

TEVELEV DEGREES IN GROMOV-WITTEN THEORY

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In memory of Bumsig Kim

ABSTRACT. For a nonsingular projective variety X , the virtual Tevelev degree in Gromov-Witten theory is defined as the virtual degree of the morphism from $\overline{\mathcal{M}}_{g,n}(X, d)$ to the product $\overline{\mathcal{M}}_{g,n} \times X^n$. After proving a simple formula for the virtual Tevelev degree in the (small) quantum cohomology ring of X using the quantum Euler class, we provide several exact calculations for flag varieties and complete intersections. In the cominuscule case (including Grassmannians, Lagrangian Grassmannians, and maximal orthogonal Grassmannians), the virtual Tevelev degrees are calculated in terms of the eigenvalues of an associated self-adjoint linear endomorphism of the quantum cohomology ring. For complete intersections of low degree (compared to dimension), we prove a product formula. The calculation for complete intersections involves the primitive cohomology. Virtual Tevelev degrees are better behaved than arbitrary Gromov-Witten invariants, and, by recent results of [LP21], are much more likely to be enumerative.

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1. INTRODUCTION

1.1. **Virtual Tevelev degrees.** Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of Deligne-Mumford stable curves over \mathbb{C} of genus g with n marked points. The moduli space $\overline{\mathcal{M}}_{g,n}$ is nonsingular (as a stack), irreducible, and satisfies

$$\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n.$$

The stability condition implies $2g - 2 + n > 0$.

Let X be a nonsingular, projective, algebraic variety over \mathbb{C} of dimension r , and let $d \in H_2(X, \mathbb{Z})$. The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, d)$ has virtual

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dimension

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X, d) = \int_d c_1(T_X) + (r-3)(1-g) + n,$$

which equals the dimension of $\overline{\mathcal{M}}_{g,n} \times X^n$ if and only if

$$(1) \quad \int_d c_1(T_X) = r(n+g-1).$$

If the dimension constraint (1) holds, we expect to find a finite number of maps from a fixed curve (C, p_1, \dots, p_n) of genus g to X of curve class d that send the marked points p_i to fixed general points in X . Tevelev degrees in Gromov-Witten theory are defined to be the corresponding virtual count.

Definition 1.1. Let $g \geq 0$, $n \geq 0$, and $d \in H_2(X, \mathbb{Z})$ satisfy $2g - 2 + n > 0$ and the dimension constraint (1). Let

$$\tau : \overline{\mathcal{M}}_{g,n}(X, d) \rightarrow \overline{\mathcal{M}}_{g,n} \times X^n$$

be the canonical morphism obtained from the domain curve and the evaluation maps. The **virtual Tevelev degree** $\text{vTev}_{g,d,n}^X \in \mathbb{Q}$ is defined by

$$\tau_*([\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}}) = \text{vTev}_{g,d,n}^X \cdot [\overline{\mathcal{M}}_{g,n} \times X^n] \in A^0(\overline{\mathcal{M}}_{g,n} \times X^n).$$

Here, $[\]^{\text{vir}}$ and $[\]$ denote the virtual and usual fundamental classes, respectively. The invariant $\text{vTev}_{g,d,n}^X$ is zero if the class $d \in H_2(X, \mathbb{Z})$ is not effective, since then the moduli space $\overline{\mathcal{M}}_{g,n}(X, d)$ is empty. If $g, n \geq 0$ and $d \in H_2(X, \mathbb{Z})$ do not satisfy the dimension constraint (1), we define $\text{vTev}_{g,d,n}^X$ to vanish.

For $g, n, k \geq 0$ such that $2g - 2 + n + k > 0$, let

$$\tau : \overline{\mathcal{M}}_{g,n+k}(X, d) \rightarrow \overline{\mathcal{M}}_{g,n+k} \times X^n$$

be the morphism obtained from the domain curve and evaluations at the first n marked points. When the dimension constraint (1) holds, the more general virtual degree $\text{vTev}_{g,d,n,k}^X \in \mathbb{Q}$,

$$(2) \quad \tau_*([\overline{\mathcal{M}}_{g,n+k}(X, d)]^{\text{vir}}) = \text{vTev}_{g,d,n,k}^X \cdot [\overline{\mathcal{M}}_{g,n+k} \times X^n] \in A^0(\overline{\mathcal{M}}_{g,n+k} \times X^n),$$

is proven in [Section 2](#) to be independent of k , so

$$\text{vTev}_{g,d,n,k}^X = \text{vTev}_{g,d,n}^X$$

in the stable case $2g - 2 + n > 0$. The definition of the virtual Tevelev degree can be naturally extended to the four¹ unstable cases where $2g - 2 + n \leq 0$ by

$$\text{vTev}_{g,d,n}^X = \text{vTev}_{g,d,n,k}^X$$

for any sufficiently large k .

While general Gromov-Witten invariants of varieties can be complicated to compute, we will see that the virtual Tevelev degrees are much better behaved.

¹ $(g, n) = (0, 0), (0, 1), (0, 2), (1, 0)$.

1.2. **Quantum cohomology.** Let $\{\gamma_j\}$ be any basis² of $H^*(X)$, and let $\{\gamma_k^\vee\}$ be the dual basis defined by

$$\int_X \gamma_j \cup \gamma_k^\vee = \delta_{j,k}.$$

Let $QH^*(X)$ be the small quantum cohomology ring of X defined via the 3-point Gromov-Witten invariants in genus 0,

$$\gamma_i \star \gamma_j = \sum_{d \in H_2(X, \mathbb{Z})} \sum_k \langle \gamma_i, \gamma_j, \gamma_k^\vee \rangle_{0,d}^X \cdot q^d \gamma_k \in QH^*(X),$$

see [FP97] for an introduction. Let

$$\Delta = \sum_j \gamma_j^\vee \otimes \gamma_j \in H^*(X \times X)$$

be the standard Künneth decomposition³ of the diagonal class.

Definition 1.2. The **quantum Euler class** of X is

$$E = \sum_j \gamma_j^\vee \star \gamma_j \in QH^*(X).$$

The classical ($q = 0$) part of the quantum Euler class is determined by the usual topological Euler characteristic $\chi(X)$,

$$E = \chi(X) \cdot P + q\text{-corrections},$$

where $P \in H^{2r}(X)$ is the point class. Since E is the image of Δ under the canonical multiplication map

$$H^*(X) \otimes H^*(X) \xrightarrow{\star} QH^*(X),$$

E is independent of basis choice. The quantum Euler class was first introduced⁴ by Abrams in [Abr00], see also [CMP10, Section 8].

Our first result is that virtual Tevelev degrees can be computed in $QH^*(X)$ using the point class and the quantum Euler class.

Theorem 1.3. For $g, n \geq 0$ and $d \in H_2(X, \mathbb{Z})$ we have

$$\text{vTev}_{g,d,n}^X = \text{Coeff}(P^{\star n} \star E^{\star g}, q^d P).$$

The notation $A^{\star n}$ denotes the power of $A \in QH^*(X)$ with respect to the quantum product. A canonical way to write the coefficient in [Theorem 1.3](#) is

$$\text{Coeff}(P^{\star n} \star E^{\star g}, q^d P) = \int_X \text{Coeff}(P^{\star n} \star E^{\star g}, q^d).$$

Tevelev degrees have been studied in several contexts [CL21, CPS21, FL21, LP21, Tev20] starting with $X = \mathbb{P}^1$. Almost always, virtual Tevelev degrees are much better behaved than enumerative Tevelev degrees [LP21] and general Gromov-Witten invariants. Our results concern exact calculations of virtual Tevelev degrees in two main cases: cominuscule flag varieties and low degree complete intersections in projective spaces. An asymptotic equality between virtual and enumerative Tevelev degrees for certain Fano varieties (including flag varieties and low degree hypersurfaces) is proved in [LP21], so many of our calculations are actual curve counts.

²Cohomology and quantum cohomology will always be taken here with \mathbb{Q} -coefficients.

³The order matters!

⁴The definition there differs slightly but is, in fact, equivalent to ours.

1.3. Virtual Tevelev degrees: hypersurfaces. Using [Theorem 1.3](#) and properties of the quantum cohomology of hypersurfaces, we obtain the following results:

- The projective space case $X = \mathbb{P}^r$ has a particularly simple answer:

$$(3) \quad \text{vTev}_{g,d,n}^{\mathbb{P}^r} = (r+1)^g$$

whenever the dimension constraint [\(1\)](#) is satisfied.

- The case of a quadric hypersurface $Q^r \subset \mathbb{P}^{r+1}$ takes a special form. Let

$$\delta = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases}$$

Theorem 1.4. *For nonsingular quadrics Q^r of dimension $r \geq 3$,*

$$\text{vTev}_{g,d,n}^{Q^r} = \frac{(2r)^g + (-1)^d (2\delta)^g}{2}$$

whenever the dimension constraint [\(1\)](#) is satisfied.

We index here the curve classes of $Q^r \subset \mathbb{P}^{r+1}$ for $r \geq 3$ by their associated degree d in \mathbb{P}^{r+1} . For fixed genus g , using the enumerativity results of [\[LP21\]](#), the virtual Tevelev degrees for Q^r in sufficiently high degree d are actual curve counts.

The precise statement in enumerative geometry from [\[LP21\]](#) is the following. Let C be a fixed general curve of genus g . The dimension constraint for Q^r is

$$d = n + g - 1.$$

For d (and hence n) sufficiently large, let (C, p_1, \dots, p_n) be defined by general points of C . Let $x_1, \dots, x_n \in Q^r$ be general points. Then, the actual count of maps

$$f : C \rightarrow Q^r \quad \text{satisfying } f(p_i) = x_i$$

where f has degree d , is equal to $\text{vTev}_{g,d,n}^{Q^r}$. Whether the counting problem for quadrics can be solved directly by classical techniques is an interesting question.

The excluded $r = 2$ case of the quadric surface $Q^2 \subset \mathbb{P}^3$ has additional curve classes since $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. The virtual Tevelev degrees of Q^2 are determined by a product rule in [Section 2.2](#).

- For low degree hypersurfaces, we also have a complete calculation. The curve classes of hypersurfaces of dimension at least 3 are indexed by their associated degree in projective space.

Theorem 1.5. *Let $X_e \subset \mathbb{P}^{r+1}$ be a nonsingular hypersurface of degree $e \geq 3$ and dimension $r \geq 2e - 3$. Then, for $g + n \geq 2$ we have*

$$\text{vTev}_{g,d,n}^{X_e} = ((e-1)!)^n \cdot (r+2-e)^g \cdot e^{(d-n)e-g+1}$$

whenever the dimension constraint [\(1\)](#) is satisfied.

Notice that for any nonsingular projective Fano variety X , the virtual Tevelev degrees $\text{vTev}_{g,d,n}^X$ for which $g + n \leq 1$ are given by

$$\text{vTev}_{0,d,0}^X = 0, \quad \text{vTev}_{0,d,1}^X = \delta_{d,0}, \quad \text{vTev}_{1,d,0}^X = \delta_{d,0} \chi(X),$$

so they can safely be left out of [Theorem 1.5](#).

Most of the previous work on the quantum cohomology of the hypersurface $X_e \subset \mathbb{P}^{r+1}$ concerns the subalgebra of classes restricted from \mathbb{P}^{r+1} . However,

the definition of the quantum Euler class \mathbf{E} involves *all* of the cohomology of X_e . Controlling the contributions of the primitive cohomology is perhaps the most interesting⁵ aspect of the proof of [Theorem 1.5](#).

Our techniques apply unchanged to the case of complete intersections of low degree. The results for complete intersections take a very similar form and are stated in [Theorem 5.19](#) of [Section 5.6](#). Various patterns are presented beyond our degree/dimension ranges for hypersurfaces and complete intersections in [Section 5](#). There are many open questions.

1.4. Virtual Tevelev degrees: cominuscule flag varieties. The formulas for the virtual Tevelev degrees for projective spaces \mathbb{P}^r and quadrics Q^r have a different flavor than for the complete intersections of [Theorem 1.5](#). For fixed genus g , the virtual Tevelev degree of \mathbb{P}^r and Q^r depend upon the degree d at most by a parity condition (no dependence on d at all for \mathbb{P}^r and mod 2 dependence for Q^r). Since d is related to g and n by the dimension constraint, we can also view the dependence as periodic in n . We prove in [Section 3](#) that all cominuscule⁶ flag varieties have such simple behavior.

Theorem 1.6. *Let X be a cominuscule flag variety. When the dimension constraint (1) is satisfied, $\text{vTev}_{g,d,n}^X$ depends only upon g and*

$$n \pmod{\text{ord}(\mathbf{P})},$$

where $\text{ord}(\mathbf{P})$ is the finite order of the point class \mathbf{P} in the group of units of the ring $QH^*(X)/\langle q-1 \rangle$.

Cominuscule flag varieties X include Grassmannians, Lagrangian Grassmannians, and maximal orthogonal Grassmannians. In genus 0 and 1, we find complete closed forms for the virtual Tevelev degrees of these spaces. For general g , we calculate the virtual Tevelev degrees for all cominuscule flag varieties in terms of the eigenvalues and eigenvectors of a basic operator

$$[\mathbf{E}/\mathbf{P}]_0 : QH^*(X)_{q,0} \rightarrow QH^*(X)_{q,0}$$

on the degree 0 part of the localized quantum cohomology. The formula for virtual Tevelev degrees of arbitrary genus is presented in [Theorem 4.8](#) of [Section 4](#).

The operator \mathbf{E}/\mathbf{P} is defined via quantum multiplication by \mathbf{E} followed by the inverse (after localization) of quantum multiplication by the point class \mathbf{P} , and $[\mathbf{E}/\mathbf{P}]_0$ is the restriction to the degree 0 part. A crucial property of \mathbf{E}/\mathbf{P} is symmetry with respect to the strange duality involution of [\[Pos05, CMP07\]](#).

As an example of the effectivity of our results, the virtual Tevelev degrees of the Grassmannian $\text{Gr}(2, 5)$, the first Grassmannian which is not a projective space or a quadric, are

$$\text{vTev}_{g,d,n}^{\text{Gr}(2,5)} = \frac{5 - \sqrt{5}}{10} \left(\frac{25 + 5\sqrt{5}}{2} \right)^g + \frac{5 + \sqrt{5}}{10} \left(\frac{25 - 5\sqrt{5}}{2} \right)^g$$

when the dimension constraint is satisfied. For fixed genus g , using the enumerativity results of [\[LP21\]](#), the virtual Tevelev degrees for all flag varieties in sufficiently

⁵See also [\[ABPZ21, Hu15\]](#) for recent results on the Gromov-Witten theory of complete intersections which also confront the primitive cohomology.

⁶The full definition is reviewed in [Section 3](#).

high degree d are actual curve counts. Several further calculations are presented in [Section 4.4](#) and [Section 4.5](#)

While our formulas for virtual Tevelev degrees of cominuscule flag varieties and complete intersections look different, they share the feature that they rely only on a small and well-behaved subring of the quantum cohomology of X . For cominuscule varieties, the calculation occurs in the subring $QH^*(X)_{q,0}$. For complete intersections, the calculation is dictated to a surprising extent by the subring defined by classes restricted from the ambient projective space.

[Theorem 1.3](#) implies that the virtual Tevelev degrees of all flag varieties are non-negative integers, and [Theorem 5.19](#) shows that virtual Tevelev degrees of low-degree complete intersections are non-negative integers. [Example 5.24](#) shows that $\text{vTev}_{1,2,1}^X = -64$ when $X \subset \mathbb{P}^7$ is a complete intersection of three general quadrics, so virtual Tevelev degrees can be negative (X is just out of the bounds required by [Theorem 5.19](#)). On the other hand, we have not been able to find a virtual Tevelev degree that is not an integer. It would be interesting to know if non-integer virtual Tevelev degrees exist.

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2. REDUCTION TO QUANTUM COHOMOLOGY

2.1. Proof of [Theorem 1.3](#). Let X be a nonsingular complex projective variety of dimension r , and let $d \in H_2(X, \mathbb{Z})$. Let $g, n, k \geq 0$ be non-negative integers satisfying $2g - 2 + n + k > 0$. If the dimension constraint [\(1\)](#) is satisfied, we have by definition [\(2\)](#),

$$\text{vTev}_{g,d,n,k}^X \cdot [\overline{\mathcal{M}}_{g,n+k} \times X^n] = \tau_* \left([\overline{\mathcal{M}}_{g,n+k}(X, d)]^{\text{vir}} \right) \in A^0(\overline{\mathcal{M}}_{g,n+k} \times X^n),$$

where $\tau = (\pi, \text{ev}_{[n]})$ is the morphism

$$\tau : \overline{\mathcal{M}}_{g,n+k}(X, d) \rightarrow \overline{\mathcal{M}}_{g,n+k} \times X^n$$

constructed from the factors

$$\pi : \overline{\mathcal{M}}_{g,n+k}(X, d) \rightarrow \overline{\mathcal{M}}_{g,n+k} \quad \text{and} \quad \text{ev}_{[n]} : \overline{\mathcal{M}}_{g,n+k}(X, d) \rightarrow X^n,$$

where $\text{ev}_{[n]}$ denotes evaluation at the first n marked points.

