COMBINATORIAL K-THEORY

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1. Introduction

Let X be an algebraic variety. The *Grothendieck ring* $K(X)$ of X is defined as the free abelian group generated by isomorphism classes $[F]$ of algebraic vector bundles on X, modulo the relations $[F] = [F'] + [F'']$ whenever there exists a short exact sequence $0 \to F' \to F \to F'' \to 0$. Multiplication is defined by the tensor product, $[F_1] \cdot [F_2] = [F_1 \otimes F_2]$. Any morphism $f : X \to X'$ of varieties defines a pullback ring homomorphism $f^* : K(X') \to K(X)$ given by $f^*([E]) = [f^*E]$.

In the following, we will assume that X is non-singular and quasi-projective. This implies that any coherent $\mathcal{O}_X$-module $\mathcal{G}$ has a resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathcal{G} \to 0$$

by locally free sheaves on X. One can therefore define the class of $\mathcal{G}$ in the Grothendieck ring by

$$[\mathcal{G}] = \sum_{i=0}^{n} (-1)^i [F_i] \in K(X)\,.$$

In particular, if $Y \subset X$ is a closed subvariety (or subscheme), we define the *Grothendieck class* of Y to be the class $[\mathcal{O}_Y]$ of its structure sheaf.

Suppose Z is another closed subvariety of X. In intersection theory one studies the geometry of the intersection $Y \cap Z$. This can be done using (Chow) cohomology by examining the product $[Y]\cdot[Z] \in H^*(X)$, or with K-theory by studying the product of Grothendieck classes $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] \in K(X)$. When Y and Z meet sufficiently transversally, these products give the respective classes of the intersection, in the sense that $[Y \cap Z] = [Y] \cdot [Z] \in H^*(X)$ and $[\mathcal{O}_{Y\cap Z}] = [\mathcal{O}_Y] \cdot [\mathcal{O}_Z] \in K(X)$. For example, one can show that these identities hold when Y and Z are Cohen-Macaulay, and all components of $Y \cap Z$ have dimension $\dim(Y) + \dim(Z) - \dim(X)$.

The Grothendieck ring has a *topological filtration* $K(X) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ by ideals. The ideal $\mathcal{F}_i$ is generated by all classes $[\mathcal{O}_Y]$ for which $Y \subset X$ has codimension at least i. There is a group homomorphism $H^*(X) \to \mathrm{gr}(K(X)) = \bigoplus_{i\geq 0} \mathcal{F}_i/\mathcal{F}_{i+1}$ from the Chow ring to the associated graded ring of $K(X)$ given by $[Y] \mapsto [\mathcal{O}_Y]$, and this map becomes an isomorphism of rings after tensoring with $\mathbb{Q}$ (see [11, Ex. 15.2.16]). The geometric meaning of this is that intersection theory in the cohomology ring only captures the lowest graded piece of information available in the Grothendieck ring.

Example 1. Let $C \subset \mathbb{P}^2$ be a curve of degree 3. Since C can be degenerated into a union of three lines, its cohomology class is $[C] = 3[\text{line}]$. The Grothendieck class is finer and can "see" that these lines overlap in three points. Thus $[\mathcal{O}_C] =$

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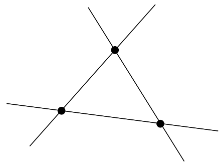
$3[\mathcal{O}_{\text{line}}] - 3[\mathcal{O}_{\text{point}}]$. This example generalizes easily to a hypersurface of any degree in $\mathbb{P}^n$.



The goal of these notes is to explain the combinatorics that arises in the study of K-theoretic intersection theory of some concrete varieties and degeneracy loci. The material is mostly taken from the papers [5, 4], as well as from joint work [8] with A. Kresch, H. Tamvakis, and A. Yong.

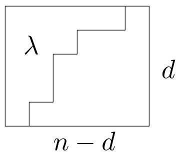
Our notes are based on a lecture series presented at the *Mini-School on Schubert Varieties* at the Banach Center in the Spring of 2003. We thank Piotr Pragacz, Andrzej Weber, and the Banach Center for their kind invitation and hospitality. We also thank R. Vakil for private communication concerning a symmetry property of K-theoretic Schubert structure constants.

2. K-theory of Grassmannians

Let $X = \mathrm{Gr}(d,n)$ be the Grassmann variety of d-dimensional subspaces of $\mathbb{C}^n$. For each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$ with $\lambda_1 \leq n-d$, there is a Schubert variety

$$X_\lambda = \{V \in X \mid \dim(V \cap \mathbb{C}^{n-d+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq d\}\,.$$

Here $\mathbb{C}^k \subset \mathbb{C}^n$ denotes the subspace of vectors where the last $n-k$ coordinates are zero. The codimension of X_λ in X is equal to the weight $|\lambda| = \sum_{i=1}^d \lambda_i$ of λ. We will identify the partition λ with its *Young diagram* of boxes, which has λ_1 boxes in the top row, λ_2 boxes in the second row, etc. For example, we write $(3,1,0) =$. If we let $R = (n-d)^d$ denote a rectangular partition with d rows and $n-d$ columns, then the Schubert varieties in X correspond to the Young diagrams λ contained in R.

$R =$ 

We will write $\mathcal{O}_\lambda = [\mathcal{O}_{X_\lambda}]$ for the Grothendieck class of the Schubert variety X_λ. In the notation of Brion's lectures we have $\mathcal{O}_\lambda = \mathcal{O}^I$ for $I = \{\lambda_d + 1, \dots, \lambda_1 + d\}$ (see [2, Ex. 3.4.3]). These classes form a basis for the Grothendieck ring of X:

$$K(X) = \bigoplus_{\lambda \subset R} \mathbb{Z} \cdot \mathcal{O}_\lambda\,.$$

As a first example of the multiplicative structure in this ring, we compute the Grothendieck class of the intersection of two general translates of the divisor $X_{(1)}$.

Example 2. Set $Y = X_{(1)} = X_{\square}$ and $Z = \{V \in X \mid V \cap M \neq 0\}$, where $M = \mathbb{C}^{n-d-1} \oplus 0 \oplus \mathbb{C} \oplus 0^{d-1} \subset \mathbb{C}^n$ is the sum of $\mathbb{C}^{n-d-1}$ and the line spanned by the $(n-d+1)$st standard basis vector in $\mathbb{C}^n$. Since Z is a translate of Y by the action of $\mathrm{GL}_n(\mathbb{C})$ on X, it follows that $[\mathcal{O}_Y] = [\mathcal{O}_Z] = \mathcal{O}_{\square}$. By the definitions we have $Y \cap Z = X_{\ydiagram{2}} \cup X_{\ydiagram{1,1}}$. Since we have a reduced intersection $X_{\ydiagram{2}} \cap X_{\ydiagram{1,1}} = X_{\ydiagram{2,1}}$, we obtain a short exact sequence

$$0 \to \mathcal{O}_{Y\cap Z} \to \mathcal{O}_{X_{\ydiagram{2}}} \oplus \mathcal{O}_{X_{\ydiagram{1,1}}} \to \mathcal{O}_{X_{\ydiagram{2,1}}} \to 0\,,$$

where the first map is given by $f \mapsto (f,f)$ and the second by $(f,g) \mapsto f-g$. Since Y and Z are Cohen-Macaulay, it follows that $\mathcal{O}_{\ydiagram{1}} \cdot \mathcal{O}_{\ydiagram{1}} = [\mathcal{O}_Y]\cdot[\mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}] = \mathcal{O}_{\ydiagram{2}} + \mathcal{O}_{\ydiagram{1,1}} - \mathcal{O}_{\ydiagram{2,1}}$ in $K(X)$. We note that the geometric construction in this example is a special case of the degeneration techniques used by Vakil to compute cohomology products on Grassmannians [27].

