EQUIVARIANT RIGIDITY OF RICHARDSON VARIETIES

ANDERS S. BUCH, PIERRE–EMMANUEL CHAPUT, AND NICOLAS PERRIN

Abstract. We prove that Schubert varieties in flag manifolds are uniquely determined by their equivariant cohomology classes, as well as a stronger result that replaces Schubert varieties with closures of Bialynicki-Birula cells under suitable conditions. This is used to prove a conjecture from [BCP23], stating that any two-pointed curve neighborhood representing a quantum cohomology product with a Seidel class is a Schubert variety.

1. Introduction

A Schubert variety Ω in a flag manifold $X = G/P$ is called *rigid* if it is uniquely determined by its class $[\Omega]$ in the cohomology ring $H^*(X)$. More precisely, if $Z \subset X$ is any irreducible closed subvariety such that $[Z]$ is a multiple of $[\Omega]$ in $H^*(X)$, then Z is a G-translate of Ω. This problem has been studied in numerous papers, see e.g. [Hon05, Hon07, Cos11, RT12, CR13, Cos14, Cos18, HM20] and the references therein.

In this paper we show that all Schubert varieties are *equivariantly rigid*. In other words, if $T \subset G$ is a maximal torus, $\Omega \subset X$ is a T-stable Schubert variety, and $Z \subset X$ is a (non-empty) T-stable closed subvariety such that the T-equivariant class $[Z] \in H_T^*(X)$ is a multiple of $[\Omega]$, then $Z = \Omega$. We use this result to prove a conjecture from [BCP23], stating that a two-pointed curve neighborhood corresponding to a quantum cohomology product with a Seidel class, is an explicitly determined Schubert variety. This conjecture was known in some cases when X is cominuscule, in all cases when X is a flag variety of type A [LLSY22, Tar23], and for $X = \mathrm{SG}(2, 2n)$ [BPX]

More generally, let T be an algebraic torus over an algebraically closed field, let X be a non-singular projective T-variety with finite fixed point set X^T, and assume that all fixed points $p \in X^T$ are *fully definite*, in the sense that all T-weights of the Zariski tangent space T_pX belong to a strict half-space of the character lattice of T. Assume also that $X^T = X^{\mathbb{G}_m}$ holds for some 1-parameter subgroup $\mathbb{G}_m \subset T$, such that the associated Bialynicki-Birula decomposition $X = \bigcup X_p^+$ is a *stratification*, in the sense that each cell closure $\overline{X_p^+}$ is a union of cells. In this situation we prove the following result.

Theorem. Let $Z \subset X$ be a T-stable closed subvariety such that the T-equivariant Chow class of Z is a multiple of the class of a cell closure $\overline{X_p^+}$. Then $Z = \overline{X_p^+}$.

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In addition to flag varieties, this result applies to a class of horospherical varieties, which includes all non-singular horospherical varieties of Picard rank 1 [Pas09]. If X is defined over the field of complex numbers, the Chow class of Z may be replaced with its class in the T-equivariant singular cohomology ring $H_T^*(X)$. In fact, we only use the restrictions $[Z]_p \in H_T^*(\text{point})$ of this class to T-fixed points $p \in X^T$, which do not depend on the chosen cohomology theory.

To prove the theorem, we first show that the fixed point set of Z is given by $Z^T = \{p \in X^T : [Z]_p \neq 0\}$. Under the assumptions of the theorem, this implies that Z and $\overline{X_p^+}$ have the same T-fixed points. We then observe that $Z^T \subset \overline{X_p^+}$ implies $Z \subset \overline{X_p^+}$ when the Bialynicki-Birula decomposition of X is a stratification.

Our paper is organized as follows. In Section 2 we recall some basic facts and notation related to torus actions. In Section 3 we prove that the restricted class $[Z]_p$ is non-zero for each fixed point $p \in Z^T$, and more generally that the equivariant local class $\eta_p Z$ is non-zero when p is a fully definite T-fixed point of Z. This is used to prove the above theorem in Section 4 and Section 5. Section 6 interprets the theorem for flag varieties, which is used in Section 7 to prove the conjecture about curve neighborhoods from [BCP23]. Finally, Section 9 interprets our theorem for certain horospherical varieties.

sec:actions

2. Torus actions

We work with varieties over a fixed algebraically closed field $\mathbb{K}$. Varieties are reduced but not necessarily irreducible. A point will always mean a closed point. The multiplicative group of $\mathbb{K}$ is denoted $\mathbb{G}_m = \mathbb{K} \smallsetminus \{0\}$. An (algebraic) torus is a group variety isomorphic to $(\mathbb{G}_m)^r$ for some $r \in \mathbb{N}$.

Let $T = (\mathbb{G}_m)^r$ be an algebraic torus. Any rational representation V of T is a direct sum $V = \bigoplus_\lambda V_\lambda$ of weight spaces $V_\lambda = \{v \in V \mid t.v = \lambda(t)v \ \forall t \in T\}$ defined by characters $\lambda : T \to \mathbb{G}_m$. The *weights* of V are the characters λ for which $V_\lambda \neq 0$. The group of all characters of T is called the *character lattice* and is isomorphic to $\mathbb{Z}^r$. Given a T-variety X, we let $X^T \subset X$ denote the closed subvariety of T-fixed points. A subvariety $Z \subset X$ is called T-*stable* if $t.z \in Z$ for all $t \in T$ and $z \in Z$. In this case Z is itself a T-variety.

The T-equivariant (operational) Chow cohomology ring of X will be denoted $H_T^*(X)$, see [Ful98, Ch. 17] and [AF24]. This is an algebra over the ring $H_T^*(\text{point})$, which may be identified with the symmetric algebra of the character lattice of T. Given a class $\sigma \in H_T^*(X)$ and a T-fixed point $p \in X^T$, we let $\sigma_p \in H_T^*(\text{point})$ denote the pullback of σ along the inclusion $\{p\} \to X$. When X is defined over $\mathbb{K} = \mathbb{C}$, Chow cohomology can be replaced with singular cohomology. In fact, our arguments will only depend on equivariant classes $[Z]_p \in H_T^*(\text{point})$ obtained by restricting the class of a T-stable closed subvariety $Z \subset X$ to a fixed point, and these restrictions are independent of the chosen cohomology theory. Similarly, we can use cohomology with coefficients in either $\mathbb{Z}$ or $\mathbb{Q}$.

defn:extremal **Definition 2.1.** The T-fixed point $p \in X$ is *non-degenerate* in X if T acts with non-zero weights on the Zariski tangent space T_pX. The point p is *fully definite* if all T-weights of T_pX belong to a strict half-space of the character lattice of T.