Let $I_{g,d,n+k}^X : H^*(X)^{\otimes(n+k)} \rightarrow H^*(\overline{\mathcal{M}}_{g,n+k})$ denote the Gromov-Witten class defined by

$$I_{g,d,n+k}^X(\alpha) = \pi_* \left(\text{ev}_{[n+k]}^*(\alpha) \cap [\overline{\mathcal{M}}_{g,n+k}(X, d)]^{\text{vir}} \right).$$

By the projection formula, we obtain

$$\text{vTev}_{g,d,n,k}^X \cdot [\overline{\mathcal{M}}_{g,n+k}] = I_{g,d,n+k}^X(\mathbf{P}^{\otimes n} \otimes 1^{\otimes k}),$$

where $\mathbf{P} \in H^{2r}(X)$ is the point class. By the identity insertion axiom in Gromov-Witten theory, $\text{vTev}_{g,d,n,k}^X$ is independent of k (whenever $2g - 2 + n + k > 0$). Moreover, $\text{vTev}_{g,d,n}^X = \text{vTev}_{g,d,n,k}^X$ is given by

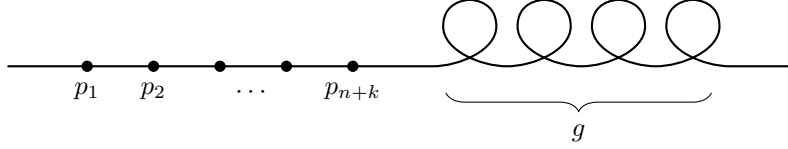
$$(4) \quad \text{vTev}_{g,d,n}^X = \int_{\overline{\mathcal{M}}_{g,n+k}} I_{g,d,n+k}^X(\mathbf{P}^{\otimes n} \otimes 1^{\otimes k}) \cap \mathbf{P}_{\overline{\mathcal{M}}_{g,n+k}},$$

whether or not the dimension constraint (1) is satisfied. Here, $\mathbf{P}_{\overline{\mathcal{M}}_{g,n+k}}$ denotes the point class in $H^*(\overline{\mathcal{M}}_{g,n+k})$.

By the genus reduction axiom of Gromov-Witten theory (see [Beh97]) we have

$$\psi^* I_{g,d,n+k}^X(\mathbf{P}^{\otimes n} \otimes 1^{\otimes k}) = I_{0,d,n+k+2g}^X(\mathbf{P}^{\otimes n} \otimes 1^{\otimes k} \otimes \Delta^{\otimes g}),$$

where $\Delta \in H^*(X \times X)$ is the diagonal class and $\psi : \overline{\mathcal{M}}_{0,n+k+2g} \rightarrow \overline{\mathcal{M}}_{g,n+k}$ sends the $(n+k+2g)$ -pointed rational curve $[\mathbb{P}^1, p_1, p_2, \dots, p_{n+k}, z_1, z'_1, \dots, z_g, z'_g]$ to the $(n+k)$ -pointed curve with g nodes obtained by identifying z_i with z'_i for $1 \leq i \leq g$:



We therefore obtain from (4) that

$$(5) \quad \text{vTev}_{g,d,n}^X = \langle \mathbf{P}^{\otimes n} \otimes 1^{\otimes k} \otimes \Delta^{\otimes g} \rangle_{\odot, d, n+k+2g}^X,$$

where, for each $m \geq 3$, we let $\langle - \rangle_{\odot, d, m}^X : H^*(X)^{\otimes m} \rightarrow \mathbb{Q}$ be the map defined by

$$\langle \alpha \rangle_{\odot, d, m}^X = \int_{\overline{\mathcal{M}}_{0,m}} I_{0,d,m}^X(\alpha) \cap \mathbf{P}_{\overline{\mathcal{M}}_{0,m}}.$$

The invariants $\langle \alpha \rangle_{\odot, d, m}^X$ should be considered *small* m -pointed Gromov-Witten invariants by the following Proposition (which played a central role in Bertram's computation of the quantum cohomology of Grassmannians [Ber97]).

Proposition 2.1. *Let $m \geq 2$. Then, the m -fold small quantum multiplication map*

$$H^*(X)^{\otimes m} \rightarrow QH^*(X)$$

is given by

$$\alpha \mapsto \sum_{d,k} \langle \alpha \otimes \gamma_k^\vee \rangle_{\odot, d, m+1}^X \cdot q^d \gamma_k.$$

Proof. The identity for $m = 2$ is the definition of the small quantum product. We proceed by induction on m .

Given $\alpha_1 \in H^*(X)^{\otimes m}$ and $\alpha_2 \in H^*(X)$, the splitting axiom of Gromov-Witten theory implies

$$\langle \alpha_1 \otimes \alpha_2 \otimes \gamma_k^\vee \rangle_{\odot, d, m+2}^X = \sum_{d_1+d_2=d} \sum_{k'} \langle \alpha_1 \otimes \gamma_{k'}^\vee \rangle_{\odot, d_1, m+1}^X \cdot \langle \gamma_{k'} \otimes \alpha_2 \otimes \gamma_k^\vee \rangle_{\odot, d_2, 3}^X.$$

By using the induction hypothesis, we see that the map of the Proposition is given by iterating the small quantum product. \square

By equation (5) for the virtual Tevelev degree and Proposition 2.1, we obtain

$$\mathrm{vTev}_{g,d,n}^X = \langle \mathbf{P}^{\otimes n} \otimes 1^{\otimes k} \otimes \Delta^{\otimes g} \rangle_{\odot, d, n+k+2g}^X = \mathrm{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^d \mathbf{P}),$$

which completes the proof of Theorem 1.3. \square

Example 2.2. We illustrate Theorem 1.3 by computing the virtual Tevelev degrees of projective space $X = \mathbb{P}^r$. The small quantum cohomology ring is given by

$$QH^*(\mathbb{P}^r) = \mathbb{Q}[H, q]/\langle H^{r+1} - q \rangle,$$

and we have

$$\mathbf{P} = H^r \quad \text{and} \quad \mathbf{E} = (r+1)H^r.$$

We identify $H_2(\mathbb{P}^r, \mathbb{Z})$ with \mathbb{Z} . The dimension constraint (1) on $g, d, n \geq 0$ is then

$$(r+1)d = r(n+g-1).$$

When satisfied, we have

$$\mathrm{vTev}_{g,d,n}^{\mathbb{P}^r} = \mathrm{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^d \mathbf{P}) = (r+1)^g.$$

2.2. Product rule. The virtual Tevelev degrees of a product of varieties are obtained from the virtual Tevelev degrees of the factors as follows.

Proposition 2.3. *Let X and Y be nonsingular projective varieties. For $g, n \geq 0$, $d_X \in H_2(X, \mathbb{Z})$, and $d_Y \in H_2(Y, \mathbb{Z})$ we have*

$$\mathrm{vTev}_{g, (d_X, d_Y), n}^{X \times Y} = \mathrm{vTev}_{g, d_X, n}^X \cdot \mathrm{vTev}_{g, d_Y, n}^Y.$$

Proof. The result is a direct consequence of the product rule in Gromov-Witten theory [Beh99]. Choose $k \geq 0$ such that $2g - 2 + n + k > 0$. Let $\mathbf{P}_X \in H^*(X)$ and $\mathbf{P}_Y \in H^*(Y)$ be the point classes. Then $\mathbf{P}_X \otimes \mathbf{P}_Y$ is the point class of $X \times Y$. By the product rule we have

$$I_{g, (d_X, d_Y), n+k}^{X \times Y} ((\mathbf{P}_X \otimes \mathbf{P}_Y)^{\otimes n} \otimes 1^{\otimes k}) = I_{g, d_X, n+k}^X (\mathbf{P}_X^{\otimes n} \otimes 1^{\otimes k}) \cdot I_{g, d_Y, n+k}^Y (\mathbf{P}_Y^{\otimes n} \otimes 1^{\otimes k})$$

in $H^*(\overline{\mathcal{M}}_{g, n+k})$. The claim therefore follows from (4). \square

The dimension constraint (1) may be satisfied for $\mathrm{vTev}_{g, (d_X, d_Y), n}^{X \times Y}$, but fail for $\mathrm{vTev}_{g, d_X, n}^X$ and $\mathrm{vTev}_{g, d_Y, n}^Y$. Then, all three invariants are zero.

Example 2.4. The virtual Tevelev degrees of the quadric surface $Q^2 \subset \mathbb{P}^3$ are easily determined by Proposition 2.3 since $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. If $2d_1 = 2d_2 = n + g - 1$, then

$$\mathrm{vTev}_{g, (d_1, d_2), n}^{Q^2} = 4^g,$$

where (d_1, d_2) denotes the bidegree of the curve class. In all other cases, the virtual Tevelev degree vanishes.

3. FLAG VARIETIES

3.1. Schubert varieties. Let G be a connected semi-simple linear algebraic group over \mathbb{C} and fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = T$. Let $W = N_G(T)/T$ be the Weyl group of G , and let Φ be the root system, with positive roots Φ^+ and simple roots Φ^S . Each root $\alpha \in \Phi$ defines a reflection $s_\alpha \in W$.

A flag variety of G is a projective variety X with a transitive G -action. Every flag variety X contains a unique B -stable point $\mathfrak{p}_B \in X$. We let $P_X \subset G$ denote the parabolic subgroup stabilizing \mathfrak{p}_B . We then have the identifications

$$X = G/P_X = \{g.P_X \mid g \in G\}, \quad \mathfrak{p}_B = 1.P_X.$$

Let $W_X = N_{P_X}(T)/T$ denote the Weyl group of P_X , and let $W^X \subset W$ be the set of minimal length representatives of the cosets in W/W_X . Every element $u \in W$ defines a B -stable Schubert variety $X_u = \overline{Bu.P_X}$ and an (opposite) B^- -stable Schubert variety $X^u = \overline{B^-u.P_X}$. For $u \in W^X$, we have

$$\dim(X_u) = \text{codim}(X^u, X) = \ell(u).$$

The set of Schubert classes $\{[X^u] \mid u \in W^X\}$ is a basis of the cohomology ring $H^*(X)$. The dual basis is $\{[X_u] \mid u \in W^X\}$, in the sense that $\int_X [X_u] \cdot [X^v] = \delta_{u,v}$. Let w_0^X denote the longest element in W^X . Then $[X^{w_0^X}]$ is the class of a point in X , and $\ell(w_0^X) = \dim(X)$. When no confusion is possible, we will also write $\mathbf{P} = [X^{w_0^X}]$ for the class of a point.

3.2. Quantum cohomology. Let $QH^*(X)_q$ denote the localized small quantum cohomology ring of X . This ring is an algebra over the Laurent polynomial ring

$$\mathbb{Q}[q^{\pm 1}] = \mathbb{Q}[q_\alpha^{\pm 1} \mid \alpha \in \Phi^S \setminus \Phi_X],$$

where $\Phi_X \subset \Phi$ is the root system of P_X .⁷ We have $QH^*(X)_q = H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q^{\pm 1}]$ as a $\mathbb{Q}[q^{\pm 1}]$ -module, and multiplication in $QH^*(X)_q$ is defined by

$$[X^u] \star [X^v] = \sum_{w,d} \langle [X^u], [X^v], [X_w] \rangle_d q^d [X^w],$$

where the sum is over all $w \in W^X$ and effective degrees $d \in H_2(X, \mathbb{Z})$. Here, we use the notation $q^d = \prod q_\alpha^{d_\alpha}$ for each $d \in H_2(X, \mathbb{Z})$, where $d_\alpha = \int_{X_{s_\alpha}} d$. The invariants $\langle [X^u], [X^v], [X_w] \rangle_d$ are enumerative, in particular they are non-negative integers (see [FP97]). By [FW04, Thm. 9.1], every quantum product $[X^u] \star [X^v]$ of Schubert classes is non-zero. The quantum Euler class of a flag variety is naturally expressed using the Schubert basis:

$$(6) \quad \mathbf{E} = \sum_{u \in W^X} [X_u] \star [X^u].$$

The ring $QH^*(X)_q$ has the \mathbb{Q} -basis

$$\mathcal{B} = \{q^d [X^u] \mid d \in H_2(X) \text{ and } u \in W^X\}.$$

For every $\mathbf{A} \in QH^*(X)$ and basis element $q^d [X^u] \in \mathcal{B}$, let

$$\text{Coeff}(\mathbf{A}, q^d [X^u])$$

denote the coefficient of $q^d [X^u]$ when \mathbf{A} is expanded in the basis \mathcal{B} . The usual (small) quantum cohomology ring $QH^*(X)$ is the subring $H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q] \subset QH^*(X)_q$ spanned by all elements $q^d [X^u]$ for which $d \in H_2(X, \mathbb{Z})$ is an effective degree.

Proposition 3.1. *For any flag variety X , we have $\text{vTev}_{g,d,n}^X \in \mathbb{Z}_{\geq 0}$ for all $g, n \geq 0$ and $d \in H_2(X, \mathbb{Z})$.*

⁷The quantum cohomology ring of a flag variety can also be defined with \mathbb{Z} -coefficients.

Proof. Using that the quantum product $[X^u] \star [X^v]$ of any two Schubert classes is a non-negative integer combination of the basis \mathcal{B} , it follows from (6) that $\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}$ is a non-negative integer combination of \mathcal{B} . The claim follows since $v\text{TeV}_{g,d,n}^X$ is one of the coefficients in the expansion of $\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}$ by [Theorem 1.3](#). \square

3.3. Seidel classes. Let $QH^*(X)/\langle q-1 \rangle$ denote the quotient of $QH^*(X)$ by the ideal generated by $q_\alpha - 1$ for $\alpha \in \Phi^S \setminus \Phi_X$.

Definition 3.2. A Schubert class $[X^v]$ is called a *Seidel class* if

$$[X^v]^{\star p} = q^d$$

for some integer $p \geq 1$ and $d \in H_2(X)$. A Seidel class $[X^v]$ is an element of finite order in the group of units $(QH^*(X)/\langle q-1 \rangle)^\times$. Let $\text{ord}([X^v])$ denote the order of a Seidel class.

Lemma 3.3. *Let $[X^v]$ be a Seidel class of X . We have the following properties:*

- (a) $[X^v] \star [X^u] \in \mathcal{B}$ for all $u \in W^X$,
- (b) $[X_v] \star [X^v] = \mathbf{P}$, the point class of X ,
- (c) $[X^v]$ is invertible in $QH^*(X)_q$ with inverse $[X^v]^{\star(-1)} \in \mathcal{B}$,
- (d) $\text{Coeff}([X^v] \star \mathbf{A}, [X^v] \star \mathbf{B}) = \text{Coeff}(\mathbf{A}, \mathbf{B})$ for all $\mathbf{A} \in QH^*(X)_q$ and $\mathbf{B} \in \mathcal{B}$.

Proof. Let $p = \text{ord}([X^v])$ and write $[X^v]^{\star p} = q^d$. Part (a) follows because any quantum product $[X^v] \star [X^w]$ is a non-zero linear combination of \mathcal{B} with non-negative integer coefficients, and $[X^v]^{\star p} \star [X^u] = q^d [X^u]$ is a single element of \mathcal{B} . Part (b) follows from (a) because $[X_v] \cdot [X^v] = \mathbf{P}$ in $H^*(X)$. Part (c) holds because $q^{-d} [X^v]^{\star(p-1)}$ is an inverse of $[X^v]$, and part (d) follows from (a) and (c). \square

3.4. Cominuscule flag varieties. A simple root $\gamma \in \Phi^S$ is called *cominuscule* if, when the highest root of Φ is written as a linear combination of simple roots, the coefficient of γ is one. The non-trivial Seidel classes of X correspond to cominuscule simple roots of G by some remarkable relationships.