More generally, given two partitions $\lambda, \mu \subset R$, we can write

$$\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu c^\nu_{\lambda\mu} \mathcal{O}_\nu$$

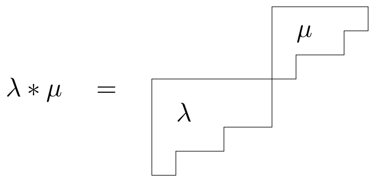
for unique structure constants $c^\nu_{\lambda\mu} \in \mathbb{Z}$. These structure constants depend only on the partitions λ, μ, ν, and not on our choice of Grassmann variety $X = \mathrm{Gr}(d, \mathbb{C}^n)$. Our first goal will be to describe these constants, which determine the K-theoretic intersection theory on X. Using the isomorphism $H^*(X) \cong \mathrm{gr}(K(X))$ it follows that the integer $c^\nu_{\lambda\mu}$ is non-zero only when $|\nu| \geq |\lambda| + |\mu|$. When we have equality $|\nu| = |\lambda| + |\mu|$, the coefficient $c^\nu_{\lambda\mu}$ is a structure constant in the cohomology ring. These cohomology constants are called *Littlewood-Richardson coefficients* and play an important role in numerous fields, including representation theory, geometry, and combinatorics, see e.g. [13]. Lenart [22] has proved that the K-theoretic Pieri coefficients $c^\nu_{(p),\mu}$ can be expressed as plus or minus a binomial coefficient.

Set-valued tableaux. We will formulate a general rule for the structure constants $c^\nu_{\lambda\mu}$, and for this purpose we introduce some notation. Define a *set-valued tableau* to be a labeling of the boxes of a skew diagram with finite non-empty sets of positive integers, so that the numbers in the diagram increase weakly along rows and increase strictly down columns. More precisely, the maximum number of any box must be less than or equal to the minimum of the box to the right of it, and strictly smaller than the minimum of the box below it. For example

$$T = \begin{ytableau} \none & \none & 1 & 2\,3 \\ \none & 1\,2 & 2\,3\,4 \\ 2 & 3\,5 & 7 \end{ytableau}$$

is a set-valued tableau whose shape is the skew diagram $(4,3,3)/(2,1)$. Define the *content* of a set-valued tableau to be the sequence $(c_1, c_2, \dots)$, where c_i is the number of boxes containing the integer i. We also define the *word* of a set-valued tableau to be the sequence of integers in its boxes when read from left to right then bottom to top; the integers of a single box are arranged in increasing order. The above tableau T has content $(2,4,3,1,1,0,1)$ and word $(2,3,5,7,1,2,2,3,4,1,2,3)$. A sequence of integers is called a *reverse lattice word* if every integer $i \geq 2$ in the sequence is followed by more $i-1$'s than i's.

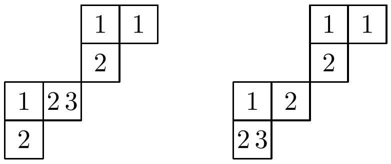
Given two partitions λ and μ, we let $\lambda * \mu$ denote the skew shape obtained by attaching the Young diagrams for λ and μ corner to corner as shown.



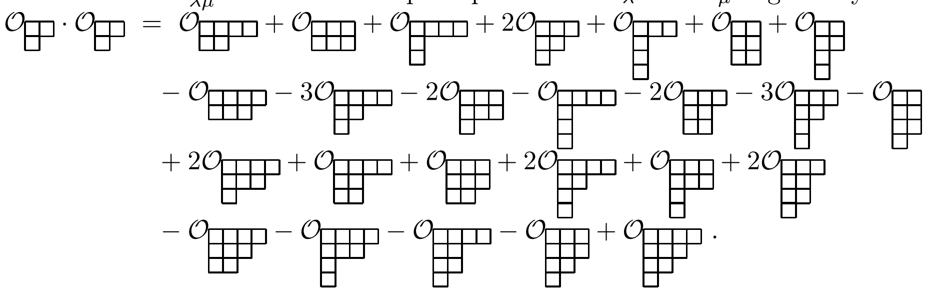
The following theorem from [5] is a K-theoretic generalization of the classical Littlewood-Richardson rule [23].

Theorem 1. *The structure constant $c^{\nu}_{\lambda\mu}$ is equal to $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of set-valued tableaux of shape $\lambda * \mu$, such that the word is a reverse lattice word with content ν.*

Example 3. Let $\lambda = \mu = (2,1) =$ and $\nu = (3,3,1) =$. Then there are two set-valued tableaux of shape $\lambda * \mu$ such that the word is a reverse lattice word with content ν.



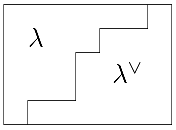
It follows that $c^{\nu}_{\lambda\mu} = -2$. The complete product of $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ is given by



It follows from the theorem that K-theoretic structure constants on Grassmannians have signs that alternate with codimension, that is $(-1)^{|\nu|-|\lambda|-|\mu|} c^{\nu}_{\lambda\mu} \geq 0$. This alternation in sign has in fact been established by Brion for all homogeneous spaces G/P, where G is a simply connected semisimple algebraic group [3]. This is one of the topics discussed in Brion's lectures [2]. One can also derive from Theorem 1 that, if $\mathcal{O}_\nu$ occurs with non-zero coefficient in $\mathcal{O}_\lambda \cdot \mathcal{O}_\mu$, then we have $X_\lambda \cap X_\mu \supset X_\nu \supset \bigcup_\rho X_\rho$, where the last union is over all partitions ρ such that $[X_\rho]$ occurs in the cohomology product $[X_\lambda] \cdot [X_\mu] \in H^*(X)$. In Example 3 above, this says that all shapes occurring in the square of $\mathcal{O}_{(2,1)}$ are contained in the union of the shapes in the first row of terms.

Another consequence of the theorem is the following symmetry property of the K-theoretic Schubert structure constants $c^{\nu}_{\lambda\mu}$. For a partition $\lambda \subset R$ we let $\lambda^\vee =$

$(n-d-\lambda_d, \dots, n-d-\lambda_1)$ be the dual partition.



Corollary 1. *For any three partitions $\lambda, \mu, \nu \subset R$ we have $c^\nu_{\lambda\mu} = c^{\mu^\vee}_{\lambda\nu^\vee}$.*

Notice that when $|\nu| = |\lambda| + |\mu|$, this is immediate from the identity $c^\nu_{\lambda\mu} = \int_X [X_\lambda] \cdot [X_\mu] \cdot [X_{\nu^\vee}]$ for cohomological structure constants.

The proof given below was produced in response to a question of R. Vakil, who had observed that the corollary follows from a conjectured K-theoretic puzzle rule of A. Knutson and T. Tao. We note that this conjecture has now been proved by Knutson and Vakil by translating K-theoretic puzzles to set-valued tableaux [17].

