FROBENIUS MORPHISMS MODULO p^2

ANDERS BUCH, JESPER F. THOMSEN, NIELS LAURITZEN, AND VIKRAM MEHTA

Abstract. Let X be a projective normal algebraic variety over a perfect field k of characteristic $p > 0$ with a flat lift to a scheme X' over the Witt vectors $W_2(k)$ of length two. Let $\tilde{\Omega}^i_{X/k}$ be the sheaf of Zariski i-forms on X. If the absolute Frobenius morphism $F : X \to X$ lifts to a morphism $F' : X' \to X'$, we prove that $\mathrm{H}^i(X, \tilde{\Omega}^j_{X/k} \otimes L) = 0$, where L is an ample line bundle on X and $i > 0$. When X is a toric variety, Frobenius lifts to $W_2(k)$ and we get a simple proof of the Bott-Steenbrink-Danilov vanishing theorem and the degeneration of the Danilov spectral sequence [2].

Morphismes de Frobenius modulo p^2

Résumé - Soient k un corps parfait de caractéristique $p > 0$ et X une k-variété projective normale admettant un relèvement plat X' sur l'anneau $W_2(k)$ des vecteurs de Witt de longueur 2. Notons $\tilde{\Omega}^i_{X/k}$ le faisceau des formes différentielles de Zariski de degré i. Si le morphisme de Frobenius $F : X \to X$ se relève sur X', nous prouvons que $\mathrm{H}^i(X, \tilde{\Omega}^j_{X/k} \otimes L) = 0$, pour L un faisceau inversible ample sur X et $i > 0$. Nous montrons que l'hypothèse est vérifiée si X est une variété torique. On obtient ainsi une démonstration simple du théorème d'annulation de Bott-Steenbrink-Danilov et de la dégénérescence de la suite spectrale de Danilov [2].

Version française abrégée - Soient k un corps parfait de caractéristique $p > 0$ et X une variété lisse sur k, de dimension n, admettant un relèvement plat X' sur l'anneau $W_2(k)$ des vecteurs de Witt de longueur 2. Si le morphisme de Frobenius $F : X \to X$ se relève en $F' : X' \to X'$, on obtient un morphisme de complexes ([3], Remarques 2.2(ii))

$$\sigma : \bigoplus_{i \geq 0} \Omega^i_{X/k}[-i] \to F_* \Omega^\bullet_{X/k}$$

induisant l'isomorphisme de Cartier C^{-1} sur la cohomologie. Utilisant la dualité parfaite $\Omega^i_{X/k} \otimes \Omega^{n-i}_{X/k} \to \Omega^n_{X/k}$ nous prouvons que σ est scindable.

Supposons maintenant que X est une variété projective normale admettant un relèvement plat sur $W_2(k)$. Supposons aussi que le morphisme de Frobenius de X se relève en $W_2(k)$. Notons $\tilde{\Omega}^i_{X/k} = j_* \Omega^i_{U/k}$ le faisceau des formes différentielles de Zariski de degré i, et soit j l'immersion du lieu lisse U dans X. On obtient un relèvement de Frobenius de U en $W_2(k)$. Le scindage de $\sigma : \oplus_{i \geq 0} \Omega^i_{U/k}[-i] \to F_* \Omega^\bullet_{U/k}$ montre d'une part que $\mathrm{H}^i(X, \tilde{\Omega}^j_{X/k} \otimes L) = 0$, pour L un faisceau inversible ample sur X et $i > 0$ et, d'autre part, que la suite spectrale d'hypercohomologie $E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}^p_{X/k}) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}^\bullet_{X/k})$ dégénère en E_1.

Si X est une variété torique nous montrons, de façon explicite, que le morphisme de Frobenius se relève sur $W_2(k)$. On obtient ainsi une démonstration simple du théorème d'annulation de Bott-Steenbrink-Danilov (ceci semble être la première démonstration complète publiée de ce théorème (voir [1], p. 294)), et aussi la dégénérescence de la suite spectrale de Danilov [2].

1. Preliminaries

Let k be a perfect field of characteristic $p > 0$ and X a smooth k-variety of dimension n. By Ω_X we denote the sheaf of k-differentials on X and $\Omega_X^j = \wedge^j \Omega_X$. The (absolute) Frobenius morphism $F : X \to X$ is the morphism on X, which is the identity on the level of points and given by $F^\#(f) = f^p : \mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$ on the level of functions. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we define the $\mathcal{O}_X$-module $F_*\mathcal{F}$, which is $\mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module multiplication on $F_*\mathcal{F}$ is changed according to the homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$.

1.1. The Cartier operator. The universal derivation $d : \mathcal{O}_X \to \Omega_X$ gives rise to a family of k-homomorphisms $d^j : \Omega_X^j \to \Omega_X^{j+1}$ making $\Omega_X^\bullet$ into a complex of k-modules which is called the de Rham complex of X. By applying F_* to the de Rham complex, we obtain a complex $F_*\Omega_X^\bullet$ of $\mathcal{O}_X$-modules. Let $B_X^i \subseteq Z_X^i \subseteq F_*\Omega_X^i$ denote the coboundaries and cocycles in degree i. The following theorem on the cohomology of $F_*\Omega_X^\bullet$ is due to Cartier.