A flag variety $Y = G/P_Y$ is called *cominuscule* if $P_Y \subset G$ is a maximal parabolic subgroup defined by excluding a cominuscule simple root,

$$\Phi_Y = \Phi^S \setminus \{\gamma\}.$$

It is proved in Bourbaki that the set

$$W^{\text{comin}} = \{w_0^Y \mid Y \text{ is a cominuscule flag variety of } G\} \cup \{1\}$$

is a subgroup of the Weyl group W . Furthermore, it is proved in [\[CMP09\]](#) that

$$[X^{w_0^Y}] \star [X^u] = q^{d(Y,u)} [X^{w_0^Y u}] \in QH^*(X)$$

for every $w_0^Y \in W^{\text{comin}}$ and $u \in W^X$, where the degree $d(Y,u)$ is explicitly given in terms of root data. In particular, the group W^{comin} acts on $QH^*(X)/\langle q-1 \rangle$ by $w_0^Y \cdot [X^u] = [X^{w_0^Y}] \star [X^u]$. We state the following special case.

Theorem 3.4 (Chaput, Manivel, Perrin). *Let X be any flag variety of G , and let Y be any cominuscule flag variety of G . Then, $[X^{w_0^Y}]$ is a Seidel class of X , and $\text{ord}([X^{w_0^Y}])$ is equal to the order of w_0^Y in W^{comin} .*

The following table shows the complete list of cominuscule flag varieties Y as well as $\text{ord}(w_0^Y)$ in each case.

| Name | Y | $\text{ord}(w_0^Y)$ |
|---------------------------------|--------------------|--------------------------|
| Grassmannian of type A | $\text{Gr}(m, N)$ | $N/\text{gcd}(m, N)$ |
| Lagrangian Grassmannian | $\text{LG}(N, 2N)$ | 2 |
| Maximal orthogonal Grassmannian | $\text{OG}(N, 2N)$ | 2 if N is even, else 4 |
| Quadric hypersurface | Q^r | 2 |
| Cayley Plane | E_6/P_6 | 3 |
| Freudenthal variety | E_7/P_7 | 2 |

If X is a cominuscule flag variety, then the point class $\mathbf{P} \in H^*(X)$ is a Seidel class by [Theorem 3.4](#).

Corollary 3.5. *Let X be any cominuscule flag variety. The virtual Tevelev degree*

$$\text{vTev}_{g,d,n}^X = \text{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^d \mathbf{P}) = \text{Coeff}(\mathbf{E}^{\star g}, q^d \mathbf{P}^{\star(1-n)})$$

depends on n modulo $\text{ord}(\mathbf{P}) = \text{ord}(w_0^X)$.

Corollary 3.6. *Let X be any cominuscule flag variety of dimension r . The virtual Tevelev degrees of X of genus 0 are determined by*

$$\text{vTev}_{0,d,n}^X = \begin{cases} 1 & \text{if } n \equiv 1 \text{ modulo } \text{ord}(\mathbf{P}) \text{ and } \int_d c_1(T_X) = r(n-1), \\ 0 & \text{otherwise.} \end{cases}$$

Some coefficients of the quantum Euler class \mathbf{E} of an arbitrary flag variety have the following combinatorial description.

Theorem 3.7. *Let X be any flag variety and assume that $[X_v]$ is a Seidel class. Then we have*

$$\text{Coeff}(\mathbf{E}, q^d [X^v]) = \#\{u \in W^X \mid [X_v] \star [X^u] = q^d [X^u]\}.$$

Proof. We have

$$\begin{aligned} \text{Coeff}(\mathbf{E}, q^d [X^v]) &= \sum_{u \in W^X} \text{Coeff}([X_u] \star [X^u], q^d [X^v]) \\ &= \sum_{u \in W^X} \langle [X_u], [X^u], [X_v] \rangle_d \\ &= \sum_{u \in W^X} \text{Coeff}([X_v] \star [X^u], q^d [X^u]). \end{aligned}$$

Since $[X_v]$ is a Seidel class, we have $[X_v] \star [X^u] \in \mathcal{B}$, hence

$$\text{Coeff}([X_v] \star [X^u], q^d [X^u]) = \begin{cases} 1 & \text{if } [X_v] \star [X^u] = q^d [X^u], \\ 0 & \text{otherwise,} \end{cases}$$

which proves the identity. \square

Corollary 3.8. *Let X be any cominuscule flag variety. The virtual Tevelev degrees of X of genus 1 are determined by*

$$\text{vTev}_{1,d,n}^X = \text{Coeff}(\mathbf{E}, q^d \mathbf{P}^{\star(1-n)}) = \#\{u \in W^X \mid \mathbf{P}^{\star n} \star [X^u] = q^d [X^u]\}.$$

Proof. The point class \mathbf{P} is a Seidel class of X by [Theorem 3.4](#). Write

$$q^d \mathbf{P}^{*(1-n)} = q^e [X^v] \in QH^*(X)_q.$$

Then, $[X^v]$ is a Seidel class. Since $[X_v] \star [X^v] = \mathbf{P}$, $[X_v]$ is also a Seidel class. The calculation

$$q^{-d} \mathbf{P}^{*n} \star q^e [X^v] = q^{-d} \mathbf{P}^{*n} \star q^d \mathbf{P}^{*(1-n)} = \mathbf{P} = q^{-e} [X_v] \star q^e [X^v]$$

shows that $q^{-d} \mathbf{P}^{*n} = q^{-e} [X_v]$. We therefore obtain

$$\begin{aligned} \text{Coeff}(\mathbf{P}^{*n} \star \mathbf{E}, q^d \mathbf{P}) &= \text{Coeff}(\mathbf{E}, q^e [X^v]) \\ &= \#\{u \in W^X \mid [X_v] \star [X^u] = q^e [X^u]\} \\ &= \#\{u \in W^X \mid \mathbf{P}^{*n} \star [X^u] = q^d [X^u]\}, \end{aligned}$$

as required. \square

3.5. Quadric hypersurfaces. Let $X = Q^r \subset \mathbb{P}^{r+1}$ be a quadric hypersurface of dimension $r \geq 3$. Then, Q^r is a cominuscule flag variety. The quantum cohomology ring $QH^*(Q^r)$ is a basic example in the subject, see e.g. [\[CMP08\]](#). Structure theorems for this ring are also special cases of results about the quantum cohomology of complete intersections proved in [Section 5](#). We have $\deg(q) = \deg(\mathbf{P})$, and the relation $\mathbf{P}^{*2} = q^2$ holds in $QH^*(Q^r)$ since three general points of Q^r define a plane which cuts Q^r in a conic. Define the constant

$$\delta = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases}$$

Then, $H^*(Q^r)$ has rank $r + \delta$. For $u \in W^X$, we have⁸

$$[X_u] \star [X^u] = \begin{cases} \mathbf{P} & \text{if } \ell(u) \in \{0, r/2, r\}, \\ \mathbf{P} + q & \text{otherwise.} \end{cases}$$

This follows by noting that $[X^u]$ is a Seidel class when $\ell(u) = r/2$ by [Theorem 3.4](#), and from the quantum Chevalley formula [\[FW04, Thm. 10.1\]](#) when $\ell(u) \neq r/2$. We compute the quantum Euler class of Q^r as

$$\mathbf{E} = \sum_{u \in W^X} [X_u] \star [X^u] = (r + \delta) \mathbf{P} + (r - \delta) q.$$

Theorem 3.9. *Let $g, d, n \geq 0$ satisfy the constraint $d = n + g - 1$. Then,*

$$\text{vTev}_{g,d,n}^{Q^r} = \frac{(2r)^g + (-1)^d (2\delta)^g}{2}.$$

Proof. Using the binomial formula

$$\mathbf{E}^{*g} = \sum_{i=0}^g \binom{g}{i} (r + \delta)^i (r - \delta)^{g-i} \mathbf{P}^{*i} q^{g-i}$$

and the relation $\mathbf{P}^{*2} = q^2$, we obtain

$$\text{vTev}_{g,d,n}^{Q^r} = \text{Coeff}(\mathbf{E}^{*g}, q^d \mathbf{P}^{*(1-n)}) = \sum_{\substack{0 \leq i \leq g \\ n+i \text{ odd}}} \binom{g}{i} (r + \delta)^i (r - \delta)^{g-i},$$

⁸The quantum multiplication here can also be deduced from the complete intersection analysis in [Section 5](#).

which matches the expansion of

$$\frac{((r + \delta) + (r - \delta))^g - (-1)^{n+g} ((r + \delta) - (r - \delta))^g}{2},$$

as required. \square

3.6. Grassmannians of type A. For the remaining cominuscule flag varieties we give simple formulas for the virtual Tevelev degrees of genus 1 based on [Theorem 3.7](#). More generally, we give formulas for all numbers $\text{Coeff}(\mathbf{E}, q^e[X^v])$ for which $[X^v]$ is a Seidel class.

Let $X = \text{Gr}(m, N)$ be the Grassmannian of m -planes in \mathbb{C}^N . The dimension of X is $r = m(N - m)$. The elements of W^X can be identified with partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_m \geq 0)$ for which $\lambda_1 \leq N - m$. The corresponding opposite Schubert variety is given by

$$X^\lambda = \{V \in X \mid \dim(V \cap \mathbb{C}^{N-m+i-\lambda_i}) \geq i \text{ for } 1 \leq i \leq m\},$$

where $\mathbb{C}^k \subset \mathbb{C}^N$ denotes the B^- -stable subspace of dimension k . The point class is $\mathbf{P} = [X^{((N-m)^m)}]$, where (a^b) denotes the partition (a, \dots, a) with b copies of a .

The quantum cohomology ring $QH^*(X)$ was computed in [[Wit95](#), [Ber97](#)]. Elementary proofs of the facts we need can be found in [[Buc03](#)]. The grading of $QH^*(X)$ is determined by

$$\deg[X^\lambda] = 2|\lambda| = 2 \sum \lambda_i$$

and $\deg(q) = 2N$. The subgroup of Seidel classes in $(QH^*(X)/(q-1))^\times$ is generated by the class $[X^{(1^m)}]$, the top Chern class of the dual of the tautological subbundle on X . We have $\text{ord}([X^{(1^m)}]) = N$, and the powers of $[X^{(1^m)}]$ are given by

$$[X^{(1^m)}]^{*k} = \begin{cases} [X^{(k^m)}] & \text{if } 0 \leq k \leq N - m, \\ q^{k-N+m} [X^{((N-m)^{N-k})}] & \text{if } N - m \leq k \leq N. \end{cases}$$

Theorem 3.10. *Let $X = \text{Gr}(m, N)$, and let $k, d \in \mathbb{Z}$ satisfy $Nd + mk = r$. Then,*

$$\text{Coeff}(\mathbf{E}, q^d [X^{(1^m)}]^{*k}) = \binom{cN/m}{c},$$

where $c = \gcd(d, m)$.

Corollary 3.11. *Let $n, d \geq 0$ satisfy $Nd = rn$. Then,*

$$\text{vTev}_{1,d,n}^{\text{Gr}(m,N)} = \binom{cN/m}{c},$$

where $c = \gcd(d, m)$.

Proof. The result follows from [Theorem 3.10](#) since

$$\text{Coeff}(\mathbf{P}^{*n} \star \mathbf{E}, q^d \mathbf{P}) = \text{Coeff}(\mathbf{E}, q^d \mathbf{P}^{*(1-n)}) = \text{Coeff}(\mathbf{E}, q^d [X^{(1^m)}]^{*k}),$$

where $k = (N - m)(1 - n)$. \square

We will prove [Theorem 3.10](#) using Postnikov's cylindrical model of the basis

$$\mathcal{B} = \{q^d [X^\lambda]\}$$

of $QH^*(X)_q$. Define the set

$$\mathcal{P}_X = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq N - m\}.$$

We identify \mathcal{P}_X with a rectangle of boxes with m rows and $N-m$ columns. The pair (i, j) represents the box in row i and column j , where the row number i increases from top to bottom and the column number j increases from left to right. Let \leq be the *north-west to south-east* partial order on \mathcal{P}_X , defined by $(i', j') \leq (i'', j'')$ if and only if $i' \leq i''$ and $j' \leq j''$. A partition λ as above corresponds to the (lower) order ideal in \mathcal{P}_X of boxes (i, j) for which $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i$.

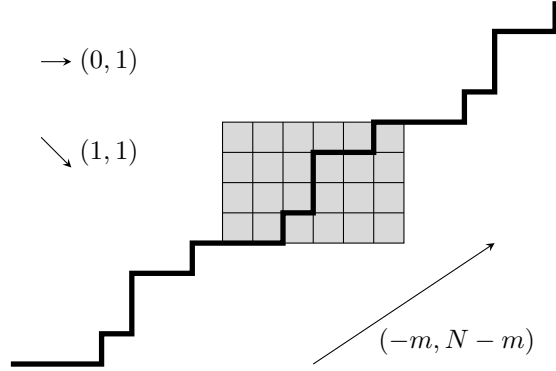
More generally, the elements $q^d[X^\lambda]$ of \mathcal{B} correspond to (proper, non-empty, lower) order ideals in Postnikov's *cylinder* [Pos05],

$$\widehat{\mathcal{P}}_X = \mathbb{Z}^2 / \mathbb{Z}(-m, N-m).$$

In other words, we extend the rectangle \mathcal{P}_X to the plane \mathbb{Z}^2 , but identify two boxes if they differ by an integer multiple of the vector $(-m, N-m)$. The order ideal of $1 \in \mathcal{B}$ is the set $I_0 = \{(i, j) \in \widehat{\mathcal{P}}_X \mid (i, j) \not\leq (1, 1)\}$. Equivalently, I_0 is the set of boxes of $\widehat{\mathcal{P}}_X \setminus \mathcal{P}_X$ that are smaller than some box in \mathcal{P}_X . The order ideal of $[X^\lambda]$ is the union of I_0 with the order ideal of λ in \mathcal{P}_X .

An order ideal of $\widehat{\mathcal{P}}_X$ is determined by its border, which is a path of horizontal and vertical line segments of unit length, with the property that the path is invariant under translation by the vector $(-m, N-m)$, see Figure 1. The order ideal of $q^d[X^\lambda]$ is obtained by translating the border of the order ideal of $[X^\lambda]$ by the vector (d, d) . In particular, multiplication by q corresponds to translation by $(1, 1)$. Similarly, quantum multiplication by $[X^{(1^m)}]$ corresponds to translation by $(0, 1)$.

FIGURE 1. The border of the order ideal of $[X^{(5,3,3,2)}]$ in the cylinder $\widehat{\mathcal{P}}_X$ of $X = \text{Gr}(4, 6)$. The boxes of \mathcal{P}_X are colored gray.



Proof of Theorem 3.10. Since $[X^{(1^m)}]^{*(N-m-k)} \star [X^{(1^m)}]^{*k} = \mathbf{P}$, it follows from Theorem 3.7 that $\text{Coeff}(\mathbf{E}, q^d[X^{(1^m)}]^{*k})$ is equal to the number of Schubert classes $[X^\lambda]$ for which

$$q^{-d}[X^{(1^m)}]^{*(N-m-k)} \star [X^\lambda] = [X^\lambda].$$

Since multiplication by q^{-d} corresponds to translation of paths by the vector $(-d, -d)$, and multiplication by $[X^{(1^m)}]^{*(N-m-k)}$ corresponds to translation by $(0, N-m-k)$, we deduce that $\text{Coeff}(\mathbf{E}, q^d[X^{(1^m)}]^{*k})$ is equal to the number of paths in the plane that go through the upper-right corner of \mathcal{P}_X and are invariant under translation by both of the vectors $(-d, N-m-k-d)$ and $(-m, N-m)$.

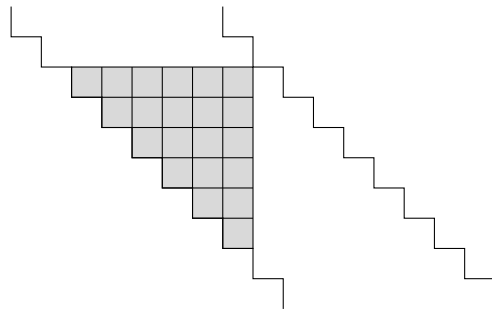
Notice that the constraint $Nd + mk = m(N - m)$ says that these vectors are parallel. We therefore must count the number of paths through the upper-right corner of \mathcal{P}_X that are invariant under translation by the greatest common divisor of the two vectors, which is the vector $(-c, (N - m)c/m)$, where $c = \gcd(d, m)$. Since such a path is determined by the first cN/m steps, and exactly c of these steps must be vertical, there are $\binom{cN/m}{c}$ such paths. \square

3.7. Lagrangian Grassmannians. Let $X = \text{LG}(N, 2N)$ be the Lagrangian Grassmannian of maximal isotropic subspaces in a symplectic vector space of dimension $2N$ over \mathbb{C} . The quantum cohomology ring $QH^*(X)$ was computed in [KT03], and elementary proofs can be found in [BKT03]. We have $\dim(X) = \frac{1}{2}N(N + 1)$, $\deg(q) = 2N + 2$, and $\mathbf{P}^{\star 2} = q^N$ in $QH^*(X)$. Since the only Seidel classes of X are 1 and \mathbf{P} , all coefficients of \mathbf{E} at Seidel classes are given by the following result.