Proof. Let $\rho : X \to \{\text{point}\}$ be a map to a point. The corresponding pushforward map $\rho_* : K(X) \to K(\text{point}) = \mathbb{Z}$ is the linear map given by $\rho_*(\mathcal{O}_\lambda) = 1$ for all $\lambda \subset R$. Define the bilinear form

$$K(X) \otimes K(X) \to \mathbb{Z} \quad ; \quad \alpha \otimes \beta \mapsto \rho_*(\alpha \cdot \beta)\,.$$

We claim that this is a perfect pairing, and that the dual basis element of $\mathcal{O}_\lambda$ equals $t\mathcal{O}_{\lambda^\vee}$, where $t = 1 - \mathcal{O}_{(1)}$. (This class t represents the top exterior power of the tautological subbundle on X.) The corollary is immediate from this claim since $c^\nu_{\lambda\mu} = \rho_*(\mathcal{O}_\lambda \cdot \mathcal{O}_\mu \cdot t\mathcal{O}_{\nu^\vee}) = c^{\mu^\vee}_{\lambda\nu^\vee}$.

It follows from Theorem 1 that $t\mathcal{O}_\lambda = \mathcal{O}_\lambda - \mathcal{O}_{(1)} \cdot \mathcal{O}_\lambda = \sum_\nu (-1)^{|\nu/\lambda|} \mathcal{O}_\nu$, where the sum is over all partitions ν such that $\lambda \subset \nu \subset R$ and ν/λ has at most one box in any row or column. This implies that $\rho_*(t\mathcal{O}_\lambda) = \delta_{\lambda,R}$, and we obtain

$$\rho_*(t\mathcal{O}_\lambda \cdot \mathcal{O}_\mu) = \sum_\nu c^\nu_{\lambda\mu}\, \rho_*(t\mathcal{O}_\nu) = c^R_{\lambda\mu}\,.$$

Finally, it is an easy exercise to check from Theorem 1 that $c^R_{\lambda\mu} = \delta_{\lambda,\mu^\vee}$. $\square$

A geometric proof that the bases $\{\mathcal{O}_\lambda\}$ and $\{t\mathcal{O}_\lambda\}$ of $K(X)$ are dual to each other can be found in Brion's lectures [2] (see Thm. 3.4.1 and Ex. 3.4.3). In the notation used there, we have $t = [L^{-1}]$ where L is the ample generator of the Picard group of X.

3. The bialgebra of stable Grothendieck polynomials

For a set-valued tableau T with content $(c_1, c_2, \dots, c_r)$, we set $x^T = x_1^{c_1} x_2^{c_2} \cdots x_r^{c_r}$. For any partition λ we define a *stable Grothendieck polynomial* for this partition by

$$G_\lambda = \sum_{\text{sh}(T)=\lambda} (-1)^{|T|-|\lambda|}\, x^T \in \mathbb{Z}[\![x_1, x_2, \dots]\!]\,.$$

The sum is over all set-valued tableaux T of shape λ. This is a special case of the stable Grothendieck polynomials studied by Fomin and Kirillov [9] (see section 6). One can check from the definition that the power series G_λ is symmetric in the variables x_i.

The term of lowest total degree in G_λ is the Schur function s_λ. This power series can be expressed as the determinant $s_\lambda = \det(e_{\lambda'_i+j-i})_{1\le i,j\le \ell(\lambda')}$ where λ' is the conjugate partition of λ, defined so that λ'_i is the number of boxes in the ith column of λ, and e_i is the elementary symmetric function of degree i, i.e. the sum of all monomials which are products of i distinct variables. Given a vector bundle E on a variety X, one defines a characteristic class $s_\lambda(E) = \det(c_{\lambda'_i+j-i}(E)) \in H^*(X)$ by substituting the i'th Chern class $c_i(E)$ for e_i. When $X = \mathrm{Gr}(d, \mathbb{C}^n)$ is a Grassmann variety and $S^\vee$ is the dual of the tautological subbundle $S \subset \mathbb{C}^n \times X$, then the classical Giambelli formula says that $s_\lambda(S^\vee)$ is equal to the cohomology class of the Schubert variety X_λ. As a consequence, the cohomology ring of a Grassmann variety is a quotient of the ring of symmetric functions $\Lambda = \bigoplus_\lambda \mathbb{Z} s_\lambda$.

The stable Grothendieck polynomials G_λ provide a K-theoretic analogue of this construction. For a vector bundle which can be written as a direct sum of line bundles, $E = L_1 \oplus \cdots \oplus L_e$, we define

$$G_\lambda(E) = G_\lambda(1 - L_1^{-1}, \dots, 1 - L_e^{-1}, 0, 0, \dots) \in K(X)\,.$$

Since G_λ is symmetric, this can be written as a polynomial in the exterior powers of the dual bundle $E^\vee$, so the definition also makes sense when E is not a direct sum of line bundles. With this notation, it follows from [21] or [14, Thm. 3] that $\mathcal{O}_\lambda = G_\lambda(S^\vee)$ in the Grothendieck ring of $\mathrm{Gr}(d, \mathbb{C}^n)$. We explain more about this in section 6.

Example 4. We compute $G_{(2,1)}(1-a_1, 1-a_2, 1-a_3) = 1 - a_1a_2 - a_1a_3 - a_2a_3 + a_1^2a_2a_3 + a_1a_2^2a_3 + a_1a_2a_3^2 - a_1^2a_2^2a_3^2 = 1 - (a_1a_2 + a_1a_3 + a_2a_3) + (a_1 + a_2 + a_3)(a_1a_2a_3) - (a_1a_2a_3)^2$. It follows that for a vector bundle E of rank 3, we have $G_{(2,1)}(E) = 1 - [\bigwedge^2 E^\vee] + [E^\vee \otimes \bigwedge^3 E^\vee] - [\bigwedge^3 E^\vee]^2 \in K(X)$.

Since the lowest term of G_λ is the Schur function s_λ, it follows that the power series G_λ are linearly independent. We let Γ denote their linear span.

$$\Gamma = \bigoplus_\lambda \mathbb{Z}\, G_\lambda \subset \mathbb{Z}[\![x_1, x_2, \dots]\!]$$

The following result generalizes Theorem 1.

Theorem 2. *For any partitions λ and μ we have*

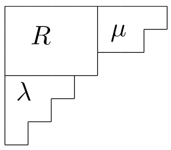
$$G_\lambda \cdot G_\mu = \sum c^\nu_{\lambda\mu}\, G_\nu$$

where the sum is over all partitions ν, and the constants $c^\nu_{\lambda\mu}$ are given by Theorem 1.

It follows from this theorem that each product $G_\lambda \cdot G_\mu$ is a finite linear combination of stable Grothendieck polynomials G_ν. In particular, Γ is a subring of the power series ring $\mathbb{Z}[\![x_1, x_2, \dots]\!]$. This ring provides a combinatorial model for the Grothendieck ring of a Grassmann variety. The topological filtration of the Grothendieck ring $K(X)$ corresponds to ideals $I_j = \bigoplus_{|\lambda| \geq j} \mathbb{Z} G_\lambda$ of Γ, and the ring of symmetric functions is recovered as the associated graded ring $\Lambda = \bigoplus_{j \geq 0} I_j / I_{j+1}$ of Γ.