Theorem 1. There is a uniquely determined graded $\mathcal{O}_X$-algebra isomorphism

$$C^{-1} : \Omega_X^\bullet \to \mathcal{H}^\bullet(F_*\Omega_X^\bullet)$$

which in degree 1 is given locally as

$$C^{-1}(da) = a^{p-1}da$$

Proof. [5], Theorem 7.2. □

With some abuse of notation, we let C denote the natural homomorphism $Z_X^i \to \Omega_X^i$, which after reduction modulo B_X^i gives the inverse isomorphism to C^{-1}.

1.2. Witt vectors. Let $W(k)$ be the ring of Witt vectors for k and put $W_n(k) = W(k)/p^n$. The ring $W_n(k)$ is flat over $\mathbb{Z}/p^n$, there is an isomorphism $W_n(k)/pW_n(k) \cong k$ and $W(k) = \varprojlim_n W_n(k)$. The ring homomorphism on $W_2(k)$ given by $(a, b) \mapsto (a^p, b^p)$ reduces to the Frobenius homomorphism modulo p.

2. Liftings of Frobenius to $W_2(k)$

Assume that there is a flat scheme $X^{(2)}$ over $\operatorname{Spec} W_2(k)$ such that $X \cong X^{(2)} \times_{W_2(k)} k$. We say that the Frobenius morphism F lifts to $W_2(k)$ if there exists a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$, which reduces to F modulo p.

Theorem 2. If the Frobenius morphism on X lifts to $W_2(k)$, then there is a split quasi-isomorphism

$$\bigoplus_{i \geq 0} \Omega_X^i[-i] \xrightarrow{\sigma} F_*\Omega_X^\bullet$$

inducing C^{-1} on cohomology.

Proof. The construction of σ is well known ([3], Remarques 2.2(ii)). We give a splitting $\eta_i : F_*\Omega^i_X \to \Omega^i_X$ of $\sigma_i : \Omega^i_X \to F_*\Omega^i_X$. Now $\sigma_0 : \mathcal{O}_X \to F_*\mathcal{O}_X$ is the Frobenius homomorphism and σ_i ($i > 0$) splits $C : Z^i_X \to \Omega^i_X$. The natural perfect pairing $\Omega^i_X \otimes \Omega^{n-i}_X \to \Omega^n_X$ gives an isomorphism between $\mathcal{H}om_X(\Omega^{n-i}_X, \Omega^n_X)$ and Ω^i_X. It is easy to check that the homomorphism η_i

$$F_*\Omega^i_X \to \mathcal{H}om_X(\Omega^{n-i}_X, \Omega^n_X) \cong \Omega^i_X$$

given by $\eta_i(\omega)(z) = C(\sigma_{n-i}(z) \wedge \omega)$, splits σ_i. □

2.1. Bott-Steenbrink-Danilov vanishing. Let X be a normal variety and let j denote the inclusion of the smooth locus $U \subseteq X$. If the Frobenius morphism lifts to $W_2(k)$ on X, then the Frobenius morphism on U also lifts to $W_2(k)$. Define the Zariski sheaf $\tilde{\Omega}^i_X$ of i-forms on X as $j_*\Omega^i_U$. Since Ω^i_U is locally free and $\operatorname{codim}(X - U) \geq 2$ it follows that $\tilde{\Omega}^i_X$ is a coherent sheaf on X.

Theorem 3. Let X be a projective normal variety such that F lifts to $W_2(k)$. Then

$$\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) = 0$$

for $s > 0$ and L an ample line bundle.

Proof. Let U be the smooth locus of X and let j denote the inclusion of U into X. On U we have by Theorem 2 a split sequence $0 \to \Omega^r_U \to F_*\Omega^r_U$ which pushes down to the split sequence (F commutes with j) $0 \to \tilde{\Omega}^r_X \to F_*\tilde{\Omega}^r_X$ Now tensoring with L and using the projection formula we get injections for $s > 0$: $\mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L) \hookrightarrow \mathrm{H}^s(X, \tilde{\Omega}^r_X \otimes L^p)$. Iterating these injections and using that the Zariski sheaves are coherent one gets the desired vanishing theorem by Serre's theorem. □

2.2. Degeneration of the Hodge to de Rham spectral sequence. Let X be a projective normal variety with smooth locus U. Associated with the complex $\tilde{\Omega}^\bullet_X$ there is a spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \tilde{\Omega}^p_X) \implies \mathrm{H}^{p+q}(X, \tilde{\Omega}^\bullet_X)$$

where $\mathrm{H}^\bullet(X, \tilde{\Omega}^\bullet_X)$ denotes the hypercohomology of the complex $\tilde{\Omega}^\bullet_X$. This is the Hodge to de Rham spectral sequence for Zariski sheaves.