Equivalently, $p \in X^T$ is fully definite in X if and only if there exists a 1-parameter subgroup $\rho : \mathbb{G}_m \to T$ such that $\mathbb{G}_m$ acts with strictly positive weights on T_pX though ρ. For example, if $X = G/P$ is a flag variety and $T \subset G$ is a maximal torus, then all points of X^T are fully definite in X (see Section 6). Any non-degenerate T-fixed point must be isolated in X^T.

Remark 2.2. If X is a normal quasi-projective T-variety, then $X^{\mathbb{G}_m} = X^T$ holds for all general 1-parameter subgroups $\rho : \mathbb{G}_m \to T$. Here a 1-parameter subgroup is called general if it avoids finitely many hyperplanes in the lattice of all 1-parameter subgroups. This follows because X admits an equivariant embedding $X \subset \mathbb{P}(V)$, where V is a rational representation of T [Kam66, Mum65, Sum74].

sec:local

3. Equivariant local classes

Let Z be a T-variety, fix $p \in Z^T$, and let $\mathfrak{m} \subset \mathcal{O}_{Z,p}$ be the maximal ideal in the local ring of p. Then the tangent cone $C_p Z = \operatorname{Spec}(\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1})$ is a T-stable closed subscheme of the Zariski tangent space $T_p Z = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \operatorname{Spec}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2))$. The *local class* of Z at p is defined by (see [AF24, §17.4])

$$\eta_p Z = [C_p Z] \ \in \ H_T^*(T_p Z) = H_T^*(\text{point})\,. \tag{1}$$

When p is a non-singular point of Z, we have $\eta_p Z = 1$.

prop:local

Proposition 3.1. *Let Z be a T-variety and let $p \in Z^T$ be fully definite in Z. Then $\eta_p Z \neq 0$ in $H_T^*(\text{point})$.*

Proof. We may assume that p is a singular point of Z, so that $C_p Z$ has positive dimension. Choose $\mathbb{G}_m \subset T$ such that $\mathbb{G}_m$ acts with positive weights on $T_p Z$. It suffices to show that the class of $C_p Z$ is non-zero in $H^*_{\mathbb{G}_m}(T_p Z)$. Let $\{v_1, \dots, v_n\}$ be a basis of $T_p Z$ consisting of eigenvectors of $\mathbb{G}_m$. Then the action of $\mathbb{G}_m$ is given by $t.v_i = t^{a_i} v_i$ for positive integers $a_1, \dots, a_n > 0$. Set $A = \prod_{i=1}^n a_i$, and let $\mathbb{G}_m$ act on $U = \mathbb{K}^n$ by $t.u = t^A u$. Then the map $\phi : T_p Z \to U$ defined by

$$\phi(c_1 v_1 + \dots + c_n v_n) = (c_1^{A/a_1}, \dots, c_n^{A/a_n})$$

is a finite $\mathbb{G}_m$-equivariant morphism. By [EG98, Thm. 4] we obtain

$$H^*_{\mathbb{G}_m}(U \smallsetminus \{0\}) \otimes \mathbb{Q} = H^*(\mathbb{P}U) \otimes \mathbb{Q}\,,$$

where $\mathbb{P}U = (U \smallsetminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^{n-1}$ is the projective space of lines in U, and

$$\phi_*[C_p Z]\,|_{U \smallsetminus \{0\}} = \deg(\phi)\,[\phi(C_p Z \smallsetminus \{0\})/\mathbb{G}_m] \ \in H^*(\mathbb{P}U) \otimes \mathbb{Q}\,.$$

The result now follows from the fact that every non-empty closed subvariety of projective space defines a non-zero Chow class. □

cor:local

Corollary 3.2. *Let X be a T-variety, $Z \subset X$ a T-stable closed subvariety, and $p \in Z^T$ a T-fixed point of Z. If p is non-singular and non-degenerate in X, and p is fully definite in Z, then $[Z]_p \neq 0 \in H_T^*(\text{point})$.*

Proof. By [AF24, Prop. 17.4.1] we have $[Z]_p = c_m(T_p X / T_p Z) \cdot \eta_p Z$, where $m = \dim T_p X - \dim T_p Z$. The result therefore follows from Proposition 3.1, noting that T acts with non-zero weights on $T_p X / T_p Z$. □

The following example rules out some potential generalizations of Corollary 3.2.

Example 3.3. Let $\mathbb{G}_m$ act on $\mathbb{A}^4$ by

$$t.(a, b, c, d) = (ta,\, tb,\, t^{-1}c,\, t^{-1}d)\,.$$

Set $Z = V(ad - bc) \subset \mathbb{A}^4$, and let $p = (0,0,0,0)$ be the origin in $\mathbb{A}^4$. Then $T_p Z = T_p \mathbb{A}^4 = \mathbb{A}^4$ and $C_p Z = Z$. Since $\mathbb{G}_m$ acts trivially on the equation $ad - bc$, we have $\eta_p Z = [Z] = 0$ in $H^*_{\mathbb{G}_m}(\mathbb{A}^4)$ (see [AF24, §2.3]).

sec:rigidity

4. Rigidity of fixed point inclusive subvarieties

Let T be an algebraic torus and let X be a T-variety. We will show in Section 6 that Schubert varieties and Richardson varieties in a flag variety X satisfy the following two definitions.

defn:rigid **Definition 4.1.** A T-stable closed subvariety $\Omega \subset X$ is *T-equivariantly rigid* if it is uniquely determined by its T-equivariant cohomology class up to a constant. More precisely, if $Z \subset X$ is any T-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H_T^*(X)$ for some $0 \neq c \in \mathbb{Q}$, then $Z = \Omega$.

defn:fpi **Definition 4.2.** A T-stable closed subvariety $\Omega \subset X$ is *T-fixed point inclusive* if, for any T-stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$.