For partitions λ, μ, and ν, we set $d^\nu_{\lambda\mu} = c^\rho_{\nu R}$, where R is a rectangular partition which is wider than λ and taller than μ, and $\rho = (R+\mu, \lambda)$ is the partition obtained by attaching λ and μ to the sides of R.

$$\rho \quad = \quad (R + \mu, \lambda) \quad =$$



It is not hard to see that $d^{\nu}_{\lambda\mu}$ is independent of the choice of the rectangle R. Given a finite set of variables $x_1, \dots, x_n$, we write $G_\lambda(x_1, \dots, x_n) = G_\lambda(x_1, \dots, x_n, 0, 0, \dots)$. The coefficients $d^{\nu}_{\lambda\mu}$ have the following interpretation.

Theorem 3. *For any partition ν and integers $0 < m < n$ we have*

$$G_\nu(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu} G_\lambda(x_1, \dots, x_m) \cdot G_\mu(x_{m+1}, \dots, x_n)\,.$$

Theorem 3 implies that we can define a coassociative and cocommutative coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ on Γ by setting

$$\Delta(G_\nu) = \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu}\, G_\lambda \otimes G_\mu\,.$$

Example 5. Let $\nu = (2,1) = \ydiagram{2,1}$. Using Theorem 2 we compute

$$\begin{aligned} G_{\ydiagram{2,1}} \cdot G_{\ydiagram{2,2}} = {} & G_{\ydiagram{4,3}} + G_{\ydiagram{4,2,1}} + G_{\ydiagram{3,3,1}} + G_{\ydiagram{3,2,2}} + G_{\ydiagram{3,2,1,1}} + G_{\ydiagram{2,2,2,1}} \\ & - 2G_{\ydiagram{4,3,1}} - G_{\ydiagram{4,2,2}} - G_{\ydiagram{3,3,2}} - G_{\ydiagram{4,2,1,1}} - G_{\ydiagram{3,3,1,1}} - 2G_{\ydiagram{3,2,2,1}} \\ & + G_{\ydiagram{4,3,2}} + G_{\ydiagram{4,3,1,1}} + G_{\ydiagram{4,2,2,1}} + G_{\ydiagram{3,3,2,1}} - G_{\ydiagram{4,3,2,1}}\,. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta(G_{\ydiagram{2,1}}) = {} & 1 \otimes G_{\ydiagram{2,1}} + G_{\ydiagram{1}} \otimes G_{\ydiagram{2}} + G_{\ydiagram{1}} \otimes G_{\ydiagram{1,1}} + G_{\ydiagram{2}} \otimes G_{\ydiagram{1}} + G_{\ydiagram{1,1}} \otimes G_{\ydiagram{1}} + G_{\ydiagram{2,1}} \otimes 1 \\ & - 2G_{\ydiagram{1}} \otimes G_{\ydiagram{2,1}} - G_{\ydiagram{2}} \otimes G_{\ydiagram{2}} - G_{\ydiagram{2}} \otimes G_{\ydiagram{1,1}} - G_{\ydiagram{1,1}} \otimes G_{\ydiagram{2}} - G_{\ydiagram{1,1}} \otimes G_{\ydiagram{1,1}} - 2G_{\ydiagram{2,1}} \otimes G_{\ydiagram{1}} \\ & + G_{\ydiagram{2}} \otimes G_{\ydiagram{2,1}} + G_{\ydiagram{1,1}} \otimes G_{\ydiagram{2,1}} + G_{\ydiagram{2,1}} \otimes G_{\ydiagram{2}} + G_{\ydiagram{2,1}} \otimes G_{\ydiagram{1,1}} - G_{\ydiagram{2,1}} \otimes G_{\ydiagram{2,1}}\,. \end{aligned}$$

The coproduct Δ corresponds to the pullback along the embedding

$$\iota : \mathrm{Gr}(d_1, n_1) \times \mathrm{Gr}(d_2, n_2) \to \mathrm{Gr}(d_1 + d_2, n_1 + n_2)$$

which maps a pair $(V_1 \subset \mathbb{C}^{n_1}, V_2 \subset \mathbb{C}^{n_2})$ to $V_1 \oplus V_2 \subset \mathbb{C}^{n_1+n_2}$. In fact, if we let S_1, S_2, and S denote the tautological subbundles on $\mathrm{Gr}(d_1, n_1)$, $\mathrm{Gr}(d_2, n_2)$, and $\mathrm{Gr}(d_1 + d_2, n_1 + n_2)$, respectively, then $\iota^*(G_\nu(S)) = G_\nu(\iota^* S) = G_\nu(S_1 \oplus S_2) = \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu} G_\lambda(S_1) \cdot G_\mu(S_2)$. Here we have identified S_1 and S_2 with their pullbacks to the product $\mathrm{Gr}(d_1, n_1) \times \mathrm{Gr}(d_2, n_2)$.

The coproduct Δ makes Γ into a commutative and cocommutative bialgebra. The linear map $\Gamma \to \mathbb{Z}$ which sends $1 = G_{(0)}$ to one and G_λ to zero for $|\lambda| > 0$ furthermore gives a counit. The bialgebra Γ also has an involution $\Gamma \to \Gamma$ defined by $G_\lambda \mapsto G_{\lambda'}$. This involution corresponds to the duality isomorphism $\mathrm{Gr}(d, n) \cong \mathrm{Gr}(n - d, n)$ of Grassmannians. Equivalently, the structure constants of Γ satisfy the identities $c^{\nu}_{\lambda\mu} = c^{\nu'}_{\lambda'\mu'}$ and $d^{\nu}_{\lambda\mu} = d^{\nu'}_{\lambda'\mu'}$.

4. Geometric specializations of stable Grothendieck polynomials

Given two vector bundles E and F over the variety X and a partition ν, we define

$$G_\nu(F - E) \;=\; \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu}\, G_\lambda(F) \cdot G_{\mu'}(E^\vee) \;\in\; K(X)$$

where $E^\vee$ is the dual bundle of E and μ' is the conjugate partition of μ. It is proved in [5] that $G_\lambda(F-E)$ is a specialization of the double stable Grothendieck polynomials of Fomin and Kirillov [9]. It therefore follows from a super symmetry property of the latter polynomials that, for any vector bundle H we have

$$G_\lambda(F \oplus H - E \oplus H) = G_\lambda(F-E)\,. \tag{1}$$

In other words, G_λ gives a well defined linear map $G_\lambda : K(X) \to K(X)$. Since the coproduct Δ is coassociative, it follows from this that

$$G_\nu(F-E) = \sum_{\lambda,\mu} d_{\lambda\mu}^{\nu}\, G_\lambda(F-H) \cdot G_\mu(H-E)\,. \tag{2}$$

Another useful identity is $G_\lambda(F-E) = G_{\lambda'}(E^\vee - F^\vee)$, which was first proved by Fomin (see [5, Lemma 3.4].)

