

$$[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] \;=\; q^{d(w,u)}\,[\mathcal{O}_{w^{-1}.X^{wu}}] \quad \textit{and} \quad [\mathcal{O}_{X^w}] \star [\mathcal{O}_{w.X^u}] \;=\; q^{d(w,u)}\,[\mathcal{O}_{X^{wu}}]$$

in $\mathrm{QK}_T(X)$.

The two identities in Conjecture 8.1 are equivalent by Lemma 6.6. The non-equivariant case of this conjecture was proved in [BCP23, Cor. 3.7] when X is a cominuscule flag variety. Using Theorem 7.1, we can extend this result to equivariant quantum K-theory.

Theorem 8.2. *Conjecture 8.1 is true when X is any cominuscule flag variety.*

Proof. Since $q^{d(w,u)}$ is the only power of q appearing in the quantum cohomology product $[X_{w_0 w}] \star [X^u]$, it follows from [BCMP22, Thm. 8.3 and Remark 8.15] that the same holds for the quantum K-theory product $[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}]$, noting that this product has no exceptional degree by the inequality in [BCMP22, Def. 8.2]. Since $\Gamma_{d(w,u)-1}(X_{w_0 w}, X^u) = \emptyset$, we obtain $[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)}\,[\mathcal{O}_{\Gamma_{d(w,u)}(X_{w_0 w}, X^u)}] = q^{d(w,u)}\,[\mathcal{O}_{w^{-1}.X^{wu}}]$ by Theorem 7.1. □

A morphism $\pi : Z \to Y$ is called *cohomologically trivial* if $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_Z = 0$ for $j \geq 1$. We propose the following generalization of Theorem 7.1.

Conjecture 8.3. *Let $X = G/P$ be a flag variety, $w \in W^{\mathrm{comin}}$, $u \in W$, and let $e \in H_2(X, \mathbb{Z})$ be any effective degree.*

(a) *We have $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u) = \Gamma_e(w^{-1}.X^{wu})$.*

(b) *The evaluation map $\mathrm{ev}_3 : M_{d(w,u)+e}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is cohomologically trivial.*

Conjecture 8.3 is true for $e = 0$; part (a) is equivalent to Theorem 7.1, and part (b) holds because the map $\mathrm{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is birational by [Bel04, CMP09], and $M_{d(w,u)}(X_{w_0 w}, X^u)$ has rational singularities by [BCMP13, Cor. 3.1]. For $e \geq 0$, Theorem 7.1 implies that

$$\Gamma_e(w^{-1}.X^{wu}) \;=\; \Gamma_e(\Gamma_{d(w,u)}(X_{w_0 w}, X^u)) \;\subset\; \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)\,, \tag{6}$$

and $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is irreducible by [BCMP13, Cor. 3.8]. Conjecture 8.3(a) is therefore true if and only if $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ and $\Gamma_e(X^{wu})$ have the same dimension.

The general case of Conjecture 8.3 can be seen as a variant of the quantum-equals-classical theorem for Gromov-Witten invariants as stated in [BCMP18b, Thm. 4.1]. The conjecture immediately implies the identity

$$\mathrm{ev}_{3,*}[\mathcal{O}_{M_{d(w,u)+e}(X_{w_0 w}, X^u)}] \;=\; [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}] \tag{7}$$

in $K_T(X)$. By using the projection formula along ev_3, this implies that the K-theoretic Gromov-Witten invariants of X associated to Seidel products can be computed in the ordinary K-theory ring $K_T(X)$ by

$$\begin{aligned} I_e([\mathcal{O}_{X_{w_0 w}}],[\mathcal{O}_{X^u}],\mathcal{F}) &= \chi_{M_e}(\mathrm{ev}_1^*[\mathcal{O}_{X_{w_0 w}}]\cdot \mathrm{ev}_2^*[\mathcal{O}_{X^u}]\cdot \mathrm{ev}_3^*(\mathcal{F})) \\ &= \begin{cases} \chi_X([\mathcal{O}_{\Gamma_{e-d(w,u)}(w^{-1}.X^{wu})}]\cdot \mathcal{F}) & \text{if } e \geq d(w,u), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\mathcal{F} \in K_T(X)$ is an arbitrary K-theory class, and $\chi_X : K_T(X) \to \Gamma$ is the sheaf Euler characteristic map.

