# A FORMULA FOR NON-EQUIORIENTED QUIVER ORBITS OF TYPE $A$

ANDERS SKOVSTED BUCH AND RICHÁRD RIMÁNYI

Abstract. We prove a positive combinatorial formula for the equivariant class of an orbit closure in the space of representations of an arbitrary quiver of type $A$. Our formula expresses this class as a sum of products of Schubert polynomials indexed by a generalization of the minimal lace diagrams of Knutson, Miller, and Shimozono. The proof is based on the interpolation method of Fehér and Rimányi. We also conjecture a more general formula for the equivariant Grothendieck class of an orbit closure.

## 1. Introduction

A *quiver* is an oriented graph $Q = (Q_0, Q_1)$ consisting of a set of vertices $Q_0$ and a set of arrows $Q_1$. Each arrow $a \in Q_1$ has a tail $t(a) \in Q_0$ and a head $h(a) \in Q_0$. In this paper we will consider a quiver $Q$ of type $A$, i.e. a chain of vertices with arrows between them. We identify the vertex and arrow sets with integer intervals, $Q_0 = \{0, 1, 2, \dots, n\}$ and $Q_1 = \{1, 2, \dots, n\}$, such that $\{t(a), h(a)\} = \{a-1, a\}$ for each $a \in Q_1$. We also set $\delta(a) = h(a) - t(a)$, which equals $-1$ for a *leftward* arrow and $+1$ for a *rightward* arrow.

Fix a dimension vector $e = (e_0, e_1, \dots, e_n)$ of non-negative integers, and set $E_i = \mathbb{C}^{e_i}$ for each $i$. The set of quiver representations of dimension vector $e$ form the affine space

$$V = \operatorname{Hom}(E_{t(1)}, E_{h(1)}) \oplus \cdots \oplus \operatorname{Hom}(E_{t(n)}, E_{h(n)})\,,$$

which has a natural action (with finitely many orbits) of the group $G = \mathrm{GL}(E_0) \times \cdots \times \mathrm{GL}(E_n)$ given by $(g_0, \dots, g_n).(\phi_1, \dots, \phi_n) = (g_{h(1)} \phi_1 g_{t(1)}^{-1}, \dots, g_{h(n)} \phi_n g_{t(n)}^{-1})$. The goal of this paper is to prove a formula for the $G$-equivariant cohomology class of an orbit closure for this action. We note that Poincaré duality in equivariant cohomology was introduced by Kazarian [14], but simpler methods can be used to define the classes of Zariski closed subsets of $V$ [9, 11]. Our formula can also be interpreted as a formula for degeneracy loci defined by a quiver of vector bundles and bundle maps over a complex variety. This application relies on Bobiński and Zwara's proof that orbit closures of type $A$ are Cohen-Macaulay [4].

The quiver $Q$ is *equioriented* if all arrows have the same direction. A formula for the orbit closures for such a quiver was proved by Buch and Fulton [8]. Notice also that the problem specializes to the classical Thom-Porteous formula when $n = 1$. The formula proved in this paper generalizes a different formula for equioriented orbit closures, called the *component formula*, which was conjectured by Knutson, Miller, and Shimozono and proved in [15] and [7].

---

*Date*: January 19, 2006.
2000 *Mathematics Subject Classification.* 14N10; 57R45, 05E15, 14M12.

For an arbitrary quiver of Dynkin type, the *interpolation method* of Fehér and Rimányi makes it possible to compute the class of an orbit closure as the unique solution to a system of linear equations, which say that this class must vanish when restricted to a disjoint orbit [9, §2]. The proof of our formula relies on this method, as well as on a simplification of the ideas from [7].