If the bundles E and F have ranks e and f, and if λ and μ are partitions such that $\ell(\lambda) \leq f$ and $\mu_1 \leq e$, we also have the factorization formula

$$G_{(e^f)+\lambda,\mu}(F-E) = G_\lambda(F)\, G_{(e^f)}(F-E)\, G_\mu(-E) \tag{3}$$

where $(e^f)+\lambda,\mu$ denotes the partition $(e+\lambda_1,\dots,e+\lambda_f,\mu_1,\mu_2,\dots)$.

Given a proper morphism $\pi : X \to X'$ of varieties, there is a corresponding pushforward homomorphism $\pi_* : K(X) \to K(X')$, which is defined by $\pi_*([\mathcal{G}]) = \sum_{j\geq 0}(-1)^j [R^j\pi_*(\mathcal{G})]$ for any coherent sheaf $\mathcal{G}$ on X. The factorization formula (3) is a special case of the following Gysin formula from [4]. It generalizes Pragacz's cohomological Gysin formula from [25] and [15, (4.4)] to K-theory.

Theorem 4. *Let E and F be vector bundles of ranks e and f over X, and let $\pi : \mathrm{Gr}(d,F) \to X$ denote the Grassmann bundle of subbundles of F of rank d, with universal exact sequence $0 \to S \to \pi^*F \to Q \to 0$. Set $q = f-d$. Let λ and μ be partitions such that $\ell(\lambda) \leq q$, and $\lambda_q \geq \max(e, \mu_1 + d)$. Then we have in $K(X)$ that*

$$\pi_*(G_\lambda(Q - \pi^*E) \cdot G_\mu(S - \pi^*E)) = G_{(\lambda_1 - d,\dots,\lambda_q - d,\mu_1,\mu_2,\dots)}(F-E)\,.$$

Like Pragacz's cohomological Gysin formula, this result remains true with slightly weaker conditions. Namely, one may drop the requirement that $\lambda_q - d \geq \mu_1$. This means that the sequence $I = (\lambda_1 - d,\dots,\lambda_q - d,\mu_1,\mu_2,\dots)$ is not necessarily a partition; the definition of G_I when I is an arbitrary sequence of integers is given in [4, §3], but will not be discussed here. It is curious to note that the same formula is also true in cohomology when λ and μ are arbitrary partitions such that $\ell(\lambda) \leq q$ and $\ell(\mu) \leq d$ [25], [15, (4.3)], but this case of the formula is false in K-theory. For example, if $F = \mathcal{O}_X^{\oplus 2}$ is a trivial bundle of rank 2 and $d = q = 1$, then one can check that $\pi_*(G_2(Q-\pi^*E)G_1(S-\pi^*E)) = G_{(1,1)}(-E) + G_{(3)}(-E) - G_{(3,1)}(-E)$, and this agrees with $G_{(1,1)}(F-E)$ only when E has rank at most two.

5. Degeneracy loci

Let $\phi : E \to F$ be a map of vector bundles of ranks e and f over X. Given an integer $t \leq \min(e,f)$, there is a degeneracy locus

$$\Omega_t = \Omega_t(\phi) = \{x \in X \mid \mathrm{rank}(\phi_x : E(x) \to F(x)) \leq t\}\,.$$

This locus has a natural structure of subscheme of X, given as the zero-section of the map $\wedge^{t+1}\phi : \wedge^{t+1}E \to \wedge^{t+1}F$. The expected (and maximal possible) codimension of Ω_t is the number $(e-t)(f-t)$. When this codimension is attained, the classical Thom-Porteous formula expresses the cohomology class of Ω_t as a Schur determinant (see [11, §14.4]). This formula generalizes to K-theory as follows.

Theorem 5. *If Ω_t has codimension $(e-t)(f-t)$ in X then its Grothendieck class is given by*
$$[\mathcal{O}_{\Omega_t}] = G_{(e-t)^{f-t}}(F - E) \in K(X)\,.$$

The Thom-Porteous degeneracy locus Ω_t can be generalized as follows. Let $E_0 \to E_1 \to \cdots \to E_n$ be a sequence of vector bundles and bundle maps over X. A set of *rank conditions* for this sequence is a collection $r = \{r_{ij}\}$ of non-negative integers, for $0 \leq i < j \leq n$. This data defines the *quiver variety*
$$\Omega_r = \Omega_r(E_\bullet) = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij} \ \forall i < j\}\,.$$
We will demand that the rank conditions can occur, i.e. there exists a sequence of vector spaces and linear maps $V_0 \to V_1 \to \cdots \to V_n$ such that $\dim(V_i) = \operatorname{rank}(E_i)$ and $\operatorname{rank}(V_i \to V_j) = r_{ij}$ for all $i < j$. If we set $r_{ii} = \operatorname{rank}(E_i)$ for each i, then this condition is equivalent to demanding that $r_{ij} \leq \min\{r_{i,j-1}, r_{i+1,j}\}$ for all $0 \leq i < j \leq n$ and $r_{ij} + r_{i-1,j+1} \geq r_{i-1,j} + r_{i,j+1}$ for all $0 < i \leq j < n$ [1]. In this case, the expected codimension of the quiver variety Ω_r is equal to
$$d(r) = \sum_{i<j} (r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})\,.$$
In work with Fulton [7], we proved a formula for the cohomology class of a quiver variety Ω_r. The following theorem from [4] generalizes this formula to K-theory.

Theorem 6. *Let $E_0 \to E_1 \to \cdots \to E_n$ be a sequence of vector bundle maps such that the quiver variety $\Omega_r(E_\bullet)$ has codimension $d(r)$ in X. Then the Grothendieck class of this quiver variety is given by*
$$[\mathcal{O}_{\Omega_r}] = \sum c_\mu(r)\, G_{\mu_1}(E_1 - E_0) \cdot G_{\mu_2}(E_2 - E_1) \cdots G_{\mu_n}(E_n - E_{n-1}) \in K(X)$$
where the sum is over sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions μ_i, and the coefficients $c_\mu(r)$ are integers depending on the rank conditions.

The coefficients $c_\mu(r)$ of this formula are called *quiver coefficients*. They are uniquely determined by the fact that the theorem is true for all varieties X, as well as the requirement that $c_\mu(r)$ is non-zero only if $\ell(\mu_i) \leq r_{ii}$ for all i. These coefficients can furthermore be computed by an explicit combinatorial algorithm based on the operations of the bialgebra Γ, see [4, §4].

A quiver coefficient $c_\mu(r)$ is non-zero only when $\sum |\mu_i| \geq d(r)$. When $\sum |\mu_i| = d(r)$, the coefficient $c_\mu(r)$ also appears in the formula for the cohomology class of Ω_r; these coefficients are called *cohomological quiver coefficients*. It was conjectured in [7] and proved by Knutson, Miller, and Shimozono in [16] that cohomological quiver coefficients are non-negative. The K-theoretic quiver coefficients were conjectured to have alternating signs, in the following sense.[1]

Conjecture 1. *For any rank conditions r and sequence of partitions μ, we have*
$$(-1)^{\sum |\mu_i| - d(r)}\, c_\mu(r) \geq 0\,.$$

[1]After these lectures were presented, this conjecture from [4] has been proved in the papers [6] and [24].