Theorem 3.12. *Let $n, d \geq 0$ satisfy $2d = nN$. Then,*

$$\text{vTev}_{1,d,n}^{\text{LG}(N,2N)} = \begin{cases} \text{Coeff}(\mathbf{E}, \mathbf{P}) = 2^N & \text{if } n \text{ is even;} \\ \text{Coeff}(\mathbf{E}, q^{N/2}) = 2^{N/2} & \text{if } n \text{ is odd.} \end{cases}$$

FIGURE 2. The partially ordered set $\widehat{\mathcal{P}}_X$ for $X = \text{LG}(6, 12)$, with the boxes of \mathcal{P}_X colored gray.



We use a generalization of Postnikov's cylinder constructed in [BCMP21]. The Schubert classes of $\text{LG}(N, 2N)$ correspond to order ideals in a triangular set of boxes $\mathcal{P}_X = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq j \leq N\}$, where the row number i increases from top to bottom, and the column number j increases from left to right. The partial order is the north-west to south-east order, defined by $(i', j') \leq (i'', j'')$ if and only if $i' \leq i''$ and $j' \leq j''$. The elements of \mathcal{B} correspond to (non-empty, proper, lower) order ideals in the larger set $\widehat{\mathcal{P}}_X = \{(i, j) \in \mathbb{Z}^2 \mid i \leq j \leq i + N\}$, see Figure 2. Multiplication by q moves the border of an order ideal one unit diagonally in south-east direction. Given an order ideal $\lambda \subset \mathcal{P}_X$, the order ideal of $\mathbf{P} \star [X^\lambda]$ is obtained by reflecting λ in a diagonal and attaching it to the right side of \mathcal{P}_X , see Figure 3.

Proof. If n is even, the formula states that $H^*(X)$ has rank 2^N . Assume that n is odd. Then N must be even, and $\text{Coeff}(\mathbf{E}, q^{N/2})$ is the number of order ideals $\lambda \subset \mathcal{P}_X$ for which $q^{-N/2} \mathbf{P} \star [X^\lambda] = [X^\lambda]$. This holds if and only if the border of λ is a path from the upper-right corner of \mathcal{P}_X to the middle point on the south-west side of \mathcal{P}_X , and this path must be symmetric under reflections in the diagonal, see

Figure 4. The path is therefore determined by the first $N/2$ steps, so there are $2^{N/2}$ such paths. \square

FIGURE 3. The order ideal of $\mathbf{P} \star [X^\lambda]$ is obtained from the order ideal of $[X^\lambda]$ as follows, for $\lambda = (4, 1)$.

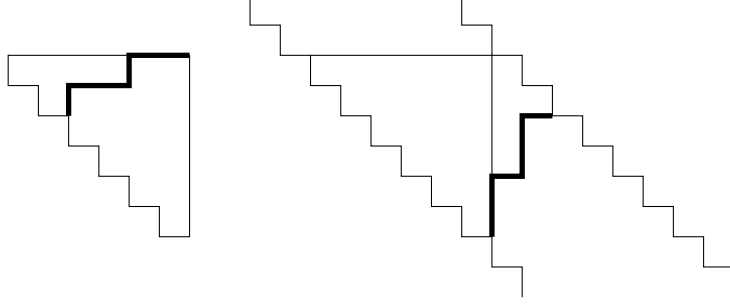
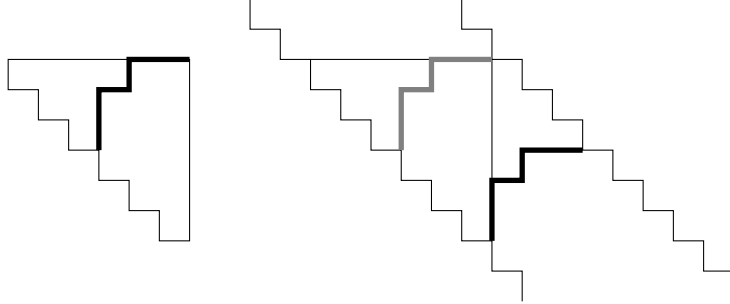


FIGURE 4. The border of the order ideal of $[X^{(4,2,1)}]$ is symmetric under reflection in the diagonal. Equivalently, $q^{-N/2} \mathbf{P} \star [X^{(4,2,1)}] = [X^{(4,2,1)}]$ in $QH^*(X)$.



3.8. Maximal orthogonal Grassmannians. Let $X = \text{OG}(N, 2N)$ be the Grassmannian parametrizing (one component of) maximal isotropic subspaces in an orthogonal complex vector space of dimension $2N$. The quantum cohomology ring $QH^*(X)$ was computed in [KT04], elementary proofs can be found in [BKT03]. We have $\dim(X) = \frac{1}{2}N(N-1)$ and $\deg(q) = 4N-4$. The Seidel classes⁹ of X are 1 , \mathbf{P} , $[X^{N-1}]$, and $\mathbf{P}/[X^{N-1}]$. Here $X^{N-1} \cong \text{OG}(N-1, 2N-2)$ denotes the Schubert variety of maximal isotropic subspaces that contain a fixed isotropic vector. We have $[X^{N-1}]^{\star 2} = q$ and $\mathbf{P}^{\star 4} = q^N$. If N is even, we furthermore have $\mathbf{P}^{\star 2} = q^{N/2}$. Let $\mathbf{E}_{q=1}$ denote the image of \mathbf{E} in $QH^*(X)/(q-1)$.

⁹We use a quotient \mathbf{A}/\mathbf{B} of cohomology classes only if \mathbf{B} is invertible in the localized quantum cohomology ring, in which case it should be interpreted as $\mathbf{A}/\mathbf{B} = \mathbf{A} \star \mathbf{B}^{\star(-1)}$.

Theorem 3.13. *Let $X = \text{OG}(N, 2N)$. We have*

$$\begin{aligned} \text{Coeff}(\mathbf{E}_{q=1}, \mathbf{P}) &= 2^{N-1}, \\ \text{Coeff}(\mathbf{E}_{q=1}, 1) &= \begin{cases} \text{Coeff}(\mathbf{E}, q^{N/4}) = 2^{N/2} & \text{if } N \equiv 0 \pmod{4}, \\ 0 & \text{otherwise,} \end{cases} \\ \text{Coeff}(\mathbf{E}_{q=1}, [X^{N-1}]) &= \begin{cases} \text{Coeff}(\mathbf{E}, q^{\frac{N-2}{4}} [X^{N-1}]) = 2^{N/2} & \text{if } N \equiv 2 \pmod{4}, \\ 0 & \text{otherwise, and} \end{cases} \\ \text{Coeff}(\mathbf{E}_{q=1}, \mathbf{P}/[X^{N-1}]) &= 0. \end{aligned}$$

Proof. The coefficient of \mathbf{P} is the topological Euler characteristic of X , and the coefficient of $\mathbf{P}/[X^{N-1}]$ is zero since $\deg [X^{N-1}]$ is not a multiple of $\deg(q)$. The two remaining cases are equivalent to the identity

$$\text{Coeff}(\mathbf{E}, [X^{N-1}]^{\star N/2}) = 2^{N/2}$$

when N is even. The elements of W^X can be identified with order ideals in

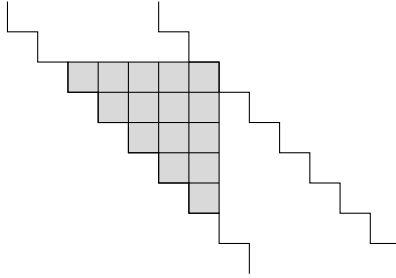
$$\mathcal{P}_X = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq j \leq N-1\},$$

and elements of \mathcal{B} correspond to order ideals in

$$\widehat{\mathcal{P}}_X = \{(i, j) \in \mathbb{Z}^2 \mid i \leq j \leq i + N - 2\},$$

see [BCMP21] and Figure 5. Multiplication by $[X^{N-1}]$ is given by translating borders diagonally by one unit, and multiplication by a point is given by the same procedure as for Lagrangian Grassmannians, except the reflected shape will be attached to \mathcal{P}_X one row lower. The number $\text{Coeff}(\mathbf{E}, [X^{N-1}]^{\star N/2})$ is equal to the number of order ideals $\lambda \subset \mathcal{P}_X$ for which $[X^{N-1}]^{\star(-N/2)} \star \mathbf{P} \star [X^\lambda] = [X^\lambda]$. This holds if and only if the border of λ is a path from the upper-right corner of \mathcal{P}_X to the middle outer corner of the south-west side of \mathcal{P}_X that is symmetric under reflection in the diagonal. There are $2^{N/2}$ such paths. \square

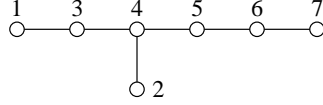
FIGURE 5. The partially ordered set $\widehat{\mathcal{P}}_X$ for $X = \text{OG}(6, 12)$, with the boxes of \mathcal{P}_X colored gray.



Corollary 3.14. *Let $n, d \geq 0$ satisfy $4d = nN$. Then,*

$$\text{vTev}_{1,d,n}^{\text{OG}(N,2N)} = \begin{cases} \text{Coeff}(\mathbf{E}, \mathbf{P}) = 2^{N-1} & \text{if } n \text{ is even;} \\ \text{Coeff}(\mathbf{E}, q^{N/4}) = 2^{N/2} & \text{if } n \text{ is odd.} \end{cases}$$

3.9. Exceptional cominuscule flag varieties. There are two exceptional cominuscule flag varieties, the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7 . The quantum cohomology of these spaces is known from [CMP08]. The following results account for the virtual Tevelev degrees of genus 1, and use the standard ordering of the simple roots:



The quantum Euler classes were obtained using the Equivariant Schubert Calculator [Buc]. More general formulas for the virtual Tevelev degrees of arbitrary genus are included among the examples in Section 4.5.

Theorem 3.15. *Let $X = E_6/P_6$ be the Cayley plane. We have $\dim(X) = 16$ and $P^{*3} = q^4$ in $QH^*(X)$. For $n, d \geq 0$ satisfying $3d = 4n$, we have*

$$\text{vTev}_{1,d,n}^X = \text{Coeff}(\mathbf{E}, \mathbf{P}) = \chi(E_6/P_6) = 27.$$

The quantum Euler class of the Cayley plane is

$$\mathbf{E} = 27 \mathbf{P} + 27 q[X^{s_2 s_4 s_5 s_6}] + 45 q[X^{s_3 s_4 s_5 s_6}].$$

Theorem 3.16. *Let $X = E_7/P_7$ be the Freudenthal variety. We have $\dim(X) = 27$ and $P^{*2} = q^3$ in $QH^*(X)$. For $n, d \geq 0$ satisfying $2d = 3n$, we have*

$$\text{vTev}_{1,d,n}^X = \text{Coeff}(\mathbf{E}, \mathbf{P}) = \chi(E_7/P_7) = 56.$$

The quantum Euler class of the Freudenthal variety is

$$\mathbf{E} = 56 \mathbf{P} + 160 q[X^{s_6 s_5 u}] + 272 q[X^{s_5 s_1 u}] + 160 q[X^{s_3 s_1 u}],$$

where $u = s_4 s_3 s_2 s_4 s_5 s_6 s_7$.

4. STRANGE SYMMETRY OF \mathbf{E}/\mathbf{P}

4.1. Overview. Let $X = G/P_X$ be a cominuscule flag variety. Quantum multiplication by \mathbf{E}/\mathbf{P} preserves the degree grading of $QH^*(X)_q$. Let

$$[\mathbf{E}/\mathbf{P}]_k : QH^*(X)_{q,k} \rightarrow QH^*(X)_{q,k}$$

denote the restriction of quantum multiplication by \mathbf{E}/\mathbf{P} to the degree k subspace $QH^*(X)_{q,k} \subset QH^*(X)_q$. A *strange inner product* on $QH^*(X)_q$ which respects the degree grading is defined by the *strange duality involution*

$$\iota : QH^*(X)_q \rightarrow QH^*(X)_q$$

of [Pos05, CMP07]. We will prove that quantum multiplication by \mathbf{E}/\mathbf{P} in $QH^*(X)_q$ is self-adjoint with respect to the strange inner product and that *all* virtual Tevelev degrees of X can be expressed in terms of the eigenvalues and eigenvectors of the degree 0 operator $[\mathbf{E}/\mathbf{P}]_0$.

4.2. Strange duality. Let $\alpha_0 \in \Phi$ denote the highest root. Given any simple root $\beta \in \Phi^S$, let $n_\beta(\alpha_0)$ denote the coefficient of β obtained when α_0 is expanded in the basis of simple roots, and define

$$\epsilon(\beta) = \begin{cases} 1 & \text{if } \beta \text{ is a long root,} \\ -1 & \text{if } \beta \text{ is a short root.} \end{cases}$$

If the root system Φ is simply-laced, then all roots are long by convention. Given a minimal representative $u \in W^X$ and a reduced expression $u = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$ we set

$$y(u) = \prod_{i=1}^{\ell} n_{\beta_i}(\alpha_0)^{\epsilon(\beta_i)}.$$

Minimal representatives of cominuscule Schubert varieties are known to be fully commutative [Ste96], in the sense that any reduced expression can be obtained from any other by interchanging commuting simple reflections. It follows that the rational number $y(u)$ does not depend on the chosen reduced expression of u . More generally, given any Weyl group element $w \in W$, we set

$$y(w) = y(u)$$

where $u \in W^X \cap wW_X$ is the unique minimal representative of the coset wW_X . If $X = \text{Gr}(m, N)$ is a Grassmannian of type A, then $y(w) = 1$ for all $w \in W$.

Let $\delta(u)$ denote the minimal degree of a rational curve in X through $1.P_X$ and $u.P_X$. When $u \in W^X$, this is the number of occurrences of the cominuscule simple root γ in any reduced expression of u [CMP08, Prop. 18], and by [CMP09, Thm. 1] we have

$$(7) \quad \mathbf{P} \star [X^u] = q^{\delta(u)} [X^{w_0^X u}].$$

Let $w_{0,X} \in W_X$ be the longest element in the Weyl group of P_X , and let $w_0^X \in W^X$ be the minimal representative of the point class. Then $w_0^X w_{0,X} = w_0$ is the longest element in W .

The involution ι of the following result is called the *strange duality involution*. It was constructed for Grassmannians of type A in [Pos05, Thm. 6.5] and generalized to all cominuscule flag varieties in [CMP07, Thm. 1.1].

Theorem 4.1. *Let X be any cominuscule flag variety. There is a well-defined ring involution $\iota : QH^*(X)_q \rightarrow QH^*(X)_q$ given by*

$$\iota(q) = y(s_{\alpha_0}) q^{-1} \quad \text{and} \quad \iota[X^u] = y(u) q^{-\delta(u)} [X^{w_{0,X} u}].$$

Given $u \in W^X$ we denote the dual Weyl group element by $u^\vee = w_{0,X} u w_0^X \in W^X$. We have $X^{u^\vee} = w_{0,X} X_u$ as subvarieties of X , in particular $[X^{u^\vee}] = [X_u]$.

Lemma 4.2. *For $u \in W^X$ we have $\iota[X^u] = y(u) [X_u]/\mathbf{P}$.*

Proof. By Theorem 4.1 and (7) we have

$$\mathbf{P} \star \iota[X^u] = y(u) q^{-\delta(u)} \mathbf{P} \star [X^{w_{0,X} u}] = y(u) [X^{w_0^X w_{0,X} u}] = y(u) [X_u],$$

which is equivalent to the Lemma. \square

Definition 4.3. For $A, B \in QH^*(X)_q$, the *strange inner product* is

$$(A, B) = \text{Coeff}(A \star \iota(B), 1).$$

The strange norm of A is the real number $|A| = \sqrt{(A, A)}$.

The strange inner product is the least strange in type A, where the Schubert basis \mathcal{B} is an orthonormal basis of $QH^*(X)_q$.