The $G$-orbits in $V$ are classified by the *lace diagrams* of Abeasis and Del Fra [1, 2]. For equioriented quivers, these diagrams were reinterpreted as sequences of permutations by Knutson, Miller, and Shimozono [15], who called a lace diagram *minimal* if the sum of the lengths of these permutations is equal to the codimension of the corresponding orbit. The component formula writes the class of an orbit closure as a sum of products of Schubert polynomials indexed by all minimal lace diagrams for the orbit. The same construction turns out to work for an arbitrary quiver of type $A$, although most definitions need to be changed to take the orientation of the arrows into account, including the definition of a minimal lace diagram. By combining our definition of non-equioriented minimal lace diagrams with certain $K$-theoretic transformations of lace diagrams from [7], we furthermore obtain a natural conjecture for the equivariant Grothendieck class of an orbit closure. This conjecture generalizes the $K$-theoretic component formulas from [6, 18].

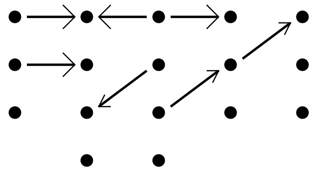
Our paper is organized as follows. In Section 2 we give the definition of minimal lace diagrams, state our formula, and prove some combinatorial properties of the formula. We also explain its interpretation as a formula for degeneracy loci. Section 3 explains the interpolation method and completes the proof of our formula. Finally, in Section 4 we pose our conjectured formula for the Grothendieck class of an orbit closure of type $A$.

We thank the referee for several helpful suggestions to our exposition.

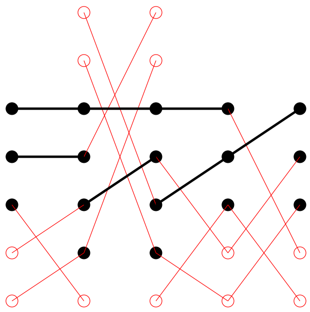
## 2. The non-equioriented component formula

2.1. **Lace diagrams.** The $G$-orbits in $V$ are classified by the lace diagrams of Abeasis and Del Fra [1]. Define a *lace diagram* for the dimension vector $e$ to be a sequence of $n+1$ columns of dots, with $e_i$ dots in column $i$, together with line segments connecting dots of consecutive columns. Each dot may be connected to at most one dot in the column to the left of it, and to at most one dot in the column to the right of it.

The quiver representations $\phi = (\phi_1, \dots, \phi_n)$ in the orbit given by a lace diagram can be obtained by identifying the dots of column $i$ with chosen basis vectors of $E_i$, and defining each linear map $\phi_a : E_{t(a)} \to E_{h(a)}$ according to the connections between the dots. In other words, if dot $j$ of column $t(a)$ is connected to dot $k$ of column $h(a)$, then $\phi_a$ maps the $j$th basis element of $E_{t(a)}$ to the $k$th basis element of $E_{h(a)}$; and if dot $j$ of column $t(a)$ is not connected to any dot in column $h(a)$, then the corresponding basis element of $E_{t(a)}$ is mapped to zero. For example, the following lace diagram represents an orbit in the space of representations of the quiver $Q = (\circ \to \circ \leftarrow \circ \to \circ \to \circ)$ of dimension vector $e = (3, 4, 4, 3, 3)$.



A lace diagram can be interpreted as a sequence of $n$ permutations as follows. For each *rightward* arrow $a \in Q_1$ we let $w_a$ be the permutation of smallest possible length such that $w_a(k) = j$ whenever the $k$th dot from the *top* of column $a$ is connected to the $j$th dot from the *top* of column $a-1$. If $a \in Q_1$ is a *leftward* arrow then we let $w_a$ be the permutation of smallest length such that $w_a(j) = k$ if the $j$th dot from the *bottom* of column $a-1$ is connected to the $k$th dot from the *bottom* of column $a$. Notice in particular that each permutation $w_a$ is read off the diagram *against* the direction of the arrow $a \in Q_1$. The lace diagram is determined by the sequence of permutations $\mathbf{w} = (w_1, \dots, w_n)$ together with the dimension vector $e$. Equivalently, the permutation sequence $\mathbf{w}$ describes the connections between the dots of an *extension* of the lace diagram, which is obtained by adding extra dots and connections below each rightward arrow and above each leftward arrow. The above displayed lace diagram corresponds to the permutation sequence $(w_1, w_2, w_3, w_4)$ where $w_1 = 12453$, $w_2 = 536412$, $w_3 = 13524$, and $w_4 = 24513$. The diagram has the following extension.