We explain how to prove Theorem 6 for a sequence of three vector bundles $E \to F \to H$ of ranks e, f, and h. In this case the quiver variety is defined as

$$\Omega = \{x \in X \mid \operatorname{rank}(E \to F) \leq r, \operatorname{rank}(F \to H) \leq s, \operatorname{rank}(E \to H) \leq t\}$$

for some non-negative integers r, s, and t. The requirement that these rank conditions can occur says that $r \leq \min(e, f)$, $s \leq \min(f, h)$, $t \leq \min(r, s)$, and $f + t \geq r + s$.

To make sure that the bundle maps are maximally generic, we start by replacing X with the bundle $\operatorname{Hom}(E, F) \oplus \operatorname{Hom}(F, H)$ and the bundle maps with the universal maps $E \to F \to H$ on this bundle. (In the rest of this section we avoid explicit notation for pullbacks of bundles.)

Form the product $Y = \operatorname{Gr}(r, F) \times_X \operatorname{Gr}(s, H)$ of Grassmann bundles with projection $\pi : Y \to X$, and let $F' \subset F$ and $H' \subset H$ denote the tautological subbundles on Y. On the subset $Z = Z(E \to F/F') \cap Z(F \to H/H') \subset Y$, the bundle map from E to F factors through F', and the map $F \to H$ factors through H'.

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr, dashed] & & F \arrow[rr] \arrow[dr, dashed] & & H \\
& F' \arrow[ur] & & H' \arrow[ur] &
\end{tikzcd}$$

This gives a map $F' \to F \to H'$ on Z, and we let $\Omega_t = \Omega_t(F' \to H') \subset Z$ denote the corresponding Thom-Porteous locus.

Since all the rank conditions are satisfied on Ω_t, it follows that $\pi(\Omega_t) \subset \Omega$. In fact, by using that the bundle maps are universal, one can show that π maps Ω_t birationally onto Ω (see [7, Lemma 1].) Since quiver varieties have rational singularities [18] and π is proper, it follows that $\pi_*(\mathcal{O}_{\Omega_t}) = \mathcal{O}_\Omega$ and $R^j\pi_*(\mathcal{O}_{\Omega_t}) = 0$ for all $j \geq 1$, so we get

$$\pi_*([\mathcal{O}_{\Omega_t}]) \;=\; [\mathcal{O}_\Omega] \;\in K(X)\,.$$

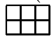
Let $R = (r-t)^{s-t}$ be a rectangle with $s - t$ rows and $r - t$ columns. By Theorem 5 and equations (2) and (3) we have

$$\begin{aligned}
[\mathcal{O}_{\Omega_t}] &= G_{(e)^{f-r}}(F/F' - E)\, G_{(f)^{h-s}}(H/H' - F)\, G_R(H' - F') \\
&= G_{(e)^{f-r}}(F/F' - E)\, G_{(f)^{h-s}}(H/H' - F) \sum_{\lambda,\mu} d^R_{\lambda\mu}\, G_\lambda(F/F')\, G_\mu(H' - F) \\
&= \sum_{\lambda,\mu} d^R_{\lambda\mu}\, G_{(e)^{f-r}+\lambda}(F/F' - E)\, G_{(f)^{h-s}}(H/H' - F)\, G_\mu(H' - F)
\end{aligned}$$

in the Grothendieck ring of Y. Using Theorem 4 we therefore obtain

$$[\mathcal{O}_\Omega] = \pi_*([\mathcal{O}_{\Omega_t}]) = \sum_{\lambda,\mu} d^R_{\lambda\mu}\, G_{(e-r)^{f-r}+\lambda}(F - E) \cdot G_{(f-s)^{h-s},\mu}(H - F)$$

in $K(X)$ as required. Notice that the quiver coefficients for a sequence of three bundles are just the coproduct coefficients $d^R_{\lambda,\mu}$. In particular, they have alternating signs.

Example 6. Suppose $\operatorname{rank}(E) = \operatorname{rank}(F) = 5$, $\operatorname{rank}(H) = 4$, $r = 3$, $s = 2$, and $t = 1$. Then we have $(e-r)^{f-r} =$ , $(f-s)^{h-s} =$ , and $R = (r-t)^{s-t} =$ .

Since $\Delta(G_R) = G_{(2)} \otimes 1 + G_{(1)} \otimes G_{(1)} + 1 \otimes G_{(2)} - G_{(2)} \otimes G_{(1)} - G_{(1)} \otimes G_{(2)}$, we obtain

$$\begin{aligned}
[\mathcal{O}_\Omega] &= G_{(4,2)}(F-E)\, G_{(3,3)}(H-F) + G_{(3,2)}(F-E)\, G_{(3,3,1)}(H-F) \\
&\quad + G_{(2,2)}(F-E)\, G_{(3,3,2)}(H-F) - G_{(4,2)}(F-E)\, G_{(3,3,1)}(H-F) \\
&\quad - G_{(3,2)}(F-E)\, G_{(3,3,2)}(H-F)\,.
\end{aligned}$$

6. Grothendieck polynomials

For each permutation $w \in S_{n+1}$, there is a *double Grothendieck polynomial* $\mathfrak{G}_w(x;y) = \mathfrak{G}_w(x_1,\dots,x_n;y_1,\dots,y_n)$ defined as follows. If $w = w_0$ is the longest permutation in S_{n+1}, then we set

$$\mathfrak{G}_{w_0}(x;y) = \prod_{i+j\le n+1} (x_i + y_j - x_i y_j)\,.$$

Otherwise there is a simple transposition $s_i \in S_{n+1}$ such that $\ell(ws_i) = \ell(w)+1$. In this case we set

$$\mathfrak{G}_w(x;y) = \pi_i(\mathfrak{G}_{ws_i})$$

where the *Demazure operator* π_i is defined by

$$\pi_i(f) = \frac{(1-x_{i+1}) f(x_1,\dots,x_i,x_{i+1},\dots,x_n) - (1-x_i) f(x_1,\dots,x_{i+1},x_i,\dots,x_n)}{x_i - x_{i+1}}\,.$$

Since the Demazure operators satisfy the Coxeter relations $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ and $\pi_i\pi_j = \pi_j\pi_i$ for $|i-j|\ge 2$, it follows that this definition is independent of the choice of s_i. The specialization $\mathfrak{G}_w(x_1,\dots,x_n) = \mathfrak{G}_w(x_1,\dots,x_n;0,\dots,0)$ is called the single Grothendieck polynomial for w.

Grothendieck polynomials were introduced by Lascoux and Schützenberger [21, 19] as representatives for the Grothendieck classes of Schubert varieties in a flag variety. More generally, the double Grothendieck polynomials have the following interpretation. Let $F_1 \subset F_2 \subset \cdots \subset F_n \to H_n \twoheadrightarrow \cdots \twoheadrightarrow H_2 \twoheadrightarrow H_1$ be a sequence of bundles on X consisting of a full flag $F_\bullet$ with a general map to a dual full flag $H_\bullet$, such that $\operatorname{rank}(F_i) = \operatorname{rank}(H_i) = i$ for all i. Given a permutation $w \in S_{n+1}$, Fulton [12] defined the degeneracy locus

$$\Omega_w = \Omega_w(F_\bullet \to H_\bullet) = \{x \in X \mid \operatorname{rank}(F_q(x) \to H_p(x)) \le r_w(p,q)\ \forall p,q\}\,,$$

where $r_w(p,q) = \#\{i \le p \mid w(i) \le q\}$. Fulton proved that the cohomology class of Ω_w is given by a double Schubert polynomial [12]. This was generalized to the following formula for the Grothendieck class in [4, Thm. 2.1] as an application of [14, Thm. 3].