When the action of T is clear from the context, we frequently drop T from the notation and write simply *equivariantly rigid* and *fixed point inclusive*. Both notions are properties of the T-equivariant embedding $\Omega \subset X$; for example, any T-variety is fixed point inclusive as a subvariety of itself, and any irreducible T-variety is equivariantly rigid as a subvariety of itself. Intersections of T-fixed points inclusive subvarieties are again T-fixed point inclusive (with the reduced scheme structure). Most of this paper concerns applications of the following observation.

thm:rigid **Theorem 4.3.** *Let X be a non-singular projective T-variety such that all fixed points $p \in X^T$ are fully definite in X. Then any irreducible T-fixed point inclusive subvariety of X is T-equivariantly rigid.*

Proof. Let $\Omega \subset X$ be irreducible and fixed point inclusive, and let $Z \subset X$ be any T-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H_T^*(X)$, with $0 \neq c \in \mathbb{Q}$. Then Corollary 3.2 shows that $Z^T = \Omega^T = \{p \in X^T : [Z]_p \neq 0\}$. Since Ω is fixed point inclusive, we obtain $Z \subset \Omega$. Finally, the assumption $[Z] = c\,[\Omega]$ implies that Z and Ω have the same dimension, so we must have $Z = \Omega$. □

sec:bbcells

5. Rigidity of Bialynicki-Birula cells

The multiplicative group $\mathbb{G}_m$ is identified with the complement of the origin in $\mathbb{A}^1$. Given a morphism of varieties $f : \mathbb{G}_m \to X$, we write $\lim_{t\to 0} f(t) = p$ if f can be extended to a morphism $\bar{f} : \mathbb{A}^1 \to X$ such that $\bar{f}(0) = p$. This limit is unique when it exists, and it always exists when X is complete.

Let X be a non-singular projective $\mathbb{G}_m$-variety such that $X^{\mathbb{G}_m}$ is finite. Then each fixed point $p \in X^{\mathbb{G}_m}$ defines the (positive) Bialynicki-Birula cell

$$X_p^+ = \{x \in X \mid \lim_{t\to 0} t.x = p\}\,.$$

A negative cell is similarly defined by $X_p^- = \{x \in X \mid \lim_{t\to 0} t^{-1}.x = p\}$. By [BB73, Thm. 4.4], these cells form a locally closed decomposition of X,

eqn:bbdecomp
$$X = \bigcup_{p \in X^{\mathbb{G}_m}} X_p^+\,, \tag{2}$$

that is, a disjoint union of locally closed subsets. In addition, each cell X_p^+ is isomorphic to an affine space.

lemma:include **Lemma 5.1.** *For any $\mathbb{G}_m$-stable closed subset $Z \subset X$, we have $Z \subset \bigcup_{p \in Z^{\mathbb{G}_m}} X_p^+$.*

Proof. For any point $x \in Z$, we have $x \in X_p^+$, where $p = \lim_{t\to 0} t.x \in Z^{\mathbb{G}_m}$. $\square$

Definition 5.2. A locally closed decomposition $X = \bigcup X_i$ is called a *stratification* if each subset X_i is non-singular and its closure $\overline{X_i}$ is a union of subsets X_j of the decomposition.

The Bialynicki-Birula decomposition (2) typically fails to be a stratification, for example when X is the blow-up of $\mathbb{P}^2$ at the point $[0,1,0]$, where $\mathbb{G}_m$ acts on $\mathbb{P}^2$ by $t.[x,y,z] = [x, ty, t^2 z]$, see [BB73, Ex. 1]. Lemma 5.1 shows that the Bialynicki-Birula decomposition is a stratification if and only if $X_q^+ \subset \overline{X_p^+}$ holds for each fixed point $q \in (\overline{X_p^+})^{\mathbb{G}_m}$. It was proved in [BB73, Thm. 5] that the decomposition is a stratification when each positive cell X_p^+ meets each negative cell X_q^- transversally. In particular, this holds when $X = G/P$ is a flag variety and $\mathbb{G}_m \subset G$ is a general 1-parameter subgroup, see [McG02, Ex. 4.2] or Lemma 6.1. When both the positive and negative Bialynicki-Birula decomposition are stratifications, all cells X_p^+ and X_q^- of complementary dimensions meet transversally, hence the positive and negative cell closures form a pair of Poincare dual bases of the cohomology ring $H^*(X)$. In this paper we utilize the following application, which follows from Lemma 5.1.

prop:bb-fpi **Proposition 5.3.** *Assume that the Bialynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+} \subset X$ is $\mathbb{G}_m$-fixed point inclusive.*

cor:bb-rigid **Corollary 5.4.** *Let T be an algebraic torus and X a non-singular projective T-variety such that all fixed points $p \in X^T$ are fully definite in X. Assume that $X^T = X^{\mathbb{G}_m}$ for some $\mathbb{G}_m \subset T$, such that the associated Bialynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+}$ is T-fixed point inclusive and T-equivariantly rigid.*

Proof. This follows from Theorem 4.3 and Proposition 5.3. $\square$

Question 5.5. We do not know whether Proposition 5.3 and Corollary 5.4 are true without the assumption that the Bialynicki-Birula decomposition of X is a stratification. It would be very interesting to settle this question.

Example 5.6. Let X be a non-singular projective toric variety, with torus $T \subset X$, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$. We show that the conclusion of Corollary 5.4 holds, even though the Bialynicki-Birula decomposition is rarely a stratification. All fixed points $p \in X^T$ are fully definite in X, as the weights of T_pX form a basis of the character lattice of T. The T-orbits $O_\tau \subset X$ correspond to the cones τ of the fan defining X, and we have $O_\sigma \subset \overline{O_\tau}$ if and only if τ is a face of σ, see [Ful93, §3.1]. In particular, the T-fixed points in X correspond to the maximal cones σ. Since X is complete, each cone τ is the intersection of the maximal cones σ corresponding to the T-fixed points in $\overline{O_\tau}$. Since all cell closures $\overline{X_p^+}$ are T-orbit closures, it suffices to show that each orbit closure $\overline{O_\tau}$ is T-fixed point inclusive. Let $Z \subset X$ be a T-stable closed subvariety such that $Z^T \subset \overline{O_\tau}$. We may assume that Z is irreducible, in which case $Z = \overline{O_\kappa}$ is also a T-orbit closure. Since κ is the intersection of the maximal cones given by the fixed points in Z^T, we obtain $\tau \subset \kappa$ and $\overline{O_\kappa} \subset \overline{O_\tau}$, as required. Now assume that X has dimension two. By [BB73, Cor. 1 of Thm. 4.5], there is a unique repulsive fixed point $b \in X^{\mathbb{G}_m}$ with $X_b^+ = \{b\}$, and a unique attractive fixed point $a \in X^{\mathbb{G}_m}$ such that X_a^+ is a dense open subset of X. For all other fixed points $p \in X^{\mathbb{G}_m} \smallsetminus \{a,b\}$, the cell $X_p^+ \cong \mathbb{A}^1$ is a line. If the Bialynicki-Birula decomposition of X is a stratification, then $b \in \overline{X_p^+}$ for all $p \in X^{\mathbb{G}_m}$. The T-fixed point b corresponds to a maximal cone σ, and b is connected to exactly two T-stable lines corresponding to the rays forming the boundary of this cone. We deduce that X contains at most four T-fixed points. Higher dimensional toric varieties for which the Bialynicki-Birula decomposition is not a stratification can be constructed by taking products. We do not know if the cell closures $\overline{X_p^+}$ are $\mathbb{G}_m$-fixed point inclusive.