Proposition 8.4. *Conjecture 8.1 is equivalent to equation (7) for all effective degrees $e \in H_2(X,\mathbb{Z})$. In particular, Conjecture 8.1 follows from Conjecture 8.3.*

Proof. Both Conjecture 8.1 and equation (7) are equivalent to the identity

$$[\mathcal{O}_{X_{w_0 w}}] \odot [\mathcal{O}_{X^u}] = \sum_{e \geq 0} q^{d(w,u)+e}\,[\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}]$$

by the definition (5) of the quantum product in $\mathrm{QK}_T(X)$. □

Theorem 8.5. *Conjecture 8.3 is true when X is a cominuscule flag variety.*

Proof. Assume that X is cominuscule. Then Conjecture 8.3(b) is a special case of [BCMP18b, Thm. 4.1], and equation (7) holds by Theorem 8.2 and Proposition 8.4. These observations together imply Conjecture 8.3(a). □

9. Horospherical varieties of Picard rank 1

In this section we interpret Theorem 4.3 and Proposition 5.3 for a class of horospherical varieties that includes all non-singular projective horospherical varieties of Picard rank 1 (except flag varieties) by Pasquier's classification [Pas09]. Let G be a connected reductive linear algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. Let V_1 and V_2 be irreducible rational representations of G, and let $v_i \in V_i$ be a highest weight vector of weight λ_i, for $i \in \{1,2\}$. We assume that $\lambda_1 \neq \lambda_2$. Define

$$X = \overline{G.[v_1+v_2]} \subset \mathbb{P}(V_1 \oplus V_2)\,.$$

If X is normal, then X is a horospherical variety of rank 1, see [Tim11, Ch. 7]. We will assume that X is non-singular and $\mathbb{K} = \mathbb{C}$, even though many claims hold more generally; this implies that X is fibered over a flag variety G/P_{12} with non-singular horospherical fibers of Picard rank 1, see Remark 9.5. Any G-translate of a B-orbit closure in X will be called a *Schubert variety*. Our next result uses the action of $T \times \mathbb{G}_m$ on X defined by $(t,z).[u_1+u_2] = t.[u_1+zu_2]$, for $u_i \in V_i$. We have $X^{T\times\mathbb{G}_m} = X^T$.

Theorem 9.1. *Any $T\times\mathbb{G}_m$-stable Schubert variety in X is $T\times\mathbb{G}_m$-fixed point inclusive and $T\times\mathbb{G}_m$-equivariantly rigid.*

Before proving Theorem 9.1, we sketch elementary proofs of some basic facts about X, which are also consequences of general results about spherical varieties, see [Tim11, Per14, Pas09] and the references therein.

Given an element $[u_1+u_2] \in \mathbb{P}(V_1\oplus V_2)$, we will always assume $u_i \in V_i$, and i will always mean an element from $\{1,2\}$. We consider $\mathbb{P}(V_i)$ as a subvariety of $\mathbb{P}(V_1\oplus V_2)$. Let $\pi_i : \mathbb{P}(V_1\oplus V_2) \smallsetminus \mathbb{P}(V_{3-i}) \to \mathbb{P}(V_i)$ denote the projection from V_{3-i}, defined by $\pi_i([u_1+u_2]) = [u_i]$. Set $X_0 = G.[v_1+v_2] \subset \mathbb{P}(V_1 \oplus V_2)$, $X_i = G.[v_i] \subset \mathbb{P}(V_i)$, and $X_{12} = G.([v_1],[v_2]) \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Since v_i is a highest weight vector, the stabilizer $P_i = G_{[v_i]}$ is a parabolic subgroup containing B. It follows that $X_i \cong G/P_i$ and $X_{12} \cong G/(P_1 \cap P_2)$ are flag varieties. In particular, X_i is closed in $\mathbb{P}(V_i)$, and X_{12} is closed in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Notice also that $X_0 \cong G/H$, where $H \subset P_1 \cap P_2$ is the kernel of the character $\lambda_1 - \lambda_2 : P_1 \cap P_2 \to \mathbb{G}_m$. This shows that X_0 is a $\mathbb{G}_m$-bundle over $G/(P_1 \cap P_2)$, so X is a non-singular projective horospherical variety of rank 1 (but not necessarily of Picard rank 1, see Remark 9.5).