Corollary 4.4. *The pairing $(-, -)$ of Definition 4.3 is an inner product on the \mathbb{Q} -vector space $QH^*(X)_q$. The set $\mathcal{B} = \{q^d[X^u] \mid u \in W^X, d \in \mathbb{Z}\}$ is an orthogonal basis of $QH^*(X)_q$, and the basis elements have norms given by*

$$|q^d[X^u]|^2 = y(s_{\alpha_0})^d y(u).$$

Proof. Since $\iota(1) = 1$ and $\iota(q^d[X^u])$ is a non-zero multiple of an element of \mathcal{B} for all $q^d[X^u] \in \mathcal{B}$, it follows that

$$\text{Coeff}(\iota(A), 1) = \text{Coeff}(A, 1)$$

for each $A \in QH^*(X)_q$. This implies that the pairing $(-, -)$ is symmetric, and the pairing is bilinear by definition. For $u, v \in W^X$ and $d, e \in \mathbb{Z}$ we have

$$\begin{aligned} (q^d[X^u], q^e[X^v]) &= \text{Coeff}(q^d[X^u] \star \iota(q^e[X^v]) \star \mathbf{P}, \mathbf{P}) \\ &= y(s_{\alpha_0})^e y(v) \text{Coeff}([X^u] \star [X^v], q^{e-d}\mathbf{P}) \\ &= y(s_{\alpha_0})^e y(v) \langle [X^u], [X^v], 1 \rangle_{0, e-d}^X \\ &= y(s_{\alpha_0})^e y(v) \delta_{u,v} \delta_{d,e}. \end{aligned}$$

This shows that \mathcal{B} is an orthogonal basis of $QH^*(X)_q$ and reveals the norms of the basis elements. \square

Lemma 4.5. *For $u \in W^X$ we have $y(u^\vee) y(u) = y(w_0^X)$.*

Proof. Let $u = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$ and $u^\vee = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ be reduced expressions. We have

$$(w_0 u^\vee w_0)^{-1} u = w_0 w_{0,X} u^{-1} w_0 w_0 u = w_0 w_{0,X} = w_0^X.$$

Since $\ell(w_0^X) = \ell(u^\vee) + \ell(u)$, we deduce that

$$w_0^X = s_{-w_0 \cdot \alpha_k} \cdots s_{-w_0 \cdot \alpha_1} s_{\beta_1} \cdots s_{\beta_\ell}$$

is a reduced expression for w_0^X . The Lemma follows from this by observing that $n_\beta(\alpha_0) = n_{-w_0 \cdot \beta}(-w_0 \cdot \alpha_0) = n_{-w_0 \cdot \beta}(\alpha_0)$ for any simple root β . \square

Proposition 4.6. *We have $\iota(\mathbf{E}/\mathbf{P}) = \mathbf{E}/\mathbf{P}$.*

Proof. For $u \in W^X$ we have by Lemma 4.2 and Lemma 4.5 that

$$\iota([X_u] \star [X^u]/\mathbf{P}) = y(u^\vee)[X^u]/\mathbf{P} \star y(u)[X_u]/\mathbf{P} \star y(w_0^X)^{-1}\mathbf{P} = [X^u] \star [X_u]/\mathbf{P}.$$

The Proposition follows from this by taking the sum over $u \in W^X$. \square

4.3. Virtual Tevelev degrees. Given any element $A \in QH^*(X)_q$, we will abuse notation and identify A with the linear endomorphism

$$A\star : QH^*(X)_q \rightarrow QH^*(X)_q$$

defined by quantum multiplication by A . Statements about diagonalizability, eigenvalues, and eigenvectors of A should be interpreted in this sense. The real numbers \mathbb{R} can be replaced by an appropriate finite extension of \mathbb{Q} in the following result.

Corollary 4.7. *The vector space $QH^*(X)_q \otimes \mathbb{R}$, endowed with the strange inner product $(-, -)$, has an orthogonal basis consisting of eigenvectors of \mathbf{E}/\mathbf{P} .*

Proof. For $A, B \in QH^*(X)_q$, [Proposition 4.6](#) shows that

$$(E/P \star A, B) = (A, E/P \star B),$$

that is, the endomorphism E/P is self-adjoint with respect to the inner product on $QH^*(X)_q$. Since E/P has degree zero in the graded ring $QH^*(X)_q$, it preserves the homogeneous components $QH^*(X)_{q,k}$ of this ring. Since these components have finite dimension, it follows that E/P is diagonalizable over \mathbb{R} by an orthogonal basis. \square

Theorem 4.8. *Let X be a cominuscule flag variety, and let*

$$A_1, A_2, \dots, A_k$$

be any orthogonal basis of $QH^(X)_{q,0} \otimes \mathbb{R}$ with respect to the strange inner product, consisting of eigenvectors of $[E/P]_0$, and let*

$$\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$$

be the corresponding eigenvalues. For all $g, d, n \geq 0$, we have

$$\begin{aligned} \text{vTev}_{g,d,n}^X &= \text{Coeff}((E/P)^{\star g}, q^d \mathbf{P}^{\star(1-n-g)}) \\ &= \sum_{i=1}^k \frac{\text{Coeff}(A_i, 1) \text{Coeff}(A_i, q^d \mathbf{P}^{\star(1-n-g)})}{|A_i|^2} \lambda_i^g. \end{aligned}$$

Proof. The vector space $QH^*(X)_{q,0} \otimes \mathbb{R}$ has an orthogonal basis of eigenvectors of $[E/P]_0$ by [Corollary 4.7](#). The formula follows from the identity

$$(E/P)^{\star g} \star 1 = (E/P)^{\star g} \star \sum_{i=1}^k \frac{(A_i, 1)}{|A_i|^2} A_i = \sum_{i=1}^k \frac{\text{Coeff}(A_i, 1)}{|A_i|^2} \lambda_i^g A_i$$

by extracting the coefficient of $q^d \mathbf{P}^{\star(1-n-g)}$ on both sides. \square

It was proved in [\[CMP10\]](#) that the ring $QH^*(X)/\langle q-1 \rangle$ is semi-simple, which implies that quantum multiplication by any element is diagonalizable over \mathbb{C} . In particular, part (b) of the following result was known.

Corollary 4.9. *Let X be a cominuscule flag variety.*

- (a) *Quantum multiplication by E/P on $QH^*(X)/\langle q-1 \rangle$ is diagonalizable over \mathbb{R} .*
- (b) *Quantum multiplication by E on $QH^*(X)/\langle q-1 \rangle$ is diagonalizable over \mathbb{C} .*
- (c) *Quantum multiplication by $E^{\text{ord}(P)}$ on $QH^*(X)/\langle q-1 \rangle$ is diagonalizable over \mathbb{R} .*

Proof. Part (a) follows from [Corollary 4.7](#). Since P is a Seidel class, quantum multiplication by P is an idempotent operation on $QH^*(X)/\langle q-1 \rangle$, hence diagonalizable over \mathbb{C} . Part (b) therefore follows from (a) and the commutativity of the ring $QH^*(X)$. Part (c) follows from (a) since $E^{\text{ord}(P)} = (E/P)^{\star \text{ord}(P)}$ in $QH^*(X)/\langle q-1 \rangle$. \square

[Theorem 4.8](#) is more efficient in practice to use than semi-simplicity since only the eigenvalues and eigenvectors of a single operator $[E/P]_0$ on $QH^*(X)_{q,0}$ are required instead of the entire system of idempotents of $QH^*(X)$.

Remark 4.10. The cominuscule flag variety X may satisfy the property that

$$(8) \quad \int_d c_1(T_X) = r(n+g-1) \quad \text{implies} \quad q^d \mathbf{P}^{*(1-n-g)} = 1$$

for all $g, d, n \in \mathbb{Z}$, where $r = \dim(X)$. If (8) holds, then the virtual Tevelev degree $\text{vTev}_{g,d,n}^X$ depends only on g when the dimension constraint (1) is satisfied. Property (8) holds for X if and only if

$$\text{ord}(P) \text{ divides } \frac{d_q}{\gcd(d_q, r)}, \quad \text{where } d_q = \frac{1}{2} \deg(q) = \int_{\text{line}} c_1(T_X),$$

with the integral over the positive generator of $H_2(X, \mathbb{Z})$. When $X = \text{Gr}(m, N)$ is a Grassmannian, the above condition is equivalent to

$$\gcd(r, N) \text{ divides } m,$$

where $r = \dim(X) = m(N-m)$. The Grassmannians of dimension $r < 20$ that do not satisfy this condition are $\text{Gr}(2, 4)$, $\text{Gr}(2, 8)$, $\text{Gr}(4, 8)$, and $\text{Gr}(3, 9)$. Property (8) fails for all quadrics Q^r , is satisfied for the Lagrangian Grassmannian $\text{LG}(N, 2N)$ if and only if N is odd, and is satisfied for the maximal orthogonal Grassmannian $\text{OG}(N, 2N)$ if and only if N is not divisible by 4.

4.4. Grassmannians. [Theorem 4.8](#) provides an effective method to calculate virtual Tevelev degrees. After revisiting the quadric Q^r , including $Q^4 = \text{Gr}(2, 4)$, we fully calculate $\text{Gr}(2, 5)$, $\text{Gr}(2, 6)$, and $\text{Gr}(3, 6)$. For $\text{Gr}(2, 8)$, $\text{Gr}(3, 8)$, and $\text{Gr}(4, 8)$, we just present the final formulas (obtained by computer algebra and using the Equivariant Schubert Calculator [\[Buc\]](#)).

Example 4.11. Let $X = Q^r$ be the quadric of dimension r . By [Section 3.5](#) we have

$$\mathbf{E}/\mathbf{P} = (r + \delta) + (r - \delta) q^{-1} \mathbf{P},$$

where $\delta = 1$ if r is odd, whereas $\delta = 2$ if r is even. An orthogonal basis of $QH^*(X)_{q,0}$ consisting of eigenvectors of \mathbf{E}/\mathbf{P} is given by

$$\mathbf{A}_1 = 1 + q^{-1} \mathbf{P} \quad \text{and} \quad \mathbf{A}_2 = 1 - q^{-1} \mathbf{P},$$

and the corresponding eigenvalues are

$$\lambda_1 = 2r \quad \text{and} \quad \lambda_2 = 2\delta.$$

Noting that

$$\text{Coeff}(\mathbf{A}_1, q^d \mathbf{P}^{*(-d)}) = 1 \quad \text{and} \quad \text{Coeff}(\mathbf{A}_2, q^d \mathbf{P}^{*(-d)}) = (-1)^d,$$

we obtain from [Theorem 4.8](#) that

$$\text{vTev}_{g,d,n}^{Q^r} = \frac{1}{2} \lambda_1^g + \frac{(-1)^d}{2} \lambda_2^g = \frac{(2r)^g + (-1)^d (2\delta)^g}{2}$$

whenever $g, d, n \geq 0$ satisfy $d = n + g - 1$.

Example 4.12. Let $X = \text{Gr}(2, 5)$. Quantum multiplication by \mathbf{E}/\mathbf{P} on the basis

$$\{1, q^{-1}[X^{(3,2)}]\}$$

of $QH^*(X)_{q,0}$ is given by

$$\begin{aligned} \mathbf{E}/\mathbf{P} \star 1 &= 10 + 5 q^{-1}[X^{(3,2)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(3,2)}] &= 5 + 15 q^{-1}[X^{(3,2)}]. \end{aligned}$$

The eigenvalues of $[\mathbf{E}/\mathbf{P}]_0$ are

$$\lambda_1 = \frac{25 + 5\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{25 - 5\sqrt{5}}{2},$$

and corresponding eigenvectors are

$$\mathbf{A}_1 = 2 + (1 + \sqrt{5}) q^{-1}[X^{(3,2)}] \quad \text{and} \quad \mathbf{A}_2 = 2 + (1 - \sqrt{5}) q^{-1}[X^{(3,2)}].$$

For $g, d, n \geq 0$ satisfying $5d = 6(n + g - 1)$, we obtain from [Theorem 4.8](#) and [Remark 4.10](#) that

$$\begin{aligned} \text{vTev}_{g,d,n}^{\text{Gr}(2,5)} &= \sum_{i=1}^2 \frac{\text{Coeff}(\mathbf{A}_i, 1)^2}{|\mathbf{A}_i|^2} \lambda_i^g \\ &= \frac{5 - \sqrt{5}}{10} \left(\frac{25 + 5\sqrt{5}}{2} \right)^g + \frac{5 + \sqrt{5}}{10} \left(\frac{25 - 5\sqrt{5}}{2} \right)^g. \end{aligned}$$

Example 4.13. Let $X = \text{Gr}(2, 6)$. Quantum multiplication by \mathbf{E}/\mathbf{P} on the Schubert basis of $QH^*(X)_{q,0}$ is given by

$$\begin{aligned} \mathbf{E}/\mathbf{P} \star 1 &= 15 + 3q^{-1}[X^{(3,3)}] + 9q^{-1}[X^{(4,2)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(3,3)}] &= 3 + 15q^{-1}[X^{(3,3)}] + 9q^{-1}[X^{(4,2)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(4,2)}] &= 9 + 9q^{-1}[X^{(3,3)}] + 27q^{-1}[X^{(4,2)}]. \end{aligned}$$

The eigenvalues and associated eigenvectors of $[\mathbf{E}/\mathbf{P}]_0$ are:

$$\begin{aligned} \lambda_1 &= 36 & \mathbf{A}_1 &= 1 + q^{-1}[X^{(3,3)}] + 2q^{-1}[X^{(4,2)}] \\ \lambda_2 &= 12 & \mathbf{A}_2 &= 1 - q^{-1}[X^{(3,3)}] \\ \lambda_3 &= 9 & \mathbf{A}_3 &= 1 + q^{-1}[X^{(3,3)}] - q^{-1}[X^{(4,2)}] \end{aligned}$$

For $g, d, n \geq 0$ satisfying $6d = 8(n + g - 1)$, we obtain from [Theorem 4.8](#) and [Remark 4.10](#) that

$$\text{vTev}_{g,d,n}^{\text{Gr}(2,6)} = \sum_{i=1}^3 \frac{\text{Coeff}(\mathbf{A}_i, 1)^2}{|\mathbf{A}_i|^2} \lambda_i^g = \frac{36^g}{6} + \frac{12^g}{2} + \frac{9^g}{3}.$$

Example 4.14. Let $X = \text{Gr}(3, 6)$. Quantum multiplication by \mathbf{E}/\mathbf{P} on the Schubert basis of $QH^*(X)_{q,0}$ is given by

$$\begin{aligned} \mathbf{E}/\mathbf{P} \star 1 &= 20 + 2q^{-1}[X^{(3,3)}] + 2q^{-1}[X^{(2,2,2)}] + 16q^{-1}[X^{(3,2,1)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(3,3)}] &= 2 + 20q^{-1}[X^{(3,3)}] + 2q^{-1}[X^{(2,2,2)}] + 16q^{-1}[X^{(3,2,1)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(2,2,2)}] &= 2 + 2q^{-1}[X^{(3,3)}] + 20q^{-1}[X^{(2,2,2)}] + 16q^{-1}[X^{(3,2,1)}], \\ \mathbf{E}/\mathbf{P} \star q^{-1}[X^{(3,2,1)}] &= 16 + 16q^{-1}[X^{(3,3)}] + 16q^{-1}[X^{(2,2,2)}] + 56q^{-1}[X^{(3,2,1)}]. \end{aligned}$$

The eigenvalues and associated orthogonal eigenvectors of $[\mathbf{E}/\mathbf{P}]_0$ are:

$$\begin{aligned} \lambda_1 &= 72 & \mathbf{A}_1 &= 1 + q^{-1}[X^{(3,3)}] + q^{-1}[X^{(2,2,2)}] + 3q^{-1}[X^{(3,2,1)}] \\ \lambda_2 &= 18 & \mathbf{A}_2 &= 2 - q^{-1}[X^{(3,3)}] - q^{-1}[X^{(2,2,2)}] \\ \lambda_3 &= 18 & \mathbf{A}_3 &= q^{-1}[X^{(3,3)}] - q^{-1}[X^{(2,2,2)}] \\ \lambda_4 &= 8 & \mathbf{A}_4 &= 1 + q^{-1}[X^{(3,3)}] + q^{-1}[X^{(2,2,2)}] - q^{-1}[X^{(3,2,1)}]. \end{aligned}$$

For $g, d, n \geq 0$ satisfying $6d = 9(n + g - 1)$, we obtain from [Theorem 4.8](#) and [Remark 4.10](#) that

$$\mathrm{vTev}_{g,d,n}^{\mathrm{Gr}(3,6)} = \sum_{i=1}^4 \frac{\mathrm{Coeff}(\mathbf{A}_i, 1)^2}{|\mathbf{A}_i|^2} \lambda_i^g = \frac{72^g}{12} + \frac{2 \cdot 18^g}{3} + \frac{8^g}{4}.$$

Example 4.15. The virtual Tevelev degrees of $\mathrm{Gr}(2, 8)$, $\mathrm{Gr}(3, 8)$, and $\mathrm{Gr}(4, 8)$ are given by the following formulas (valid when the dimension constraint (1) is satisfied):

- For $\mathrm{Gr}(2, 8)$, the virtual Tevelev degrees are

$$\frac{(2 - \sqrt{2})(64 + 32\sqrt{2})^g}{8} + \frac{(2 + \sqrt{2})(64 - 32\sqrt{2})^g}{8} + \frac{(-1)^d 32^g}{4} + \frac{(-1)^d 16^g}{4}.$$

- For $\mathrm{Gr}(3, 8)$, the virtual Tevelev degrees are

$$\frac{(3 - 2\sqrt{2})(384 + 256\sqrt{2})^g}{16} + \frac{(3 + 2\sqrt{2})(384 - 256\sqrt{2})^g}{16} + \frac{128^g}{8} + \frac{64^g}{4} + \frac{32^g}{4}.$$

- For $\mathrm{Gr}(4, 8)$, the virtual Tevelev degrees are

$$\begin{aligned} & \frac{(3 - 2\sqrt{2})(768 + 512\sqrt{2})^g}{32} + \frac{(3 + 2\sqrt{2})(768 - 512\sqrt{2})^g}{32} + \frac{128^g}{8} + \frac{64^g}{16} + \frac{16^g}{8} \\ & + \frac{(-1)^{d/2}(2 - \sqrt{2})(128 + 64\sqrt{2})^g}{8} + \frac{(-1)^{d/2}(2 + \sqrt{2})(128 - 64\sqrt{2})^g}{8}. \end{aligned}$$

For $\mathrm{Gr}(4, 8)$, the dimension constraint (1) implies that d is even.