A permutation $w$ is called a *partial permutation* from $p$ elements to $q$ elements if all descent positions of $w$ are smaller than or equal to $p$, while the descent positions of $w^{-1}$ are smaller than or equal to $q$. In other words we have $w(i) < w(i+1)$ for $p > i$ and $w^{-1}(i) < w^{-1}(i+1)$ for $i > q$. A sequence $\mathbf{w} = (w_1, \dots, w_n)$ of permutations represents a lace diagram if and only if each permutation $w_a$ is a partial permutation from $e_{h(a)}$ elements to $e_{t(a)}$ elements. In the following we identify a lace diagram with its permutation sequence $\mathbf{w}$.

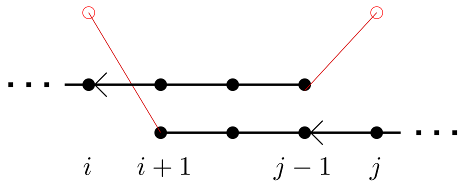
2.2. **Minimal lace diagrams.** A *strand* of a lace diagram is a maximal sequence of connected dots and line segments, and the *extension* of a strand is obtained by also including the extra line segments that it is directly connected to in the extended lace diagram. The *length* of the lace diagram $\mathbf{w} = (w_1, \dots, w_n)$ is the sum $\sum \ell(w_a)$ of the lengths of the permutations $w_a$. Equivalently, the length is equal to the total number of crossings in the extended diagram of $\mathbf{w}$.

For an orbit $\mu \subset V$ and vertices $0 \le i \le j \le n$, we define $s_{ij} = s_{ij}(\mu)$ to be the number of (non-extended) strands starting at column $i$ and terminating at column $j$ for any lace diagram representing $\mu$. We also let $r_{ij} = r_{ij}(\mu)$ denote the total number of connections from column $i$ to column $j$, i.e. $r_{ij} = \sum_{k \le i, l \ge j} s_{kl}$.

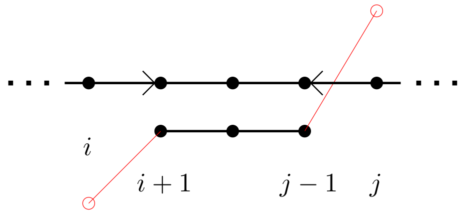
**Lemma 1.** *The length of a lace diagram representing the orbit $\mu$ is greater than or equal to the number*

$$d(\mu) = \sum_{i<j:\, \delta(i+1)=\delta(j)} (r_{i+1,j} - r_{ij})(r_{i,j-1} - r_{ij}) \;+\; \sum_{i<j:\, \delta(i+1)\neq\delta(j)} r_{ij} s_{i+1,j-1}\,.$$

*Proof.* Consider vertices $i, j \in Q_0$ with $i < j$, and assume that the arrow between $i$ and $i+1$ has the same direction as the arrow between $j-1$ and $j$, that is $\delta(i+1) = \delta(j)$. Since the left end of a strand starting at column $i+1$ is extended in the same direction (up or down) as the right end of a strand terminating at column $j-1$, it follows that (the extensions of) these strands must cross if the first strand passes through column $j$ and the second strand passes through column $i$. There are exactly $(r_{i+1,j} - r_{ij})(r_{i,j-1} - r_{ij})$ examples of this.

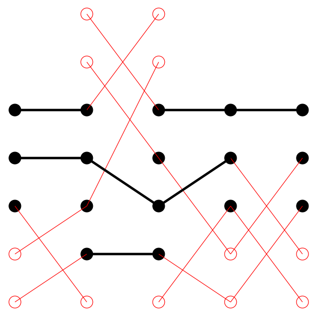



On the other hand, if $\delta(i+1) \neq \delta(j)$, then the left and right ends of a strand from column $i+1$ to column $j-1$ are extended in opposite directions, which means that such a strand must cross all strands connecting column $i$ to column $j$. This happens in $r_{ij} s_{i+1,j-1}$ examples. We have therefore identified $d(\mu)$ forced crossings in any lace diagram representing the orbit $\mu$.