Let W be the Weyl group of G, and recall the notation from Section 6.

Lemma 9.2. *We have $X = X_0 \cup X_1 \cup X_2$. The B-orbit closures in X are*

$$\overline{Bw.[v_i]} = \bigcup_{w' \leq w} Bw'.[v_i] \quad \text{for } w \in W^{P_i} \text{ and } i \in \{1,2\}, \text{ and}$$

$$\overline{Bw.[v_1+v_2]} = \bigcup_{w' \leq w} (Bw'.[v_1+v_2] \cup Bw'.[v_1] \cup Bw'.[v_2]) \quad \text{for } w \in W^{P_1 \cap P_2}.$$

Proof. Set $\mathbb{P}_0 = \mathbb{P}(V_1 \oplus V_2) \smallsetminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$. Since $\lambda_1 \neq \lambda_2$, it follows that $\overline{T.[v_1+v_2]}$ is the line through $[v_1]$ and $[v_2]$ in $\mathbb{P}(V_1 \oplus V_2)$. This implies $X_0 = (\pi_1 \times \pi_2)^{-1}(X_{12})$, hence X_0 is closed in $\mathbb{P}_0$, and $X_0 = X \cap \mathbb{P}_0$. We also have $X_i \subset X \cap \mathbb{P}(V_i) \subset \pi_i^{-1}(X_i) \cap \mathbb{P}(V_i) = X_i$, which proves the first claim. To finish the proof, it suffices to show $w'.[v_i] \in \overline{Bw.[v_1+v_2]}$ if and only if $w' \leq w$ (when $w' \in W^{P_i}$). The implication 'if' holds because $w'.[v_i] \in \overline{Tw'.[v_1+v_2]}$, and 'only if' holds because $\pi_i(\overline{Bw.[v_1+v_2]} \smallsetminus X_{3-i}) \subset \overline{Bw.[v_i]}$. □

Define an alternative action of P_i on V_{3-i} by $p \bullet u = \lambda_i(p)^{-1} p.u$, and use this action to form the space

$$G \times^{P_i} V_{3-i} = \{[g,u] : g \in G, u \in V_{3-i}\} \;/\; \{[gp,u] = [g, p \bullet u] : p \in P_i\}\,.$$

Define a morphism of varieties $\phi_i : G \times^{P_i} V_{3-i} \to \mathbb{P}(V_1 \oplus V_2)$ by $\phi_i([g,u]) = g.[v_i+u]$. This is well defined since $p.(v_i+u) = \lambda_i(p)(v_i + p \bullet u)$ holds for $p \in P_i$ and $u \in V_{3-i}$. Set $E_i = (P_i \bullet v_{3-i}) \cup \{0\} \subset V_{3-i}$. Noting that E_i is the cone over $P_i.[v_{3-i}] \cong P_i/(P_1 \cap P_2)$, it follows that E_i is closed in V_{3-i}.

Lemma 9.3. *The restricted map $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is an isomorphism of varieties. In particular, $E_i \subset V_{3-i}$ is a linear subspace.*

Proof. Assume $\phi_i([g,u]) = \phi_i([g',u'])$, and set $p = g^{-1}g'$. We obtain $p \in P_i$ and $[v_i+u] = p.[v_i+u'] = [v_i + p \bullet u']$ in $\mathbb{P}(V_1 \oplus V_2)$, hence $[g,u] = [g, p \bullet u'] = [gp, u'] = [g',u']$ in $G \times^{P_i} V_{3-i}$. We deduce that $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is bijective, so the lemma follows from Zariski's main theorem, using that $X_0 \cup X_i$ is non-singular. □

Fix a strongly dominant cocharacter $\rho : \mathbb{G}_m \to T$. For $a \in \mathbb{Z}$, define $\rho_a : \mathbb{G}_m \to T \times \mathbb{G}_m$ by $\rho_a(z) = (\rho(z), z^a)$. The resulting action of $\mathbb{G}_m$ on X is given by $\rho_a(z).[u_1+u_2] = \rho(z).[u_1 + z^a u_2]$.