sec:schubert

6. Rigidity of Richardson varieties

Let $X = G/P = \{g.P \mid g \in G\}$ be a flag variety defined by a connected reductive linear algebraic group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B^- \cap B = T$. Let Φ be the root system of non-zero weights of $T_1 G$, the tangent space of G at the identity element. The positive roots Φ^+ are the non-zero weights of $T_1 B$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the subset of minimal representatives of the cosets in W/W_P. The set of T-fixed points in X is given by $X^T = \{w.P \mid w \in W\}$, where each point $w.P$ depends only on the coset wW_P in W/W_P. Each fixed point $w.P$ defines the *Schubert varieties* $X_w = \overline{Bw.P}$ and $X^w = \overline{B^- w.P}$. For $w \in W^P$ we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$. The Bruhat order $\leq$ on W^P is defined by

$$u \leq w \quad \Leftrightarrow \quad X_u \subset X_w \quad \Leftrightarrow \quad X^u \supset X^w \quad \Leftrightarrow \quad X^u \cap X_w \neq \emptyset\,.$$

A *Richardson variety* is any non-empty intersection $X_w^u = X_w \cap X^u$ of opposite Schubert varieties in X. Any Richardson variety is reduced, irreducible, and rational, see [Deo77] and [BK05, §2].

Recall that a cocharacter $\rho : \mathbb{G}_m \to T$ is *strongly dominant* if $\langle \alpha, \rho \rangle > 0$ for all positive roots $\alpha \in \Phi^+$, where $\langle \alpha, \rho \rangle \in \mathbb{Z}$ is defined by $\alpha(\rho(t)) = t^{\langle \alpha, \rho \rangle}$ for $t \in \mathbb{G}_m$. The following lemma is well known, see e.g. [McG02, Ex. 4.2] or [BP, Cor. 3.14].

lemma:flagvar **Lemma 6.1.** *Let $\rho : \mathbb{G}_m \to T$ be a strongly dominant 1-parameter subgroup. Then the associated Bialynicki-Birula cells of X are given by $X_p^+ = B.p$, for $p \in X^T$.*

Proof. Let $\mathbb{G}_m$ act on G by conjugation through ρ. The fixed point set for this action is [Spr98, (7.1.2), (7.6.4)]

$$T = \{g \in G \mid tgt^{-1} = g \;\forall\, t \in \mathbb{G}_m\}\,,$$

and the corresponding Bialynicki-Birula cell is [Spr98, (8.2.1)]

$$B = \{g \in G \mid \lim_{t \to 0} tgt^{-1} \in T\}\,.$$

This implies $B.p \subset X_p^+$ for any fixed point $p \in X^{\mathbb{G}_m}$. We deduce from (2) that the positive Bialynicki-Birula cells in X are the B-orbits. □

cor:rigidschub **Corollary 6.2.** *Any Richardson variety X_u^v in the flag variety $X = G/P$ is T-fixed point inclusive and T-equivariantly rigid.*

Proof. It follows from Proposition 5.3 and Lemma 6.1 that Schubert varieties in X are fixed point inclusive, which in turn implies that Richardson varieties are fixed point inclusive. The B-fixed point $p = 1.P$ is fully definite in X because the weights of $T_p X$ are a subset of the negative roots of G. Since W acts transitively on X^T, this implies that all T-fixed points in X are fully definite. The result therefore follows from Theorem 4.3. $\square$

Let $E = G/B$ denote the variety of complete flags, and let $\pi : E \to X$ be the natural projection. A *projected Richardson variety* in X is the image $\Pi_w^u(X) = \pi(E_w^u)$ of a Richardson variety in E. Projected Richardson varieties in the Grassmannian $X = \mathrm{Gr}(m, n)$ of type A, obtained as images of Richardson varieties in $\mathrm{Fl}(n)$, are also called *positroid varieties*.

cor:positroid

Corollary 6.3. *Let $X = \mathrm{Gr}(m, n)$ be a Grassmannian of type A, and let $T = (\mathbb{G}_m)^n$ act on X through the diagonal action on $\mathbb{K}^n$. Then all positroid varieties in X are T-fixed point inclusive and T-equivariantly rigid.*

Proof. It was proved in [KLS13] that any positroid variety Ω is defined by Plucker equations. Equivalently, Ω is an intersection of T-stable Schubert divisors, so Ω is fixed point inclusive by Corollary 6.2 and equivariantly rigid by Theorem 4.3. $\square$

Remark 6.4. Corollary 6.3 does not hold for projected Richardson varieties in arbitrary flag varieties $X = G/P$. Each simple root β defines a projected Richardson divisor $D_\beta = \Pi_{w_0^P}^{s_\beta}(X)$, where w_0^P denotes the longest element in W^P. It frequently happens that two distinct divisors $D_{\beta'}$ and $D_{\beta''}$ have the same T-equivariant cohomology and K-theory classes, which implies that these divisors are not equivariantly rigid. For example, this is the case for the quadric hypersurfaces of dimensions 7 and 8, of Lie types B_4 and D_5, and the two-step flag variety $\mathrm{Fl}(1, 4; 5)$ of type A_4. For other flag varieties X, all projected Richardson varieties have distinct equivariant classes, but some projected Richardson divisor D_β contains all T-fixed points in X, which rules out that D_β is fixed point inclusive. For example, this is the case for the Lagrangian Grassmannian $\mathrm{LG}(2, 4)$ of type C_2 and the maximal orthogonal Grassmannian $\mathrm{OG}(4, 8)$ of type D_4. This is a special case of [BP, Lemma 3.1], which can be used to produce many more examples.