Theorem 7. *If the codimension of Ω_w equals the length of w, then*

$$[\mathcal{O}_{\Omega_w}] = \mathfrak{G}_w(x;y)$$

where $x_i = 1 - L_i^{-1} \in K(X)$ for $L_i = \ker(H_i \to H_{i-1})$, and $y_i = 1 - M_i \in K(X)$ for $M_i = F_i/F_{i-1}$.

Fulton's degeneracy locus Ω_w is a special case of a quiver variety, that is $\Omega_w = \Omega_r(F_\bullet \to H_\bullet)$ where $r = \{r_{ij}\}$ is the set of rank conditions defined by

$$r_{ij} = \begin{cases} r_w(2n+1-j, i) & \text{if } i \leq n < j \\ i & \text{if } j \leq n \\ 2n+1-j & \text{if } i \geq n+1. \end{cases}$$

It follows from this that $\mathfrak{G}_w(x;y) = [\mathcal{O}_{\Omega_r}]$, so

$$\mathfrak{G}_w(x;y) = \sum_\mu c_{w,\mu}\, G_{\mu_1}(F_2 - F_1) \cdots G_{\mu_n}(H_n - F_n) \cdots G_{\mu_{2n-1}}(H_1 - H_2) \tag{4}$$

where the constants $c_{w,\mu} = c_\mu(r)$ are quiver coefficients. Note that $G_{\mu_i}(F_{i+1} - F_i) = G_{\mu_i}(M_{i+1})$ is either zero or a power of $1 - M_{i+1}^{-1} = 1 - (1 - y_{i+1})^{-1}$, which can be expressed as a formal power series in y_{i+1}. Similarly $G_{\mu_{2n-i}}(H_i - H_{i+1}) = G_{\mu_{2n-i}}(-L_{i+1})$ is a power of $1 - (1 - x_{i+1})^{-1}$. In section 7 we describe a proof that the coefficients $c_{w,\mu}$ have alternating signs.

Let $Y = \mathrm{Fl}(n+1)$ be the variety of complete flags in $\mathbb{C}^{n+1}$, and $S_1 \subset \cdots \subset S_n \subset \mathbb{C}^{n+1} \times Y$ the tautological flag of subbundles on Y. If we let $H_\bullet$ be the dualized flag $S_n^\vee \twoheadrightarrow S_{n-1}^\vee \twoheadrightarrow \cdots \twoheadrightarrow S_1^\vee$ and let $F_\bullet \subset \mathbb{C}^{n+1} \times Y$ be the trivial flag given by $F_p = (0^{n+1-p} \oplus \mathbb{C}^p) \times Y$, then the degeneracy locus $\Omega_w(F_\bullet \to H_\bullet) \subset Y$ is identical to the Schubert variety denoted by $Y_{w_0 w}$ in Brion's lectures (see [2, Def. 1.2.2]). It follows that the single Grothendieck polynomial $\mathfrak{G}_w(x_1, \ldots, x_n)$ gives a formula for the Grothendieck class of this Schubert variety in $K(Y)$ [21].

The double Grothendieck polynomials are related to stable Grothendieck polynomials as follows. Given a permutation $w \in S_{n+1}$, we let $1^m \times w \in S_{m+n+1}$ denote the shifted permutation, which acts as the identity on the set $\{1, 2, \ldots, m\}$ and maps $m+i$ to $m + w(i)$ for $i \geq 1$. The *stable double Grothendieck polynomial* $G_w(x;y)$ of Fomin and Kirillov [9] is defined by

$$G_w(x_1, \ldots, x_m; y_1, \ldots, y_m) = \mathfrak{G}_{1^m \times w}(x_1, \ldots, x_m, 0, \ldots, 0; y_1, \ldots, y_m, 0, \ldots, 0)\,.$$

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$ be a partition. The corresponding Grassmannian permutation w_λ with descent at position d is defined by $w_\lambda(i) = i + \lambda_{d+1-i}$ for $i \leq d$ and $w_\lambda(i) < w_\lambda(i+1)$ for $i \neq d$. With the notation of section 3, we have $G_\lambda(x) = G_{w_\lambda}(x; 0)$ and

$$G_{w_\lambda}(x;y) = G_\lambda(x;y) = \sum_{\sigma,\tau} d^\lambda_{\sigma\tau} G_\sigma(x) G_{\tau'}(y)\,.$$

One can also show that the single Grothendieck polynomial for w_λ is given by $\mathfrak{G}_{w_\lambda}(x) = G_\lambda(x_1, \ldots, x_d)$, but no such identity holds for the double Grothendieck polynomial $\mathfrak{G}_{w_\lambda}(x;y)$.

Let $X = \mathrm{Gr}(d, \mathbb{C}^{n+1})$ and let $\rho : Y = \mathrm{Fl}(n+1) \to X$ be the projection that maps a complete flag $V_\bullet \subset \mathbb{C}^{n+1}$ to its subspace V_d of dimension d. Then we have $\rho^{-1}(X_\lambda) = \Omega_{w_\lambda} = \Omega_{w_\lambda}(F_\bullet \to H_\bullet)$. Since $X_\lambda \subset X$ and $\Omega_{w_\lambda} \subset Y$ have the same codimension $|\lambda| = \ell(w_\lambda)$, and since Schubert varieties are Cohen-Macaulay [26], it follows from this that $\rho^*(\mathcal{O}_\lambda) = [\mathcal{O}_{\Omega_{w_\lambda}}] = \mathfrak{G}_{w_\lambda}(x_1, \ldots, x_n)$. Since the Grothendieck classes of the Schubert varieties in Y are linearly independent, this implies that the pullback map $\rho^* : K(X) \to K(Y)$ is injective. If $S \subset \mathbb{C}^{n+1} \times X$ is the tautological subbundle on X, then we also have $\rho^*(G_\lambda(S^\vee)) = G_\lambda(\rho^* S^\vee) = G_\lambda(H_d) = G_\lambda(x_1, \ldots, x_d) = \mathfrak{G}_{w_\lambda}(x_1, \ldots, x_n)$, which shows that $\mathcal{O}_\lambda = G_\lambda(S^\vee)$ in the Grothendieck ring of X.

Every stable Grothendieck polynomial $G_w(x;y)$ can be written as a finite linear combination of the polynomials $G_\lambda(x;y)$ given by partitions [5]. We will describe a formula of Lascoux for the coefficients of this linear combination, which shows that they have alternating signs.

Given a permutation w, let r be the last descent position of w, i.e. r is maximal such that $w(r) > w(r+1)$. Set $w' = w\tau_{rk}$ where $k > r$ is maximal such that $w(r) > w(k)$. We also set $I(w) = \{i < r \mid \ell(w'\tau_{ir}) = \ell(w)\}$. Define a relation $\rhd$ on the set of all permutations as follows. If $I(w) = \emptyset$ we write $w \rhd v$ if and only if $v = 1 \times w$. Otherwise we write $w \rhd v$ if and only if there exist elements $i_1 < \cdots < i_p$ of $I(w)$, $p \geq 1$, such that $v = w'\tau_{i_1 r} \dots \tau_{i_p r}$. The following is an immediate consequence of [20, Thm. 4].