Lemma 9.4. *All T-fixed points in X are fully definite for the action of $T \times \mathbb{G}_m$.*

Proof. It follows from Lemma 9.3 that $[v_1]$ has a $T \times \mathbb{G}_m$-stable open neighborhood in X isomorphic to $B^-.[v_1] \times E_1$, where the action is given by $(t,z).(x,u) = (t.x, t \bullet zu)$. If a is sufficiently negative, then $\mathbb{G}_m$ acts through ρ_a on $T_{[v_1]}X = T_{[v_1]}X_1 \oplus E_1$ with strictly negative weights, hence $[v_1]$ is fully definite in X for the action of $T \times \mathbb{G}_m$. A symmetric argument shows that $[v_2]$ is fully definite. The result follows from this, since all T-fixed points in X are obtained from $[v_1]$ or $[v_2]$ by the action of the Weyl group W. $\square$

Proof of Theorem 9.1. For a sufficiently negative, it follows from Lemma 6.1 that the Bialynicki-Birula cells of X defined by ρ_a are

$$X^+_{w.[v_1]} = Bw.[v_1] \quad \text{and} \quad X^+_{w.[v_2]} = Bw.[v_1+v_2] \cup Bw.[v_2]\,.$$

These cells form a stratification of X by Lemma 9.2. It therefore follows from Corollary 5.4 that $\overline{Bw.[v_1]}$ and $\overline{Bw.[v_1+v_2]}$ are $T \times \mathbb{G}_m$-fixed point inclusive and $T \times \mathbb{G}_m$-equivariantly rigid for each $w \in W$. A symmetric argument applies to $\overline{Bw.[v_2]}$, which completes the proof. $\square$

Remark 9.5. The exact sequence of [Per14, Thm. 3.2.4] implies that $\mathrm{Pic}(X)$ is a free abelian group of rank equal to the rank of X (which is one) plus the number of B-stable prime divisors in X that do not contain a G-orbit. Any B-stable prime divisor meeting X_0 has the form $D = \overline{Bw_0 s_\beta.[v_1+v_2]}$, where β is a simple root, and Lemma 9.2 shows that D contains X_i if and only if β is a root of P_i. Let $P_{12} \subset G$ be the parabolic subgroup generated by P_1 and P_2. We obtain $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(G/P_{12})$. Let $\pi : X \to G/P_{12}$ be the map defined by $\pi(g.[v_1+v_2]) = \pi(g.[v_i]) = g.P_{12}$. This is a G-equivariant morphism of varieties, as its restriction to $X_0 \cup X_i$ is the composition of $\pi_i : X_0 \cup X_i \to G/P_i$ with the projection $G/P_i \to G/P_{12}$. The fibers of π are translates of $\pi^{-1}(1.P_{12}) = \overline{L.[v_1+v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where L is the Levi subgroup of P_{12} containing T. Moreover, $\pi^{-1}(1.P_{12})$ is a non-singular projective horospherical variety of Picard rank 1, so it is either a flag variety or one of the non-homogeneous spaces from Pasquier's classification [Pas09].

Question 9.6. Let X be any projective G-horospherical variety fibered over a flag variety G/P with non-singular horospherical fibers of Picard rank 1. Is it true that X is isomorphic to an orbit closure $\overline{G.[v_1+v_2]} \subset \mathbb{P}(V)$, where V is a rational representation of G, and $v_1, v_2 \in V$ are highest weight vectors?

Example 9.7. Let X be the blow-up of $\mathbb{P}^2$ at a point p, let $\pi : X \to \mathbb{P}^1$ be the morphism defined by projection from p, and set $G = \mathrm{SL}(2,\mathbb{C})$. Then X is G-horospherical and fibered over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. This variety X is isomorphic to $\overline{G.[v_1+v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where v_1 is a highest weight vector in $V_1 = \mathbb{C}^2$, and v_2 is a highest weight vector in $V_2 = \mathrm{Sym}^2(\mathbb{C}^2)$.

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