Any element $u \in W$ has a unique factorization $u = u^P u_P$ for which $u^P \in W^P$ and $u_P \in W_P$, called the *parabolic factorization* with respect to P. This factorization is *reduced* in the sense that $\ell(u) = \ell(u^P) + \ell(u_P)$. The parabolic factorization of the longest element $w_0 \in W$ is $w_0 = w_0^P w_{0,P}$, where w_0^P and $w_{0,P}$ are the longest elements in W^P and W_P, respectively. Since w_0 and $w_{0,P}$ are self-inverse, we have $w_{0,P} = w_0 w_0^P$. As preparation for the next section, we prove the following identity of Schubert varieties.

lemma:dualpoint

Lemma 6.5. *Let $Q \subset G$ be a parabolic subgroup containing B and set $w = w_0^Q$. Then $w^{-1}.X^w = X_{w_0 w}$.*

Proof. It follows from Corollary 6.2(b) that $X_{w_{0,Q}} = w_{0,Q}.X_{w_{0,Q}}$, as the T-fixed points of both Schubert varieties are $\{u.P \mid u \in W_Q\}$. By translating both sides by $w = w_0^Q$, we obtain $w.X_{w_0 w} = w_0.X_{w_0 w} = X^w$, as required. $\square$

sec:seidel

7. Seidel Neighborhoods

In this section we prove a conjecture about curve neighborhoods from [BCP23]. Since this conjecture and its proof relies on the moduli space of stable maps, we will restrict our attention to varieties defined over the field $\mathbb{K} = \mathbb{C}$ of complex numbers. As in Section 6, we let $X = G/P$ denote a flag variety.

For any effective degree $d \in H_2(X,\mathbb{Z})$, we let $M_d = \overline{\mathcal{M}}_{0,3}(X,d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of degree d and genus zero, see [FP97]. The evaluation map $\mathrm{ev}_i : M_d \to X$, defined for $1 \leq i \leq 3$, sends a stable map to the image of the i-th marked point in its domain. Given two opposite Schubert varieties X_v and X^u, the *Gromov-Witten variety* $M_d(X_v, X^u)$ is the variety of stable maps that send the first two marked points to X_v and X^u:

$$M_d(X_v, X^u) \,=\, \mathrm{ev}_1^{-1}(X_v) \cap X_2^{-1}(X^u) \,\subset M_d\,.$$

The *curve neighborhood* $\Gamma_d(X_v, X^u)$ is the union of all stable curves of degree d in X connecting X_v and X^u:

$$\Gamma_d(X_v, X^u) \,=\, \mathrm{ev}_3(M_d(X_v, X^u)) \,\subset X\,.$$

Let $\mathbb{Z}[q] = \mathrm{Span}_{\mathbb{Z}}\{q^d : d \in H_2(X,\mathbb{Z}) \text{ effective}\}$ be the semigroup ring defined by the effective curve classes on X. The equivariant quantum cohomology ring of X is an algebra over $H_T^*(\text{point}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$, which is defined by $\mathrm{QH}_T(X) = H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module. The *quantum product* of two opposite Schubert classes is given by

$$[X_v] \star [X^u] \,=\, \sum_{d \geq 0} q^d \, \mathrm{ev}_{3,*}[M_d(X_v, X^u)]\,,$$

where the sum is over all effective degrees $d \in H_2(X;\mathbb{Z})$.

A simple root $\gamma \in \Phi^+$ is called *cominuscule* if, when the highest root is written in the basis of simple roots, the coefficient of γ is one. The flag variety G/Q is cominuscule if Q is a maximal parabolic subgroup corresponding to a cominuscule simple root γ, that is, s_γ is the unique simple reflection in W^Q. Let $W^{\mathrm{comin}} \subset W$ be the subset of point representatives of cominuscule flag varieties of G, together with the identity element:

$$W^{\mathrm{comin}} \,=\, \{w_0^Q \mid G/Q \text{ is cominuscule}\} \cup \{1\}\,.$$

This is a subgroup of W, which is isomorphic to the quotient of the coweight lattice of Φ modulo the coroot lattice [Bou81, Prop. VI.2.6]. The isomorphism sends w_0^Q to the class of the fundamental coweight $\omega_\gamma^\vee$ corresponding to Q. In the following we set $d(w_0^Q, u) = \omega_\gamma^\vee - u^{-1}.\omega_\gamma^\vee \in H_2(X;\mathbb{Z})$ for any $u \in W$. Here we identify the group $H_2(X,\mathbb{Z})$ with a quotient of the coroot lattice, by mapping each simple coroot $\beta^\vee$ to the curve class $[X_{s_\beta}]$ if $s_\beta \in W^P$, and to zero otherwise.

The *Seidel representation* of W^{comin} on $\mathrm{QH}(X)/\langle q-1\rangle$ is defined by $w.[X^u] = [X^w] \star [X^u]$ for $w \in W^{\mathrm{comin}}$ and $u \in W$. In fact, we have [Sei97, Bel04, CMP09]

eqn:seidel
$$[X^w] \star [X^u] \,=\, q^{d(w,u)}\, [X^{wu}] \tag{3}$$

in the (non-equivariant) quantum ring $\mathrm{QH}(X)$. This implies that $d(w,u)$ is the unique minimal degree d for which $\Gamma_d(X_{w_0 w}, X^u)$ is not empty [FW04, BCLM20]. More generally, it was proved in [CMP09, CP23] that the identity

eqn:htseidel
$$[X^w] \star [w.X^u] \,=\, q^{d(w,u)}\, [X^{wu}] \tag{4}$$

holds in the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$. We will discuss generalizations to quantum K-theory in Section 8.

It follows from (3) and the definition of the quantum product in $\mathrm{QH}(X)$ that $[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)] = [X^{wu}]$ holds in $H^*(X)$. Conjecture 3.11 from [BCP23] asserts that $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is in fact equal to the translated Schubert variety $w^{-1}.X^{wu}$. This is proved below as a consequence of Corollary 6.2 and (4). This result was known when $X = G/P$ is cominuscule and $w = w_0^P$ [BCP23], when X is a

Grassmannian of type A and $[X^w]$ is a special Seidel class [LLSY22, Cor. 4.6], when X is any flag variety of type A [Tar23], and when X is the symplectic Grassmannian $\mathrm{SG}(2,2n)$ [BPX, Thm. 8.1].

thm:seidelnbhd

Theorem 7.1. *Let $X = G/P$ be a complex flag variety. For $w \in W^{\mathrm{comin}}$ and $u \in W$ we have $\Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1}.X^{wu}$.*

Proof. By applying w^{-1} to both sides of (4) and using Lemma 6.5, we obtain

$$[X_{w_0 w}] \star [X^u] \,=\, q^{d(w,u)}\,[w^{-1}.X^{wu}]$$

in $\mathrm{QH}_T(X)$. By definition of the quantum product, this implies that

$$[w^{-1}.X^{wu}] \,=\, \mathrm{ev}_{3,*}[M_{d(w,u)}(X_{w_0 w}, X^u)] \,=\, c\,[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)]$$

holds in $H_T^*(X)$, where c is the degree of the map $\mathrm{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u)$. The result therefore follows from Corollary 6.2. $\square$

sec:qkseidel

8. Seidel products in quantum K-theory

In this section we discuss a generalization of the Seidel multiplication formula (4) to quantum K-theory. We start by briefly recalling the definition of quantum K-theory. A more detailed discussion can be found in [BCMP18a, §2].