Formulas for $\mathrm{Gr}(2, 7)$ and $\mathrm{Gr}(3, 7)$ have complexity similar to that of the Freudenthal variety E_7/P_7 below, so they are omitted. Some virtual Tevelev degrees of these spaces are included in the following table.

| g | $\mathrm{Gr}(2, 7)$ | $\mathrm{Gr}(3, 7)$ |
|-----|-----------------------------|-------------------------------------|
| 0 | 1 | 1 |
| 1 | 21 | 35 |
| 2 | 686 | 2744 |
| 3 | 33614 | 470596 |
| 4 | 2000033 | 107884133 |
| 5 | 126825622 | 26310551764 |
| 6 | 8191782221 | 6491563697269 |
| 7 | 531900893867 | 1605160235412769 |
| 8 | 34589376715299 | 397071802007102691 |
| 9 | 2250344155712982 | 98232421880349925476 |
| 10 | 146424292089662006 | 24302307748473316398284 |
| 11 | 9527847961374037099 | 6012312236720159623681561 |
| 12 | 619985909132445247770 | 1487427484539611374221472752 |
| 13 | 40343209216871520541603 | 367985011574983125611827761985 |
| 14 | 2625182876113221414704217 | 91038368842060169714846533326833 |
| 15 | 170823979704176185099894853 | 22522614725296806700134311109583811 |

4.5. Further cominuscule flag varieties. The first Lagrangian Grassmannians do not produce new examples, $\text{LG}(1, 2) \cong \mathbb{P}^1$ and $\text{LG}(2, 4) \cong Q^3$. The following formulas are valid when the dimension constraint (1) is satisfied:

- For $\text{LG}(3, 6)$, the virtual Tevelev degrees are

$$\frac{(2 - \sqrt{2})(16 + 8\sqrt{2})^g}{4} + \frac{(2 + \sqrt{2})(16 - 8\sqrt{2})^g}{4}.$$

- For $\text{LG}(4, 8)$, the virtual Tevelev degrees are

$$\frac{(5 - 2\sqrt{5})(100 + 40\sqrt{5})^g}{20} + \frac{(5 + 2\sqrt{5})(100 - 40\sqrt{5})^g}{20} + \frac{(-1)^{d/2} 20^g}{4} + \frac{(-1)^{d/2} 4^g}{4}.$$

- For $\text{LG}(5, 10)$, the virtual Tevelev degrees are

$$\frac{(7 - 4\sqrt{3})(1008 + 576\sqrt{3})^g}{24} + \frac{(7 + 4\sqrt{3})(1008 - 576\sqrt{3})^g}{24} + \frac{144^g}{24} + \frac{48^g}{4} + \frac{16^g}{8}.$$

The first orthogonal Grassmannians do not produce new examples, $\text{OG}(2, 4) \cong \mathbb{P}^1$ and $\text{OG}(3, 6) \cong \mathbb{P}^3$. The following formulas are valid when the dimension constraint (1) is satisfied:

- For $\text{OG}(4, 8)$, the virtual Tevelev degrees are

$$\frac{12^g}{2} + \frac{(-1)^d 4^g}{2}.$$

- For $\text{OG}(5, 10)$, the virtual Tevelev degrees are

$$\frac{(2 - \sqrt{2})(32 + 16\sqrt{2})^g}{4} + \frac{(2 + \sqrt{2})(32 - 16\sqrt{2})^g}{4}.$$

- For $\text{OG}(6, 12)$, the virtual Tevelev degrees are

$$\frac{(5 - 2\sqrt{5})(200 + 80\sqrt{5})^g}{20} + \frac{(5 + 2\sqrt{5})(200 - 80\sqrt{5})^g}{20} + \frac{40^g}{4} + \frac{8^g}{4}.$$

Finally, there are two exceptional cases: the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7 . The following formulas are valid when the dimension constraint (1) is satisfied:

- For E_6/P_6 , the virtual Tevelev degrees are

$$\frac{(2 - \sqrt{3})(144 + 72\sqrt{3})^g}{6} + \frac{(2 + \sqrt{3})(144 - 72\sqrt{3})^g}{6} + \frac{9^g}{3}.$$

- For E_7/P_7 , the virtual Tevelev degrees are

$$\begin{aligned} & \frac{8^g}{4} + \frac{a_1 + a_2\zeta + a_3\zeta^{-1}}{b_1 + b_2\zeta + b_3\zeta^{-1}} (2376 + 432\zeta + 12096\zeta^{-1})^g \\ & + \frac{a_1 - a_4\zeta + a_5\zeta^{-1}}{b_1 - b_4\zeta + b_5\zeta^{-1}} (2376 - \alpha + \beta)^g + \frac{a_1 + a_6\zeta - a_7\zeta^{-1}}{b_1 + b_6\zeta - b_7\zeta^{-1}} (2376 - \alpha - \beta)^g \end{aligned}$$

where

$$\zeta = \sqrt[3]{148 + 4i\sqrt{3}}$$

is taken in the first quadrant, and the other constants are defined by:

$$\begin{aligned} \alpha &= 864\sqrt{7} \cos\left(\frac{1}{3} \arctan\left(\frac{\sqrt{3}}{37}\right)\right) & \beta &= 864\sqrt{21} \sin\left(\frac{1}{3} \arctan\left(\frac{\sqrt{3}}{37}\right)\right) \\ a_1 &= 156498345i\sqrt{3} + 1918858850 & b_1 &= 60400326564i\sqrt{3} + 740581002120 \\ a_2 &= 26312126i\sqrt{3} + 363365382 & b_2 &= 10153341360i\sqrt{3} + 140247740712 \\ a_3 &= 919199848i\sqrt{3} + 10129587384 & b_3 &= 354845016000i\sqrt{3} + 3909674626464 \\ a_4 &= 194838754i\sqrt{3} + 142214502 & b_4 &= 75200541036i\sqrt{3} + 54893858316 \\ a_5 &= 4605193768i\sqrt{3} - 6443593464 & b_5 &= 1777414805232i\sqrt{3} - 2487104837232 \\ a_6 &= 168526628i\sqrt{3} - 221150880 & b_6 &= 65047199676i\sqrt{3} - 85353882396 \\ a_7 &= 5524393616i\sqrt{3} + 3685993920 & b_7 &= 2132259821232i\sqrt{3} + 1422569789232 \end{aligned}$$

[Theorem 4.8](#) implies that all eigenvalues of E/P and all coefficients in the formula for virtual Tevelev degrees are real numbers (which is not clear from the above expressions). The following table contains some virtual Tevelev degrees of E_7/P_7 .

| g | \bar{E}_7/P_7 |
|-----|--|
| 0 | 1 |
| 1 | 56 |
| 2 | 128320 |
| 3 | 869201408 |
| 4 | 6035673223168 |
| 5 | 41931214470742016 |
| 6 | 291308765400165253120 |
| 7 | 2023810102768684733825024 |
| 8 | 14060020975152452459315593216 |
| 9 | 97679218802247250296546711830528 |
| 10 | 678607080508699448610546779756167168 |
| 11 | 4714488663641616811439032212948871282688 |
| 12 | 32752978856253489427845306031022643827703808 |
| 13 | 227544851731504006840105249380606108740637163520 |
| 14 | 1580822916191834644483662867101537620104827373617152 |
| 15 | 10982454990043024221511165369579640911620064101974147072 |

5. COMPLETE INTERSECTIONS

5.1. Classical cohomology. Let $X = V(f_1, \dots, f_L) \subset \mathbb{P}^{r+L}$ be a nonsingular complete intersection of dimension r where

$$f_i \in \Gamma(\mathbb{P}^{r+L}, \mathcal{O}(m_i))$$

for $1 \leq i \leq L$. We denote the vector of degrees by $\mathbf{m} = (m_1, \dots, m_L)$. We will always assume $r \geq 3$, so

$$H_2(X, \mathbb{Z}) = \mathbb{Z}$$

by the Lefschetz Hyperplane Theorem.

The following notation will be convenient, for any $a, b \in \mathbb{Z}$:

$$|\mathbf{m}| = \sum_{i=1}^L m_i, \quad \mathbf{m}^{am+b} = \prod_{i=1}^L m_i^{am_i+b}, \quad (am+b)! = \prod_{i=1}^L (am_i+b)! .$$

For example, the projective degree of X in \mathbb{P}^{r+L} is $\mathbf{m}^1 = \prod_{i=1}^L m_i$, and X is Fano if and only if $|\mathbf{m}| \leq r+L$.

Let $H^*(X) = H^*(X, \mathbb{Q})$ be the singular cohomology ring with rational coefficients. The restricted part of $H^*(X)$ has basis $\{1, \mathbf{H}, \dots, \mathbf{H}^r\}$, where $\mathbf{H} \in H^*(X)$ denotes the image of the hyperplane class $\mathbf{H} \in H^2(\mathbb{P}^{r+L})$ under the restriction

$$H^2(\mathbb{P}^{r+L}) \rightarrow H^2(X) .$$

The class of a point is

$$\mathbf{P} = \mathbf{m}^{-1} \mathbf{H}^r \in H^{2r}(X) .$$

The remaining cohomology of X , spanned by the primitive cohomology, is contained in the middle dimension $H^r(X)$. Let

$$\chi(X) = \int_X c_r(T_X)$$

denote the Euler characteristic of X .

5.2. Quantum cohomology. Let $X \subset \mathbb{P}^{r+L}$ be a smooth Fano complete intersection of dimension $r \geq 3$. Then the group $H_2(X, \mathbb{Z})$ is freely generated by the class of a line contained in X . A curve class in X is therefore determined by the associated projective degree, a non-negative integer.

The (small) quantum cohomology ring $QH^*(X)$ is an algebra over the polynomial ring $\mathbb{Q}[q]$ in one variable q . As a $\mathbb{Q}[q]$ -module, we have

$$QH^*(X) = H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q] .$$

The product in $QH^*(X)$ of two classes $\Gamma_1, \Gamma_2 \in H^*(X)$ is defined by

$$\Gamma_1 \star \Gamma_2 = \sum_{d \geq 0} q^d (\Gamma_1 \star \Gamma_2)_d^X ,$$

where $(\Gamma_1 \star \Gamma_2)_d^X \in H^*(X)$ is the unique class satisfying

$$\int_X (\Gamma_1 \star \Gamma_2)_d^X \cdot \Gamma_3 = \langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle_{0,d}^X = \int_{[\overline{\mathcal{M}}_{0,3}(X,d)]^{\text{vir}}} \text{ev}_1^*(\Gamma_1) \cdot \text{ev}_2^*(\Gamma_2) \cdot \text{ev}_3^*(\Gamma_3)$$

for all $\Gamma_3 \in H^*(X)$.

Since the virtual fundamental class $[\overline{\mathcal{M}}_{0,3}(X,d)]^{\text{vir}}$ has \mathbb{R} -dimension

$$2r + 2 \int_d c_1(T_X) ,$$

we see that $QH^*(X)$ is a graded ring where the elements of $H^*(X)$ have their usual real degrees and

$$\deg(q) = 2 \int_{\text{line}} c_1(T_X) = 2(r + L + 1 - |\mathbf{m}|).$$

For $A \in H^*(X)$, let $A^k \in H^*(X)$ and $A^{*k} \in QH^*(X)$ denote the k^{th} powers with respect to the classical and quantum products respectively.

In the following, we will assume $\deg(q) \geq \deg(\mathbf{H}^2)$, or equivalently,

$$(9) \quad |\mathbf{m}| \leq r + L - 1.$$

In the case (9), $QH^*(X)$ satisfies the relation [Giv98] (see also [Pan98, §3] for an exposition):

$$(10) \quad \mathbf{H}^{*(r+1)} = \mathbf{m}^m q \mathbf{H}^{*(|\mathbf{m}|-L)}.$$

By condition (9), we have $0 \leq |\mathbf{m}| - L \leq r - 1$.

We require the following result which was proved by Graber when X is a hypersurface [Pan98, Prop. 4]. The proof in the complete intersection case is the same.

Proposition 5.1 (Graber). *Let $\mathbf{R} = \text{Span}\{1, \mathbf{H}, \dots, \mathbf{H}^r\} \subset H^*(X)$ be the subspace of restricted classes. Then, $(\mathbf{R} \otimes_{\mathbb{Q}} \mathbb{Q}[q], \star)$ is a subring of $QH^*(X)$.*

Let $QH^*(X)^{\text{res}} = \mathbf{R} \otimes_{\mathbb{Q}} \mathbb{Q}[q] \subset QH^*(X)$ denote the subring of Proposition 5.1.

Lemma 5.2. *Let $\Gamma \in QH^*(X)^{\text{res}}$ be any class of degree $2r$ satisfying*

$$\Gamma \equiv a\mathbf{H}^r \pmod{q} \quad \text{and} \quad \mathbf{H} \star \Gamma = b q \mathbf{H}^{*(|\mathbf{m}|-L)},$$

for $a, b \in \mathbb{Q}$. Then, $\Gamma = a\mathbf{H}^{*r} + (b - am^m) q \mathbf{H}^{*(|\mathbf{m}|-L-1)}$.

Proof. By the definition of the quantum product, we have

$$\mathbf{H}^i \equiv \mathbf{H}^{*i} \pmod{q}$$

for $0 \leq i \leq r$. By Proposition 5.1, $\mathbf{H}^{*i} \in QH^*(X)^{\text{res}}$. Therefore, $\{1, \mathbf{H}, \mathbf{H}^{*2}, \mathbf{H}^{*3}, \dots, \mathbf{H}^{*r}\}$ is a basis of $QH^*(X)^{\text{res}}$ as a $\mathbb{Q}[q]$ -module. We can write

$$\Gamma = a\mathbf{H}^{*r} + \sum_{i=0}^{r-1} f_i(q) \mathbf{H}^{*i} \in QH^*(X)^{\text{res}}$$

where q divides f_i for all $0 \leq i \leq r - 1$. Then,

$$\mathbf{H} \star \Gamma = a\mathbf{H}^{*(r+1)} + \sum_{i=0}^{r-1} f_i(q) \mathbf{H}^{*(i+1)} \in QH^*(X)^{\text{res}}.$$

Using the second assumption and (10), we see

$$b q \mathbf{H}^{*(|\mathbf{m}|-L)} = a m^m q \mathbf{H}^{*(|\mathbf{m}|-L)} + \sum_{i=0}^{r-1} f_i(q) \mathbf{H}^{*(i+1)} \in QH^*(X)^{\text{res}},$$

which implies the desired result by extracting the $\mathbf{H}^{*(|\mathbf{m}|-L)}$ term. \square

The primitive cohomology of X is the linear subspace of $H^r(X)$ annihilated by classical multiplication by \mathbf{H} in $H^*(X)$. The next result states that primitive classes are also annihilated by quantum multiplication by \mathbf{H} .