Theorem 4. If the Frobenius morphism on X lifts to $W_2(k)$, then the spectral sequence degenerates at the E_1-term.

Proof. As complexes of sheaves of abelian groups $\tilde{\Omega}^\bullet$ and $F_*\tilde{\Omega}^\bullet$ are the same so their hypercohomology agree. Applying hypercohomology to the split injection (Theorem 2)

$$\sigma : \bigoplus_{0 \leq i} \tilde{\Omega}^i_{X/k}[-i] \to F_*\tilde{\Omega}^\bullet_X$$

we get

$$\begin{aligned}\sum_{p+q=n} \dim_k E_\infty^{pq} = \dim_k \mathrm{H}^n(X, \tilde{\Omega}^\bullet_X) &= \dim_k \mathrm{H}^n(X, F_*\tilde{\Omega}^\bullet_X) \geq \\ \sum_{p+q=n} \dim_k \mathrm{H}^q(X, \tilde{\Omega}^p_X) &= \sum_{p+q=n} \dim_k E_1^{pq}\end{aligned}$$

Since E_∞^{pq} is a subquotient of E_1^{pq}, it follows that $E_\infty^{pq} \cong E_1^{pq}$ so that the spectral sequence degenerates at E_1. □

3. Toric varieties

For specifics on the geometry toric varieties we refer to Fulton's book [4]. In this section we show by simple patching, that the Frobenius morphism lifts to $W_2(k)$ on a toric variety. The key issue is that affine toric varieties are given by k-algebras generated by monomials. Since toric varieties are normal, we can apply the results of §2 to get the Bott-Steenbrink-Danilov vanishing theorem [2] and the degeneration of the Danilov spectral sequence [2].

Let N be a lattice of rank n and M the dual lattice. Put $V = N \otimes \mathbb{R}$ and $V^* = M \otimes \mathbb{R}$. A cone σ in N is a subset of V of the form $\{r_1 v_1 + \cdots + r_s v_s | r_i \geq 0\}$, where $v_i \in N$. The dual cone $\sigma^\vee = \{u \in V^* | \forall v \in \sigma : \langle u, v\rangle \geq 0\}$ is a cone in M. A face of σ is $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$. A strongly convex cone is a cone not containing any lines. A fan Δ is a collection of strongly convex cones in N, such that if $\sigma \in \Delta$, then any face of σ is in Δ and if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of both σ and τ.

3.1. Glueing Frobenius on toric varieties. Define S_σ to be the semi group $\sigma^\vee \cap M$. Since $\sigma^\vee$ is a cone in M, S_σ is finitely generated. If k is any commutative ring the semigroup ring $k[S_\sigma]$ is a finitely generated commutative k-algebra, and $U_\sigma = \operatorname{Spec} k[S_\sigma]$ is an affine scheme of finite type over k. Let $\tau = \sigma \cap u^\perp$ be a face of σ, where $u \in S_\sigma$. Then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$, so that $k[S_\tau] = k[S_\sigma]_u$. In this way a fan Δ defines a toric variety $X(\Delta)$ by glueing affine varieties U_σ and U_τ $(\sigma, \tau \in \Delta)$ together using the common face $\sigma \cap \tau$. This construction makes sense when k is any commutative ring. In this setting the rings $k[S_\sigma]$ are free k-modules. In particular we get that $X(\Delta)$ admits a flat lift to $W_2(k)$.

Now let $e_1, \ldots, e_n$ be a basis of V and let σ be a strongly convex cone in Δ. Then $k[S_\sigma]$ is generated by monomials as a subring of $A = S(V^*)_f = k[T_1, \ldots, T_n]_f$, where $f = T_1 \ldots T_n$ and $T_i = e_i^*$. The natural lift of Frobenius to $W_2(k)$ on A given by $T_i \mapsto T_i^p$ induces lifts compatible with the glueing on $k[S_\sigma] \subseteq A$ for all $\sigma \in \Delta$. This gives a lift of Frobenius on $X(\Delta)$ to $W_2(k)$ (and in fact to $W(k)$).

In view of the results in §2 this proves the Bott-Steenbrink-Danilov vanishing theorem for toric varieties (this appears to be the first complete published proof of this theorem (see [1], p. 294)) and the degeneration of the Danilov spectral sequence (since we have these results modulo every prime number).

References

1. V. Batyrev and D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math J. **75** (1994), 293–338.
2. V. I. Danilov, *The geometry of toric varieties*, Russ. Math. Surveys **33:2** (1978), 97–154.
3. P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. math. **89** (1987), 247–270.
4. W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.
5. N. Katz, *Nilpotent connections and the monodromy theorem*, Publ. Math. IHES **39** (1970), 175–232.

(Anders Buch, Jesper F. Thomsen, Niels Lauritzen) Matematisk Institut, Aarhus Universitet, Ny Munkegade, DK-8000 Århus C, Denmark

E-mail address, Niels Lauritzen: niels@mi.aau.dk

(Vikram Mehta) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay, India

E-mail address, Vikram Mehta: vikram@tifrvax.tifr.res.in