Let $X = G/P$ be a flag variety defined over $\mathbb{K} = \mathbb{C}$. The equivariant K-theory ring $K^T(X)$ is an algebra over the representation ring $\Gamma = K^T(\text{point})$. The equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ is an algebra over the formal power series ring $\Gamma[\![q]\!] = \Gamma[\![q_\beta : s_\beta \in W^P]\!]$, which has one variable q_β for each simple reflection s_β in W^P. This ring was originally constructed by Givental and Lee [Giv00, Lee04]. As a module over $\Gamma[\![q]\!]$ we have $\mathrm{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[\![q]\!]$. The *undeformed product* of two opposite Schubert classes in $\mathrm{QK}_T(X)$ is defined by

$$[\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}] \,=\, \sum_{d \geq 0} q^d\, \mathrm{ev}_{3,*}[\mathcal{O}_{M_d(X_v, X^u)}]\,.$$

Let $\Psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ be the $\Gamma[\![q]\!]$-linear map defined by

$$\Psi([\mathcal{O}_{X^w}]) \,=\, \sum_{d \geq 0} q^d\,[\mathcal{O}_{\Gamma_d(X^w)}]\,,$$

where the curve neighborhood $\Gamma_d(X^w) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X^w))$ is defined using the evaluation maps from M_d. This curve neighborhood is a Schubert variety in X by [BCMP13, Prop. 3.2(b)], whose Weyl group element was determined in [BM15]. By [BCMP18a, Prop. 2.3], Givental's *quantum K-theory product* $\star$ is given by

eqn:qkproduct

$$[\mathcal{O}_{X_v}] \star [\mathcal{O}_{X^u}] \,=\, \Psi^{-1}([\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}])\,. \tag{5}$$

The following conjecture is the K-theoretic analogue of the Seidel multiplication formula (4) in $\mathrm{QH}_T(X)$ proved in [CMP09, CP23].

conj:qkseidel

Conjecture 8.1. *Let $X = G/P$ be a flag variety. For $w \in W^{\mathrm{comin}}$ and $u \in W$ we have*

$$[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] \,=\, q^{d(w,u)}\,[\mathcal{O}_{w^{-1}.X^{wu}}] \quad \textit{and} \quad [\mathcal{O}_{X^w}] \star [\mathcal{O}_{w.X^u}] \,=\, q^{d(w,u)}\,[\mathcal{O}_{X^{wu}}]$$

in $\mathrm{QK}_T(X)$.

The two identities in Conjecture 8.1 are equivalent by Lemma 6.5. The non-equivariant case of this conjecture was proved in [BCP23, Cor. 3.7] when X is a cominuscule flag variety. Using Theorem 7.1, we can extend this result to equivariant quantum K-theory.

Theorem 8.2. *Conjecture 8.1 is true when X is any cominuscule flag variety.*

Proof. Since $q^{d(w,u)}$ is the only power of q appearing in the quantum cohomology product $[X_{w_0 w}] \star [X^u]$, it follows from [BCMP22, Thm. 8.3 and Remark 8.15] that the same holds for the quantum K-theory product $[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}]$, noting that $d(w,u)$ is not an exceptional degree of this product. Since $\Gamma_{d(w,u)-1}(X_{w_0 w}, X^u) = \emptyset$, we obtain $[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)}\,[\mathcal{O}_{\Gamma_{d(w,u)}(X_{w_0 w}, X^u)}] = q^{d(w,u)}\,[\mathcal{O}_{w^{-1}.X^{wu}}]$ by Theorem 7.1. $\square$

A morphism $\pi : Z \to Y$ is called *cohomologically trivial* if $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_Z = 0$ for $j \geq 1$. We propose the following generalization of Theorem 7.1.

conj:seidelnbhd **Conjecture 8.3.** *Let $X = G/P$ be a flag variety, $w \in W^{\mathrm{comin}}$, $u \in W$, and let $e \in H_2(X, \mathbb{Z})$ be any effective degree.*

(a) *We have $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u) = \Gamma_e(w^{-1}.X^{wu})$.*

(b) *The evaluation map $\mathrm{ev}_3 : M_{d(w,u)+e}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is cohomologically trivial.*

Conjecture 8.3 is true for $e = 0$; part (a) is equivalent to Theorem 7.1, and part (b) holds because the map $\mathrm{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is birational by [Bel04, CMP09], and $M_{d(w,u)}(X_{w_0 w}, X^u)$ has rational singularities by [BCMP13, Cor. 3.1]. For $e \geq 0$, Theorem 7.1 implies that

$$\Gamma_e(w^{-1}.X^{wu}) = \Gamma_e(\Gamma_{d(w,u)}(X_{w_0 w}, X^u)) \subset \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)\,,$$

and $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is irreducible by [BCMP13, Cor. 3.8]. Conjecture 8.3(a) is therefore true if and only if $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ and $\Gamma_e(X^{wu})$ have the same dimension.

The general case of Conjecture 8.3 can be seen as a variant of the quantum-equals-classical theorem for Gromov-Witten invariants as stated in [BCMP18b, Thm. 4.1]. The conjecture immediately implies the identity

eqn:seidelpush
$$\mathrm{ev}_{3,*}[\mathcal{O}_{M_{d(w,u)+e}(X_{w_0 w}, X^u)}] = [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}] \tag{6}$$

in $K_T(X)$. By using the projection formula along ev_3, this implies that the K-theoretic Gromov-Witten invariants of X associated to Seidel products can be computed in the ordinary equivariant K-theory of X by

$$\begin{aligned} I_e([\mathcal{O}_{X_{w_0 w}}], [\mathcal{O}_{X^u}], \mathcal{F}) &= \chi_{M_e}(\mathrm{ev}_1^*[\mathcal{O}_{X_{w_0 w}}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X^u}] \cdot \mathrm{ev}_3^*(\mathcal{F})) \\ &= \begin{cases} \chi_X([\mathcal{O}_{\Gamma_{e-d(w,u)}(w^{-1}.X^{wu})}] \cdot \mathcal{F}) & \text{if } e \geq d(w,u), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\mathcal{F} \in K_T(X)$ is an arbitrary K-theory class, and $\chi_X : K_T(X) \to \Gamma$ is the sheaf Euler characteristic map.