Theorem 8 (Lascoux). *For any permutation w we have*

$$G_w = \sum_\lambda a_{w,\lambda}\, G_\lambda$$

where the sum is over all partitions λ, and $a_{w,\lambda}$ is equal to $(-1)^{|\lambda|-\ell(w)}$ times the number of sequences $w = w_1 \rhd w_2 \rhd \cdots \rhd w_m$ such that $w_m = w_\lambda$ is a Grassmannian permutation for λ and w_i is not Grassmannian for $i < m$.

Example 7. For the permutation $w = 2\,1\,4\,3$, we get $r = 3$, $k = 4$, $w' = 2\,1\,3\,4$, and $I(w) = \{1, 2\}$. The sequences of permutations of Theorem 8 are $w \rhd 3\,1\,2\,4 = w_{(2)}$, $w \rhd 2\,3\,1\,4 = w_{(1,1)}$, and $w \rhd 3\,2\,1\,4 \rhd 1\,4\,3\,2\,5 \rhd 2\,4\,1\,3\,5 = w_{(2,1)}$. It follows that $G_w = G_{(2)} + G_{(1,1)} - G_{(2,1)}$.

7. Alternating signs of the coefficients $c_{w,\mu}$

In this section we outline a proof that the quiver coefficients $c_{w,\mu}$ of (4) have alternating signs, based on our joint paper [8] with Kresch, Tamvakis, and Yong.

Suppose we are given a sequence of vector bundles

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset V \twoheadrightarrow H_n \twoheadrightarrow \cdots \twoheadrightarrow H_2 \twoheadrightarrow H_1$$

consisting of a full flag $F_\bullet$ of a bundle V of rank $n+1$, followed by a dual full flag $H_\bullet$ of V. Set $F_i' = V/F_i$ and $H_i' = \ker(V \twoheadrightarrow H_i)$. We then obtain a sequence

$$H_n' \subset \cdots \subset H_1' \subset V \twoheadrightarrow F_1' \twoheadrightarrow \cdots \twoheadrightarrow F_n'\,,$$

and it is an easy exercise to show that $\Omega_w(F_\bullet \to H_\bullet) = \Omega_{\widehat{w}}(H_\bullet' \to F_\bullet')$ as subschemes of X, where $\widehat{w} = w_0 w^{-1} w_0$. It follows that $\mathfrak{G}_w(x;y) = [\mathcal{O}_{\Omega_{\widehat{w}}}]$, so we obtain

$$\begin{aligned}\mathfrak{G}_w(x;y) &= \sum_\mu c_{\widehat{w},\mu} G_{\mu_1}(H_{n-1}' - H_n') \cdots G_{\mu_n}(F_1' - H_1') \cdots G_{\mu_{2n-1}}(F_n' - F_{n-1}') \\ &= \sum_\mu c_{\widehat{w},\mu} G_{\mu_1}(x_n) \cdots G_{\mu_n}(x_1; y_1) \cdots G_{\mu_{2n-1}}(0; y_n)\end{aligned}$$

where x_i and y_i are defined as in Theorem 7. Notice that $G_\lambda(x_i)$ equals x_i^p when $\lambda = (p)$ is a single row with p boxes, and is zero otherwise. Similarly $G_\lambda(0; y_i)$ is a power of y_i or zero, and furthermore $G_\lambda(x_1; y_1) = \sum_{\sigma,\tau} d_{\sigma\tau}^\lambda G_\sigma(x_1) G_{\tau'}(y_1)$.

Corollary 2. *The monomial coefficients of Grothendieck polynomials are special cases of the quiver coefficients $c_{w,\mu}$.*

The *degenerate Hecke algebra* is the free $\mathbb{Z}$-algebra $\mathcal{H}$ generated by symbols $s_1, s_2, \dots,$ modulo the relations $s_i^2 = s_i$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all i, and $s_i s_j = s_j s_i$ for $|i-j| \geq 2$. This algebra has a basis of permutations, corresponding to reduced expressions in the generators.

Now define the *universal Grothendieck polynomial* for the permutation $w \in S_n$ to be the element

$$P_w = \sum_\mu c_{w,\mu} \, G_{\mu_1} \otimes \cdots \otimes G_{\mu_{2n-1}} \in \Gamma^{\otimes 2n-1} \, .$$

The following theorem gives an explicit formula for these polynomials.

Theorem 9. *For $w \in S_{n+1}$ we have*

$$P_w = \sum (-1)^{\sum \ell(u_i) - \ell(w)} \, G_{u_1} \otimes G_{u_2} \otimes \cdots G_{u_{2n-1}} \in \Gamma^{\otimes 2n-1} \, ,$$

where the sum is over all factorizations $w = u_1 \cdot u_2 \cdots u_{2n-1}$ in the degenerate Hecke algebra $\mathcal{H}$ such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

This theorem combined with Lascoux's formula for the expansion of stable Grothendieck polynomials in the basis of Γ implies the following explicit formula for the quiver coefficients $c_{w,\mu}$.

Corollary 3. *The quiver coefficients $c_{w,\mu}$ of P_w are given by*

$$c_{w,\mu} = (-1)^{\sum |\mu_i| - \ell(w)} \sum \prod_{i=1}^{2n-1} |a_{u_i,\mu_i}| \, ,$$

where the sum is over all factorizations $w = u_1 \cdot u_2 \cdots u_{2n-1}$ in the degenerate Hecke algebra $\mathcal{H}$ such that $u_i \in S_{\min(i,2n-i)+1}$ for each i, and the constants a_{u_i,μ_i} are given by Theorem 8.

The proof of Theorem 9 is based on some identities of universal Grothendieck polynomials. For $1 \leq i \leq j \leq 2n-1$ we define

$$P_w[i,j] = \sum_{\mu : \mu_k = \emptyset \text{ for } k \notin [i,j]} c_{w,\mu} \, 1 \otimes \cdots \otimes 1 \otimes G_{\mu_i} \otimes \cdots \otimes G_{\mu_j} \otimes 1 \otimes \cdots \otimes 1 \, .$$

Using a Cauchy identity for double Grothendieck polynomials of Fomin and Kirillov [10] as well as some geometry of quiver varieties, one obtains the identity

$$P_w = \sum_{u \cdot v = w \in \mathcal{H}} (-1)^{\ell(uvw)} P_u[1,i] \cdot P_v[i+1, 2n-1] \, .$$

By iterating this formula, we obtain

$$P_w = \sum_{u_1 \cdots u_{2n-1} = w \in \mathcal{H}} (-1)^{\sum \ell(u_i) - \ell(w)} \, P_{u_1}[1,1] \cdot P_{u_2}[2,2] \cdots P_{u_{2n-1}}[2n-1, 2n-1] \, .$$

Finally we use the identity

$$P_w[i,i] = \begin{cases} 1^{\otimes i-1} \otimes G_w \otimes 1^{\otimes 2n-1-i} & \text{if } w \in S_{\min(i,2n-i)+1} \\ 0 & \text{otherwise.} \end{cases}$$

This was proved in [4] for $i = n$, and the remaining cases are easy consequences of this. Theorem 9 follows immediately from these identities.

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