Corollary 5.3. *Let $A \in H^r(X)$ satisfy $H \cdot A = 0 \in H^{r+2}(X)$. Then*

$$H \star A = 0 \in QH^*(X).$$

Proof. By [Proposition 5.1](#), we have

$$\int_X (H \star A)_d^X \cdot H^i = \langle H, A, H^i \rangle_{0,d}^X = \int_X (H \star H^i)_d^X \cdot A = 0$$

for all $d \geq 1$ and $i \geq 0$. Since $(H \star A)_d^X$ is a class of degree

$$2 + r - 2d(r + L + 1 - |m|) < r,$$

the Lefschetz theorem implies that $(H \star A)_d^X = 0$. \square

Remark 5.4. Primitive cohomology classes are not annihilated by quantum multiplication by arbitrary restricted classes. For example, it follows from [Corollary 5.9](#) below that $H^{r+L+1-|m|} = H \star H^{r+L-|m|} - m!q$, so [Corollary 5.3](#) shows that

$$H^{r+L+1-|m|} \star A = -m!qA$$

whenever $A \in H^r(X)$ is primitive.

Proposition 5.5. *Let $A, B \in H^r(X)$ satisfy $H \cdot A = H \cdot B = 0$ and $A \cdot B = P$ in $H^*(X)$. Then, we have*

$$A \star B = m^{-1}H^{*r} - m^{m-1}qH^{*(|m|-L-1)}.$$

In particular, $A \star B \in QH^(X)^{\text{res}}$.*

Proof. If $A \star B \in QH^*(X)^{\text{res}}$, then the Proposition follows immediately from [Corollary 5.3](#) and [Lemma 5.2](#) with $a = m^{-1}$ and $b = 0$.

To prove $A \star B \in QH^*(X)^{\text{res}}$, we proceed by contradiction. If $A \star B \notin QH^*(X)^{\text{res}}$, then there exists a primitive class $\gamma \in H^r(X)$ for which

$$(11) \quad \langle A, B, \gamma \rangle_{0,d}^X \neq 0$$

for some $d \geq 1$. By the dimension constraint, we have

$$3r = 2r + 2d(r + L + 1 - |m|).$$

In particular, r is even. Since $r \geq 3$ by assumption,

$$(12) \quad |m| = \left(\frac{2d-1}{2d} \right) r + L + 1 \geq L + 3.$$

By inequality (12), X cannot be a complete intersection of two quadrics (since then $|m| = 4$ and $L = 2$).

Let $V = H^r(X)_{\text{prim}} \otimes_{\mathbb{Q}} \mathbb{C}$ be the primitive cohomology of X with complex coefficients. Let $G \subset \text{GL}(V)$ denote the algebraic monodromy group defined as the Zariski closure of monodromy on primitive cohomology obtained by letting X vary in the full family of nonsingular complete intersections of degrees m in \mathbb{P}^{r+L} . Since X has dimension $r \geq 3$ and is not a complete intersection of two quadrics, G is as large as possible by results of Deligne [[Del80](#), Thm. 4.4.1], see also [[ABPZ21](#), Prop. 4.2]. More precisely $G = \text{O}(V)$ is the full orthogonal group of V with respect to the Poincaré duality pairing if r is even and $G = \text{Sp}(V)$ if r is odd. By the deformation invariance of Gromov-Witten theory, the 3-point Gromov-Witten bracket

$$\langle -, -, - \rangle_{0,d}^X : V^{\otimes 3} \rightarrow \mathbb{C}$$

is invariant under the action of G . But since $-\text{Id} \in G$ acts as -1 on 3-tensors, we deduce that $\langle -, -, - \rangle_{0,d}^X$ vanishes on $V^{\otimes 3}$, which contradicts (11). \square

Remark 5.6. Let $A, B \in H^r(X)$ be primitive classes such that $A \cdot B = 0 \in H^*(X)$. We can choose a primitive class $B' \in H^r(X)$ such that $A \cdot B' = P$. Since Proposition 5.5 implies that $A \star B' = A \star (B + B')$, we deduce that $A \star B = 0$.

5.3. **A Pieri formula modulo q^2 .** Define the polynomial $\Psi_m \in \mathbb{Z}[H_1, H_2]$ by

$$\Psi_m = \prod_{i=1}^L \prod_{j=0}^{m_i} (jH_1 + (m_i - j)H_2).$$

The total degree of Ψ_m is $L + |\mathbf{m}|$. Our calculations in $QH^*(X)$ will use the following result for the Gromov-Witten invariants of X in the class of a line,

$$\langle H^a, H^b \rangle_{0,1}^X = \int_{[\overline{\mathcal{M}}_{0,2}(X,1)]^{\text{vir}}} \text{ev}_1^*(H^a) \cdot \text{ev}_2^*(H^b).$$

Proposition 5.7. *Let $|\mathbf{m}| \leq r + L - 1$, and let $a, b \geq 0$ satisfy $a + b = 2r + L - |\mathbf{m}|$. Then, we have*

$$\langle H^a, H^b \rangle_{0,1}^X = \text{Coeff}(\Psi_m, H_1^{r+L-a} H_2^{r+L-b}).$$

Proof. The Gromov-Witten invariant in question is the same for all complete intersections of dimension r and degrees \mathbf{m} , so we may assume that X is general. This implies that the moduli space $\overline{\mathcal{M}}_{0,2}(X, 1)$ is a nonsingular projective variety of dimension $2r + L - |\mathbf{m}|$, and the virtual fundamental class $[\overline{\mathcal{M}}_{0,2}(X, 1)]^{\text{vir}}$ coincides with the usual fundamental class.

Let $E_1, E_2 \subset \mathbb{P}^{r+L}$ be general linear subspaces of codimensions a and b respectively. Using the transitive action of $\text{GL}(r + L + 1)$ on \mathbb{P}^{r+L} , Kleiman's Bertini Theorem implies that $\langle H^a, H^b \rangle_{0,1}^X$ is equal to the number of lines contained in X that meet both E_1 and E_2 . The assumptions imply that $a + b > r$, so we have

$$E_1 \cap E_2 \cap X = \emptyset.$$

Therefore, no line in X meets E_1 and E_2 in the same point. The variety of lines in X meeting E_1 and E_2 can be identified with the set

$$\{(P_1, P_2) \in E_1 \times E_2 \mid f_i(sP_1 + tP_2) = 0 \text{ for all } 1 \leq i \leq L \text{ and } (s : t) \in \mathbb{P}^1\},$$

where (f_1, \dots, f_L) are the defining equations for $X \subset \mathbb{P}^{r+L}$. We write

$$f_i(sP_1 + tP_2) = \sum_{j=0}^{m_i} f_{i,j}(P_1, P_2) s^j t^{m_i-j},$$

where $f_{i,j} \in H^0(E_1 \times E_2, \mathcal{O}(j) \boxtimes \mathcal{O}(m_i - j))$ is a section of the external tensor product of $\mathcal{O}_{E_1}(j)$ and $\mathcal{O}_{E_2}(m_i - j)$. We deduce that $\langle H^a, H^b \rangle_{0,1}^X$ is the number of points in the subscheme

$$Z = V(\{f_{i,j} : 0 \leq j \leq m_i, 1 \leq i \leq L\}) \subset E_1 \times E_2.$$

The standard construction [FP97] of $\overline{\mathcal{M}}_{0,2}(X, 1)$ for a general complete intersection X shows that the open subscheme $\text{ev}^{-1}(X \times X \setminus \Delta X)$ is isomorphic to $V(\{f_{i,j}\}) \subset \mathbb{P}^{r+L} \times \mathbb{P}^{r+L} \setminus \Delta \mathbb{P}^{r+L}$, so Z is isomorphic to $\text{ev}_1^{-1}(E_1) \cap \text{ev}_2^{-1}(E_2) \subset \overline{\mathcal{M}}_{0,2}(X, 1)$. In particular, Z is a reduced complete intersection of class

$$\Psi_m \in H^{2L+2|\mathbf{m}|}(E_1 \times E_2),$$

where the variables H_1 and H_2 are viewed as the generators of $\text{Pic}(E_1 \times E_2)$. The number of points of Z is then the integral of Ψ_m over $E_1 \times E_2$. \square

For $i \in \mathbb{Z}$, we define the following constant for notational convenience:

$$c_i = m^{-1} \text{Coeff}(\Psi_m, H_1^i H_2^{L+|m|-i}).$$

Proposition 5.8. *The constants c_i satisfy the following basic properties.*

- (a) $c_i \neq 0$ if and only if $L \leq i \leq |m|$,
- (b) $c_{L+|m|-i} = c_i$ for all $i \in \mathbb{Z}$,
- (c) $c_L = c_{|m|} = m!$,
- (d) $\sum_{i=L}^{|m|} c_i = m^m$.

Proof. Parts (a), (b), and (c) are immediate from the definition of Ψ_m , and part (d) holds because $\Psi_m(1, 1) = m^{m+1}$. \square

Corollary 5.9. *For $0 \leq i \leq r$, we have*

$$H \star H^i \equiv H^{i+1} + c_{i-r+|m|} q H^{i-r-L+|m|} \pmod{q^2}.$$

Proof. By definition of the quantum product,

$$H \star H^i \equiv H^{i+1} + q \langle H, H^i, H^{2r-i+L-|m|} \rangle_{0,1}^X m^{-1} H^{i-r-L+|m|} \pmod{q^2}.$$

By the divisor equation and the definition of c_{r+L-i} ,

$$H \star H^i \equiv H^{i+1} + c_{r+L-i} q H^{i-r-L+|m|} \pmod{q^2},$$

which is equivalent to the claim by [Proposition 5.8\(b\)](#). \square

Corollary 5.10. *Let $a, b \geq 1$ satisfy $a + b = r + L + 1 - |m|$. Then,*

$$\langle H^a, H^b, P \rangle_{0,1}^X = m!.$$

Proof. We have $H^a \star H^b = H \star H^{a-1} \star H^b = H \star H^{r+L-|m|}$, where the last equality follows since

$$a - 1 + b < r + L + 1 - |m|.$$

By [Corollary 5.9](#) and [Proposition 5.8](#),

$$\langle H^a, H^b, P \rangle_{0,1}^X = \langle H, H^{r+L-|m|}, P \rangle_{0,1}^X = c_L = m!,$$

as claimed. \square

Corollary 5.11. *We have $H^r \equiv H^{\star r} + (m! - m^m) q H^{\star(|m|-L-1)} \pmod{q^2}$.*

Proof. The result follows from [Lemma 5.2](#) and [Proposition 5.8](#), as we have

$$H \star H^r \equiv c_{|m|} q H^{|m|-L} \pmod{q^2}$$

by [Corollary 5.9](#). \square

5.4. Quantum Euler class. Let $X = V(f_1, \dots, f_L) \subset \mathbb{P}^{r+L}$ be a nonsingular complete intersection of dimension $r \geq 3$ and degrees $\mathbf{m} = (m_1, \dots, m_L)$ satisfying

$$|\mathbf{m}| \leq r + L - 1.$$

Our next results determine the quantum Euler class \mathbf{E} of X modulo q^2 .

Theorem 5.12. *We have $\mathbf{E} \in QH^*(X)^{\text{res}}$ and*

$$\mathbf{H} \star \mathbf{E} \equiv (r + L + 1 - |\mathbf{m}|) \mathbf{m}^{m-1} q \mathbf{H}^{|\mathbf{m}|-L} \pmod{q^2}.$$

Proof. [Proposition 5.5](#) implies $\mathbf{E} \in QH^*(X)^{\text{res}}$. Moreover, by [Corollary 5.3](#), we see

$$\mathbf{H} \star \mathbf{E} = \mathbf{m}^{-1} \sum_{i=0}^r \mathbf{H} \star \mathbf{H}^i \star \mathbf{H}^{r-i}.$$

By repeated application of [Corollary 5.9](#), we have

$$\mathbf{H}^{\star i} \equiv \mathbf{H}^i + \left(\sum_{j=L}^{i-1-r+|\mathbf{m}|} c_j \right) q \mathbf{H}^{i-r-L-1+|\mathbf{m}|} \pmod{q^2}.$$

Expanding modulo q^2 yields:

$$\begin{aligned} \mathbf{H} \star \mathbf{H}^i \star \mathbf{H}^{r-i} &\equiv \mathbf{H}^{\star(i+1)} \star \mathbf{H}^{r-i} - \left(\sum_{j=L}^{i-1-r+|\mathbf{m}|} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L} \\ &\equiv \left(\sum_{j=|\mathbf{m}|-i}^{|\mathbf{m}|} c_j - \sum_{j=L}^{i-1-r+|\mathbf{m}|} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L}. \end{aligned}$$

Using [Proposition 5.8\(b\)](#), we can rewrite the last result as

$$\begin{aligned} \left(\sum_{j=|\mathbf{m}|-i}^{|\mathbf{m}|} c_j - \sum_{j=L}^{i-1-r+|\mathbf{m}|} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L} &= \left(\sum_{j=L}^{L+i} c_j - \sum_{j=L}^{i-1-r+|\mathbf{m}|} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L} \\ &= \left(\sum_{j=i-r+|\mathbf{m}|}^{L+i} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L}. \end{aligned}$$

Finally, we obtain by analyzing the summation:

$$\begin{aligned} \mathbf{m}^{-1} \sum_{i=0}^r \mathbf{H} \star \mathbf{H}^i \star \mathbf{H}^{r-i} &\equiv \mathbf{m}^{-1} (r + L + 1 - |\mathbf{m}|) \left(\sum_{j=L}^{|\mathbf{m}|} c_j \right) q \mathbf{H}^{|\mathbf{m}|-L} \pmod{q^2} \\ &\equiv (r + L + 1 - |\mathbf{m}|) \mathbf{m}^{m-1} q \mathbf{H}^{|\mathbf{m}|-L} \pmod{q^2}, \end{aligned}$$

where the last equality uses [Proposition 5.8\(d\)](#). \square

Corollary 5.13. *We have*

$$\mathbf{E} \equiv \chi(X) \mathbf{m}^{-1} \mathbf{H}^{\star r} + (r + L + 1 - |\mathbf{m}| - \chi(X)) \mathbf{m}^{m-1} q \mathbf{H}^{\star(|\mathbf{m}|-L-1)} \pmod{q^2}.$$

In fact, the quantum Euler class is likely even better behaved. We conjecture the following stronger result.

Conjecture 5.14. *Let $X \subset \mathbb{P}^{r+L}$ be a nonsingular complete intersection of dimension $r \geq 3$ and degrees $\mathbf{m} = (m_1, \dots, m_L)$ satisfying*

$$|\mathbf{m}| \leq r + L - 1.$$

Then, we have

$$\mathbf{H} \star \mathbf{E} = (r + L + 1 - |\mathbf{m}|) \mathbf{m}^{\mathbf{m}-1} q \mathbf{H}^{*(|\mathbf{m}|-L)},$$

or equivalently,

$$\mathbf{E} = \chi(X) \mathbf{m}^{-1} \mathbf{H}^{\star r} + (r + L + 1 - |\mathbf{m}| - \chi(X)) \mathbf{m}^{\mathbf{m}-1} q \mathbf{H}^{*(|\mathbf{m}|-L-1)}.$$

The equivalence of the two claims in [Conjecture 5.14](#) follows from [Lemma 5.2](#) and [\(10\)](#). [Conjecture 5.14](#) is a consequence of [Theorem 5.12](#) when $\deg(q^2) > \deg(\mathbf{P})$, or equivalently, when

$$r > 2|\mathbf{m}| - 2L - 2.$$

We have verified [Conjecture 5.14](#) by computer in all cases where X is a complete intersection of dimension at most 30 or a hypersurface of dimension at most 135.

5.5. Tevelev degrees. Since $\{1, \mathbf{H}, \mathbf{H}^{\star 2}, \mathbf{H}^{\star 3}, \dots, \mathbf{H}^{\star r}\}$ is a $\mathbb{Q}[q]$ -module basis of $QH^*(X)^{\text{res}}$, we can uniquely express the point class \mathbf{P} in $QH^*(X)$ as

$$(13) \quad \mathbf{P} = \sum_{i=0}^{i_0} P_i q^i \mathbf{H}^{*(r-i(r+L+1-|\mathbf{m}|))}$$

where $P_0, \dots, P_{i_0} \in \mathbb{Q}$ and

$$i_0 = \left\lfloor \frac{r}{r + L + 1 - |\mathbf{m}|} \right\rfloor.$$

The first two coefficients

$$P_0 = \mathbf{m}^{-1}, \quad P_1 = (\mathbf{m} - 1)! - \mathbf{m}^{\mathbf{m}-1}$$

are determined by [Corollary 5.11](#).