Theorem 8.4. *Conjecture 8.1 follows from Conjecture 8.3.*

Proof. Using the identity (6), we obtain

$$[\mathcal{O}_{X_{w_0 w}}] \odot [\mathcal{O}_{X^u}] = \sum_{e \geq 0} q^{d(w,u)+e}\,[\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}] = \Psi(q^{d(w,u)}\,[\mathcal{O}_{w^{-1}.X^{wu}}])\,,$$

after which Conjecture 8.1 follows from the definition (5) of the quantum product in $\mathrm{QK}_T(X)$. $\square$

sec:horospherical

9. Horospherical varieties of Picard rank 1

In this section we interpret Theorem 4.3 and Proposition 5.3 for a class of horospherical varieties that includes all non-singular projective horospherical varieties of Picard rank 1 (except flag varieties) by Pasquier's classification [Pas09]. Let G be a connected reductive linear algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. Let V_1 and V_2 be irreducible rational representations of G, and let $v_i \in V_i$ be a highest weight vector of weight λ_i, for $i \in \{1, 2\}$. We assume that $\lambda_1 \neq \lambda_2$. Define

$$X = \overline{G.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)\,.$$

If X is normal, then X is a horospherical variety of rank 1, see [Tim11, Ch. 7]. We will assume that X is non-singular and $\mathbb{K} = \mathbb{C}$, even though many claims hold more generally; this implies that X is fibered over a flag variety G/P_{12} with non-singular horospherical fibers of Picard rank 1, see Remark 9.5. Any G-translate of a B-orbit closure in X will be called a *Schubert variety*. Our next result uses the action of $T \times \mathbb{G}_m$ on X defined by $(t, z).[u_1 + u_2] = t.[u_1 + zu_2]$, for $u_i \in V_i$. We have $X^{T \times \mathbb{G}_m} = X^T$.

thm:horo

Theorem 9.1. *Any $T \times \mathbb{G}_m$-stable Schubert variety in X is $T \times \mathbb{G}_m$-fixed point inclusive and $T \times \mathbb{G}_m$-equivariantly rigid.*

Before proving Theorem 9.1, we sketch elementary proofs of some basic facts about X, which are also consequences of general results about spherical varieties, see [Tim11, Per14, Pas09] and the references therein.

Given an element $[u_1+u_2] \in \mathbb{P}(V_1 \oplus V_2)$, we will always assume $u_i \in V_i$, and i will always mean an element from $\{1, 2\}$. We consider $\mathbb{P}(V_i)$ as a subvariety of $\mathbb{P}(V_1 \oplus V_2)$. Let $\pi_i : \mathbb{P}(V_1 \oplus V_2) \smallsetminus \mathbb{P}(V_{3-i}) \to \mathbb{P}(V_i)$ denote the projection from V_{3-i}, defined by $\pi_i([u_1 + u_2]) = [u_i]$. Set $X_0 = G.[v_1 + v_2] \subset \mathbb{P}(V_1 \oplus V_2)$, $X_i = G.[v_i] \subset \mathbb{P}(V_i)$, and $X_{12} = G.([v_1], [v_2]) \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Since v_i is a highest weight vector, the stabilizer $P_i = G_{[v_i]}$ is a parabolic subgroup containing B. It follows that $X_i \cong G/P_i$ and $X_{12} \cong G/(P_1 \cap P_2)$ are flag varieties. In particular, X_i is closed in $\mathbb{P}(V_i)$, and X_{12} is closed in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Notice also that $X_0 \cong G/H$, where $H \subset P_1 \cap P_2$ is the kernel of the character $\lambda_1 - \lambda_2 : P_1 \cap P_2 \to \mathbb{G}_m$. This shows that X_0 is a $\mathbb{G}_m$-bundle over $G/(P_1 \cap P_2)$, so X is a non-singular projective horospherical variety of rank 1 (but not necessarily of Picard rank 1, see Remark 9.5).

Let W be the Weyl group of G, and recall the notation from Section 6.

lemma:orbits

Lemma 9.2. *We have $X = X_0 \cup X_1 \cup X_2$. The B-orbit closures in X are*

$$\overline{Bw.[v_i]} = \bigcup_{w' \leq w} Bw'.[v_i] \quad \text{for } w \in W^{P_i} \text{ and } i \in \{1, 2\}, \text{ and}$$

$$\overline{Bw.[v_1 + v_2]} = \bigcup_{w' \leq w} \left(Bw'.[v_1 + v_2] \cup Bw'.[v_1] \cup Bw'.[v_2]\right) \quad \text{for } w \in W^{P_1 \cap P_2}.$$

Proof. Set $\mathbb{P}_0 = \mathbb{P}(V_1 \oplus V_2) \smallsetminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$. Since $\lambda_1 \neq \lambda_2$, it follows that $\overline{T.[v_1 + v_2]}$ is the line through $[v_1]$ and $[v_2]$ in $\mathbb{P}(V_1 \oplus V_2)$. This implies $X_0 = (\pi_1 \times \pi_2)^{-1}(X_{12})$, hence X_0 is closed in $\mathbb{P}_0$, and $X_0 = X \cap \mathbb{P}_0$. We also have $X_i \subset X \cap \mathbb{P}(V_i) \subset \pi_i^{-1}(X_i) \cap \mathbb{P}(V_i) = X_i$, which proves the first claim. To finish the proof, it suffices to show $w'.[v_i] \in \overline{Bw.[v_1+v_2]}$ if and only if $w' \leq w$ (when $w' \in W^{P_i}$). The implication 'if' holds because $w'.[v_i] \in \overline{Tw'.[v_1+v_2]}$, and 'only if' holds because $\pi_i(\overline{Bw.[v_1+v_2]} \smallsetminus X_{3-i}) \subset \overline{Bw.[v_i]}$. ☐

Define an alternative action of P_i on V_{3-i} by $p \bullet u = \lambda_i(p)^{-1} p.u$, and use this action to form the space

$$G \times^{P_i} V_{3-i} \;=\; \{[g,u] : g \in G, u \in V_{3-i}\} \;/\; \{[gp,u] = [g, p \bullet u] : p \in P_i\}\,.$$

Define a morphism of varieties $\phi_i : G \times^{P_i} V_{3-i} \to \mathbb{P}(V_1 \oplus V_2)$ by $\phi_i([g,u]) = g.[v_i + u]$. This is well defined since $p.(v_i + u) = \lambda_i(p)(v_i + p \bullet u)$ holds for $p \in P_i$ and $u \in V_{3-i}$. Set $E_i = (P_i \bullet v_{3-i}) \cup \{0\} \subset V_{3-i}$. Noting that E_i is the cone over $P_i.[v_{3-i}] \cong P_i/(P_1 \cap P_2)$, it follows that E_i is closed in V_{3-i}.

lemma:vb **Lemma 9.3.** *The restricted map $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is an isomorphism of varieties. In particular, $E_i \subset V_{3-i}$ is a linear subspace.*

Proof. Assume $\phi_i([g,u]) = \phi_i([g',u'])$, and set $p = g^{-1}g'$. We obtain $p \in P_i$ and $[v_i + u] = p.[v_i + u'] = [v_i + p \bullet u']$ in $\mathbb{P}(V_1 \oplus V_2)$, hence $[g,u] = [g, p \bullet u'] = [gp, u'] = [g', u']$ in $G \times^{P_i} V_{3-i}$. We deduce that $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is bijective, so the lemma follows from Zariski's main theorem, using that $X_0 \cup X_i$ is non-singular. ☐

Fix a strongly dominant cocharacter $\rho : \mathbb{G}_m \to T$. For $a \in \mathbb{Z}$, define $\rho_a : \mathbb{G}_m \to T \times \mathbb{G}_m$ by $\rho_a(z) = (\rho(z), z^a)$. The resulting action of $\mathbb{G}_m$ on X is given by $\rho_a(z).[u_1 + u_2] = \rho(z).[u_1 + z^a u_2]$.

lemma:horo_definite **Lemma 9.4.** *All T-fixed points in X are fully definite for the action of $T \times \mathbb{G}_m$.*

Proof. It follows from Lemma 9.3 that $[v_1]$ has a $T \times \mathbb{G}_m$-stable open neighborhood in X isomorphic to $B^-.[v_1] \times E_1$, where the action is given by $(t,z).(x,u) = (t.x, t \bullet zu)$. If a is sufficiently negative, then $\mathbb{G}_m$ acts through ρ_a on $T_{[v_1]}X = T_{[v_1]}X_1 \oplus E_1$ with strictly negative weights, hence $[v_1]$ is fully definite in X for the action of $T \times \mathbb{G}_m$. A symmetric argument shows that $[v_2]$ is fully definite. The result follows from this, since all T-fixed points in X are obtained from $[v_1]$ or $[v_2]$ by the action of the Weyl group W. ☐

Proof of Theorem 9.1. For a sufficiently negative, it follows from Lemma 6.1 that the Bialynicki-Birula cells of X defined by ρ_a are

$$X^+_{w.[v_1]} = Bw.[v_1] \quad \text{ and } \quad X^+_{w.[v_2]} = Bw.[v_1+v_2] \cup Bw.[v_2]\,.$$

These cells form a stratification of X by Lemma 9.2. It therefore follows from Corollary 5.4 that $\overline{Bw.[v_1]}$ and $\overline{Bw.[v_1+v_2]}$ are $T \times \mathbb{G}_m$-fixed point inclusive and $T \times \mathbb{G}_m$-equivariantly rigid for each $w \in W$. A symmetric argument applies to $\overline{Bw.[v_2]}$, which completes the proof. ☐

remark:pasfib **Remark 9.5.** The exact sequence of [Per14, Thm. 3.2.4] implies that $\mathrm{Pic}(X)$ is a free abelian group of rank equal to the rank of X (which is one) plus the number of B-stable prime divisors in X that do not contain a G-orbit. Any B-stable prime divisor meeting X_0 has the form $D = \overline{Bw_0 s_\beta.[v_1+v_2]}$, where β is a simple root, and Lemma 9.2 shows that D contains X_i if and only if β is a root of P_i. Let $P_{12} \subset G$ be the parabolic subgroup generated by P_1 and P_2. We obtain $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}(G/P_{12})$. Let $\pi : X \to G/P_{12}$ be the map defined by $\pi(g.[v_1+v_2]) = \pi(g.[v_i]) = g.P_{12}$. This is a G-equivariant morphism of varieties, as its restriction to $X_0 \cup X_i$ is the composition of $\pi_i : X_0 \cup X_i \to G/P_i$ with the projection $G/P_i \to G/P_{12}$.

The fibers of π are translates of $\pi^{-1}(1.P_{12}) = \overline{L.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where L is the Levi subgroup of P_{12} containing T. Moreover, $\pi^{-1}(1.P_{12})$ is a non-singular projective horospherical variety of Picard rank 1, so it is either a flag variety or one of the non-homogeneous spaces from Pasquier's classification [Pas09].

Question 9.6. Let X be any projective G-horospherical variety fibered over a flag variety G/P with non-singular horospherical fibers of Picard rank 1. Is it true that X is isomorphic to an orbit closure $\overline{G.[v_1 + v_2]} \subset \mathbb{P}(V)$, where V is a rational representation of G, and $v_1, v_2 \in V$ are highest weight vectors?

Example 9.7. Let X be the blow-up of $\mathbb{P}^2$ at a point p, let $\pi : X \to \mathbb{P}^1$ be the morphism defined by projection from p, and set $G = \mathrm{SL}(2, \mathbb{C})$. Then X is G-horospherical and fibered over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. This variety X is isomorphic to $\overline{G.[v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where v_1 is a highest weight vector in $V_1 = \mathbb{C}^2$, and v_2 is a highest weight vector in $V_2 = \mathrm{Sym}^2(\mathbb{C}^2)$.

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA
Email address: asbuch@math.rutgers.edu

Domaine Scientifique Victor Grignard, 239, Boulevard des Aiguillettes, Université de Lorraine, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France
Email address: pierre-emmanuel.chaput@univ-lorraine.fr

Centre de Mathématiques Laurent Schwartz (CMLS), CNRS, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France
Email address: nicolas.perrin.cmls@polytechnique.edu