Definition 5.15. Let $g, n \geq 0$ be non-negative integers such that

$$d = \frac{r(n+g-1)}{r+L+1-|\mathbf{m}|}$$

is a non-negative integer. There are unique rational numbers $b_i \in \mathbb{Q}$ such that

$$\mathbf{P}^{\star n} \star \mathbf{E}^{\star g} = \sum_{i=0}^{i_0} b_i q^{d+i} \mathbf{H}^{*(r-i(r+L+1-|\mathbf{m}|))}.$$

Define the **non-contributing part** of $\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}$ to be the sum

$$[\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}]^+ = \mathbf{P}^{\star n} \star \mathbf{E}^{\star g} - b_0 q^d \mathbf{H}^{\star r} = \sum_{i=1}^{i_0} b_i q^{d+i} \mathbf{H}^{*(r-i(r+L+1-|\mathbf{m}|))},$$

and define the **discrepancy** of $\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}$ to be the rational number

$$\text{Disc}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}) = \sum_{i=1}^{i_0} b_i \mathbf{m}^{-i\mathbf{m}+1}.$$

The product $\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}$ is **discrepancy-free** if $\text{Disc}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}) = 0$.

In the main cases that we will consider, the non-contributing part $[\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}]^+$ is easy to compute and has few non-zero terms. A formula for the virtual Tevelev degree,

$$\mathrm{vTev}_{g,d,n}^X = \mathrm{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^d \mathbf{P}),$$

is implied by [Conjecture 5.14](#).

Proposition 5.16. *Suppose [Conjecture 5.14](#) holds for the complete intersection*

$$X = V(f_1, \dots, f_L) \subset \mathbb{P}^{r+L},$$

and let $g, d, n \geq 0$ satisfy the dimension constraint

$$(14) \quad (r + L + 1 - |\mathbf{m}|)d = r(n + g - 1).$$

Then, we have

$$\mathrm{vTev}_{g,d,n}^X = \left(\sum_{i=0}^{i_0} P_i \mathbf{m}^{-im} \right)^n (r + L + 1 - |\mathbf{m}|)^g \mathbf{m}^{dm-g+1} - \mathrm{Disc}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}).$$

Proof. Using the relation [\(10\)](#) and [Conjecture 5.14](#), we obtain

$$\mathbf{P}^{\star n} \star \mathbf{E}^{\star g} \star \mathbf{H}^{\star(|\mathbf{m}|-L)} = \left(\sum_{i=0}^{i_0} P_i \mathbf{m}^{-im} \right)^n (r + L + 1 - |\mathbf{m}|)^g \mathbf{m}^{-g} \mathbf{H}^{\star(rn+rg+|\mathbf{m}|-L)},$$

and by [\(10\)](#) and [Definition 5.15](#) we have

$$\mathbf{P}^{\star n} \star \mathbf{E}^{\star g} \star \mathbf{H}^{\star(|\mathbf{m}|-L)} = \sum_{i=0}^{i_0} b_i \mathbf{m}^{-(d+i)\mathbf{m}} \mathbf{H}^{\star(rn+rg+|\mathbf{m}|-L)}.$$

By comparing these identities, we obtain

$$\sum_{i=0}^{i_0} b_i \mathbf{m}^{-im} = \left(\sum_{i=0}^{i_0} P_i \mathbf{m}^{-im} \right)^n (r + L + 1 - |\mathbf{m}|)^g \mathbf{m}^{dm-g}.$$

By observing that

$$\mathrm{vTev}_{g,d,n}^X = \mathrm{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^d \mathbf{P}) = \mathbf{m}^1 b_0,$$

the Proposition follows from this identity. \square

Example 5.17. Let $Q^r \subset \mathbb{P}^{r+1}$ be a quadric of dimension r and let $g, d, n \geq 0$ satisfy $d = n + g - 1$. By [Corollary 5.11](#) and [Corollary 5.13](#) we have

$$\mathbf{P} = \frac{1}{2} \mathbf{H}^{\star r} - q \quad \text{and} \quad \mathbf{E} = \frac{r + \delta}{2} \mathbf{H}^{\star r} - 2\delta q.$$

From this we obtain

$$\sum_{i=0}^{i_0} P_i \mathbf{m}^{-im} = \frac{1}{4}, \quad [\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}]^+ = (-q)^{n+g} (2\delta)^g,$$

and $\mathrm{Disc}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}) = \frac{(-1)^{n+g} (2\delta)^g}{2}.$

[Proposition 5.16](#) therefore gives

$$\mathrm{vTev}_{g,d,n}^{Q^r} = 4^{-n} r^g 2^{2d-g+1} - \mathrm{Disc}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}) = \frac{(2r)^g + (-1)^d (2\delta)^g}{2}.$$

Remark 5.18. Let $g, n \geq 0$ satisfy the condition of [Definition 5.15](#). The product $\mathbf{P}^{*n} \star \mathbf{E}^{*g}$ is always discrepancy-free when $X = \mathbb{P}^r$ is projective space, but never discrepancy-free when X is a quadric.

If [Conjecture 5.14](#) holds for X , then $\mathbf{P}^{*n} \star \mathbf{E}^{*g}$ is also discrepancy-free whenever the following inequality is satisfied:

$$(15) \quad n(r - i_0(r + L + 1 - |\mathbf{m}|)) + g(|\mathbf{m}| - L - 1) \geq |\mathbf{m}| - L.$$

In fact, this inequality implies that the expansion of $\mathbf{P}^{*n} \star \mathbf{E}^{*g}$, using the formulas of [Conjecture 5.14](#) and equation (13), is a linear combination of terms $q^e \mathbf{H}^{*p}$ with $p \geq |\mathbf{m}| - L$. Using (10), this implies that $b_i = 0$ for $i > 0$ in [Definition 5.15](#).

If X is not a quadric or \mathbb{P}^r , so that $|\mathbf{m}| > L + 1$, then inequality (15) is satisfied if either $g \geq 2$, or if $g \geq 1$, $n \geq 1$, and r is not divisible by $r + L + 1 - |\mathbf{m}|$.

Computer evidence suggests that $\mathbf{P}^{*n} \star \mathbf{E}^{*g}$ is discrepancy-free if and only if $X \cong \mathbb{P}^r$ or inequality (15) holds.

5.6. Complete intersections of low degree. When the dimension of

$$X = V(f_1, \dots, f_L) \subset \mathbb{P}^{r+L}$$

is large compared to the degrees \mathbf{m} of the defining equations,

$$\deg(q^2) > \deg(\mathbf{P}).$$

More precisely, the above condition holds if and only if

$$(16) \quad r > 2|\mathbf{m}| - 2L - 2.$$

If X is not projective space, then two consequences of (16) are:

- (i) $i_0 = 1$,
- (ii) [Conjecture 5.14](#) holds for X .

If we further assume that X is not a quadric, then (15) holds if and only if $g + n \geq 2$.

- (iii) $g + n \geq 2$ implies $\text{Disc}(\mathbf{P}^{*n} \star \mathbf{E}^{*g}) = 0$.

By putting our results together, we obtain the following formula for the virtual Tevelev degrees of X .

Theorem 5.19. *Let $|\mathbf{m}| > L + 1$ and $r > 2|\mathbf{m}| - 2L - 2$. If $g, d, n \geq 0$ satisfy*

$$(r + L + 1 - |\mathbf{m}|)d = r(n + g - 1)$$

and $g + n \geq 2$, then

$$\text{vTev}_{g,d,n}^X = ((\mathbf{m} - 1)!)^n (r + L + 1 - |\mathbf{m}|)^g \mathbf{m}^{(d-n)\mathbf{m}-g+1}.$$

Proof. Since we have (ii) and (iii), [Proposition 5.16](#) implies the result. We have $i_0 = 1$ by (i) and

$$\mathbf{P} = \mathbf{m}^{-1} \mathbf{H}^{*r} + ((\mathbf{m} - 1)! - \mathbf{m}^{\mathbf{m}-1}) q \mathbf{H}^{*(|\mathbf{m}|-L-1)}$$

by [Corollary 5.11](#). □

Example 5.20. Let $X \subset \mathbb{P}^{r+1}$ be a nonsingular cubic r -fold. The Tevelev degrees of X are given by

$$(17) \quad \text{vTev}_{g,d,n}^X = 2^n \cdot (r - 1)^g \cdot 3^{3d-3n-g+1}$$

for $g, d, n \geq 0$ satisfying $(r - 1)d = r(n + g - 1)$ and $g + n \geq 2$.

For cubic r -folds of dimension $r > 4$, the virtual Tevelev degrees of X of genus g are enumerative for all sufficiently large d by [[LP21](#), Thm. 11]. Formula (17)

should therefore admit a derivation (for d large) by classical projective geometry as developed in [FL21] for the case \mathbb{P}^r .

5.7. The border case. For further evidence, we show that [Conjecture 5.14](#) is also true when $X = V(f_1, \dots, f_L) \subset \mathbb{P}^{r+L}$ satisfies the condition

$$\deg(q^2) = \deg(\mathbf{P}),$$

or equivalently, when $r = 2|\mathbf{m}| - 2L - 2$.

Lemma 5.21. *Let $r = 2|\mathbf{m}| - 2L - 2$. Then, we have*

$$\langle H^{r-1}, \mathbf{P} \rangle_{0,2}^X = \frac{\mathbf{m}!(\mathbf{m}! + 2c_{L+1})}{4}, \quad H^r = H^{*r} + (\mathbf{m}! - \mathbf{m}^m) q H^{*(|\mathbf{m}|-L-1)} - \frac{(\mathbf{m}!)^2}{2} q^2.$$

Proof. Using [Corollary 5.9](#), we compute

$$\begin{aligned} H^{*(|\mathbf{m}|-L-2)} &= H^{|\mathbf{m}|-L-2}, \\ H^{*(|\mathbf{m}|-L-1)} &= H^{|\mathbf{m}|-L-1} + c_L q \\ H^{*(|\mathbf{m}|-L)} &= H^{|\mathbf{m}|-L} + (c_L + c_{L+1}) q H. \end{aligned}$$

Set $\delta = \langle H^{r-1}, \mathbf{P} \rangle_{0,2}^X$. We continue to compute

$$\begin{aligned} H^{*(r-1)} &= H^{r-1} + (\mathbf{m}^m - c_L - c_{L+1}) q H^{*(|\mathbf{m}|-L-2)}, \\ H^{*r} &= H^r + c_{L+1} q H^{|\mathbf{m}|-L-1} + 2\delta q^2 + (\mathbf{m}^m - c_L - c_{L+1}) q H^{*(|\mathbf{m}|-L-1)} \\ &= H^r + (\mathbf{m}^m - c_L) q H^{*(|\mathbf{m}|-L-1)} + (2\delta - c_L c_{L+1}) q^2, \\ H^{*(r+1)} &= c_L q H^{|\mathbf{m}|-L} + 2\delta q^2 H + (\mathbf{m}^m - c_L) q H^{*(|\mathbf{m}|-L)} + (2\delta - c_L c_{L+1}) q^2 H \\ &= \mathbf{m}^m q H^{*(|\mathbf{m}|-L)} + (4\delta - c_L^2 - 2c_L c_{L+1}) q^2 H. \end{aligned}$$

Relation [\(10\)](#) therefore implies

$$4\delta - c_L^2 - 2c_L c_{L+1} = 0,$$

which proves the identities. \square

Theorem 5.22. [Conjecture 5.14](#) is true when $\deg(q^2) = \deg(\mathbf{P})$.

Proof. Define constants $a_0, a_1, a_2 \in \mathbb{Q}$ by

$$\sum_{i=0}^r H^i \star H^{r-i} = a_0 H^{*r} + a_1 q H^{*(|\mathbf{m}|-L-1)} + a_2 q^2.$$

By [Proposition 5.5](#) and [Corollary 5.13](#), it suffices to show that $a_2 = 0$.

For $0 < i < r/2$, we have $H^i = H^{*i}$ and

$$H^{r-i} = H^{*(r-i)} - \left(\sum_{j=L}^{|\mathbf{m}|-i-1} c_j \right) q H^{*(|\mathbf{m}|-L-1-i)}$$

by [Corollary 5.9](#). Hence $H^i \star H^{r-i}$ does not contribute to a_2 for $0 < i < r/2$.

We are left to consider the two terms $H^r \star 1$ and $1 \star H^r$ together with $(H^{|\mathbf{m}|-L-1})^{*2}$. The first two terms do contribute $-(\mathbf{m}!)^2$ to a_2 by [Lemma 5.21](#). For the third term, we have

$$\begin{aligned} (H^{|\mathbf{m}|-L-1})^{*2} &= (H^{*(|\mathbf{m}|-L-1)} - \mathbf{m}! q)^{*2} \\ &= H^{*r} - 2\mathbf{m}! q H^{*(|\mathbf{m}|-L-1)} + (\mathbf{m}!)^2 q^2, \end{aligned}$$

which cancels the contribution of the first two. We conclude that $a_2 = 0$. \square

Corollary 5.23. *Assume that $\deg(q^2) = \deg(\mathbb{P})$. Then the dimension constraint holds for $g, d, n \geq 0$ if and only if $d = 2(n + g - 1)$. In this case we have*

$$\mathrm{vTev}_{g,d,n}^X = \left(\mathfrak{m}! \mathfrak{m}^m - \frac{1}{2} (\mathfrak{m}!)^2 \right)^n (|\mathfrak{m}| - L - 1)^g \mathfrak{m}^{(2g-2)\mathfrak{m}-n-g+1} - \mathrm{Disc}(\mathbb{P}^{\star n} \star \mathbb{E}^{\star g}),$$

where the discrepancy is given by

$$\mathrm{Disc}(\mathbb{P}^{\star n} \star \mathbb{E}^{\star g}) = \begin{cases} \left(-\frac{1}{2} \mathfrak{m}^{-1} (\mathfrak{m}!)^2 \right)^{n-1} \left(n \mathfrak{m}! \mathfrak{m}^{-\mathfrak{m}} - n - \frac{1}{2} \mathfrak{m}^{-2\mathfrak{m}} (\mathfrak{m}!)^2 \right) & \text{if } g = 0, \\ \left(-\frac{1}{2} \mathfrak{m}^{-1} (\mathfrak{m}!)^2 \right)^n (|\mathfrak{m}| - L - 1 - \chi(X)) & \text{if } g = 1, \\ 0 & \text{if } g \geq 2. \end{cases}$$

Proof. Using the formulas of [Lemma 5.21](#) and [Conjecture 5.14](#), we obtain

$$\begin{aligned} \sum_{i=0}^{i_0} P_i \mathfrak{m}^{-i\mathfrak{m}} &= \mathfrak{m}^{-2\mathfrak{m}-1} \left(\mathfrak{m}! \mathfrak{m}^m - \frac{1}{2} (\mathfrak{m}!)^2 \right), \\ [\mathbb{P}^{\star n}]^+ &= \left(-\frac{(\mathfrak{m}!)^2}{2\mathfrak{m}^1} q^2 \right)^{n-1} \left(\frac{n(\mathfrak{m}! - \mathfrak{m}^m)}{\mathfrak{m}^1} q \mathbb{H}^{\star(|\mathfrak{m}|-L-1)} - \frac{(\mathfrak{m}!)^2}{2\mathfrak{m}^1} q^2 \right), \text{ and} \\ [\mathbb{P}^{\star n} \star \mathbb{E}]^+ &= \left(-\frac{(\mathfrak{m}!)^2}{2\mathfrak{m}^1} q^2 \right)^n (|\mathfrak{m}| - L - 1 - \chi(X)) \mathfrak{m}^{\mathfrak{m}-1} q \mathbb{H}^{\star(|\mathfrak{m}|-L-1)}. \end{aligned}$$

The Corollary therefore follows from [Proposition 5.16](#). \square

Example 5.24. Let $X \subset \mathbb{P}^7$ be the intersection of three general quadrics, so $r = 4$ and $\mathfrak{m} = (2, 2, 2)$. For $g = n = 1$ and $d = 2$, [Corollary 5.23](#) gives

$$\mathrm{vTev}_{1,2,1}^X = 120 - 184 = -64.$$

In particular, virtual Tevelev degrees can be negative.

By [Proposition 3.1](#), all virtual Tevelev degrees for flag varieties are integers. It would be interesting to know if all virtual Tevelev degrees are integers for all varieties X . We have not seen a counterexample when X is a complete intersection satisfying $|\mathfrak{m}| \leq r + L - 1$.

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