□

We will prove later that the integer $d(\mu)$ of Lemma 1 is equal to the codimension of $\mu$ in $V$. We will call a lace diagram for $\mu$ *minimal* if its length is equal to $d(\mu)$. This extends Knutson, Miller, and Shimozono's definition of a minimal lace diagram for an equioriented quiver [15]. The following extended lace diagram is minimal and represents the same orbit as the diagrams of Section 2.1.



Notice that a lace diagram is minimal if and only if any two strands cross at most once, and not at all if they start or terminate at the same column (cf. [15, Thm. 3.8]). In fact, none of the forced crossings identified in the proof of Lemma 1 involve strands starting or terminating in the same column, and if two strands starting and terminating in different columns are not forced to cross, then they cross an even number of times.

2.3. **Schubert polynomials.** To state our formula, we need the Schubert polynomials of Lascoux and Schützenberger [16]. The *divided difference operator* $\partial_{a,b}$ with respect to two variables $a$ and $b$ is defined by

$$\partial_{a,b}(f) = \frac{f(a,b) - f(b,a)}{a-b}$$

where $f$ is any polynomial in these (and possibly other) variables. The *double Schubert polynomials* $\mathfrak{S}_w(x;y) = \mathfrak{S}_w(x_1,\dots,x_m;y_1,\dots,y_m)$ given by permutations $w \in S_m$ are uniquely determined by the identity

$$\partial_{x_i,x_{i+1}}(\mathfrak{S}_w(x;y)) = \begin{cases} \mathfrak{S}_{ws_i}(x;y) & \text{if } \ell(ws_i) < \ell(w) \\ 0 & \text{if } \ell(ws_i) > \ell(w) \end{cases} \tag{1}$$

together with the expression $\mathfrak{S}_{w_0}(x;y) = \prod_{i+j\leq m}(x_i - y_j)$ for the longest permutation $w_0$ in $S_m$. Using that $\mathfrak{S}_w(x;y) = (-1)^{\ell(w)}\mathfrak{S}_{w^{-1}}(y;x)$, the identity (1) is equivalent to

$$\partial_{y_i,y_{i+1}}(\mathfrak{S}_w(x;y)) = \begin{cases} -\mathfrak{S}_{s_iw}(x;y) & \text{if } \ell(s_iw) < \ell(w) \\ 0 & \text{if } \ell(s_iw) > \ell(w)\,. \end{cases} \tag{2}$$

For any permutations $u, w \in S_m$, the definition of Schubert polynomials implies that the specialization $\mathfrak{S}_w(y_u;y) = \mathfrak{S}_w(y_{u(1)},\dots,y_{u(m)};y_1,\dots,y_m)$ is zero unless $w \leq u$ in the Bruhat order on $S_m$, and for $u = w$ we have

$$\mathfrak{S}_u(y_{u(1)},\dots,y_{u(m)};y_1,\dots,y_m) = \prod_{i<j:\, u(i)>u(j)} (y_{u(i)} - y_{u(j)})\,. \tag{3}$$

Furthermore, if $k$ and $l$ are the last descent positions of $w$ and $w^{-1}$, respectively, then only the variables $x_1,\dots,x_k,y_1,\dots,y_l$ occur in $\mathfrak{S}_w(x;y)$.

2.4. **Statement of the formula.** For each $i \in Q_0$ we let $x^i = \{x^i_1,\dots,x^i_{e_i}\}$ be a set of $e_i$ variables. These variables are identified with the Chern roots in $H^*_T(V)$ of the $i$th factor of $G$, where $T$ is a maximal torus of $G$. Then $H^*_T(V)$ is the polynomial ring $\mathbb{Z}[x^i_j \mid 0 \leq i \leq n, 1 \leq j \leq e_i]$ in these variables, and $H^*_G(V) \subset H^*_T(V)$ is the subring of polynomials which are separately symmetric in each set of variables $x^i$. We let $\widetilde{x}^i = \{x^i_{e_i},\dots,x^i_1\}$ denote the variables $x^i$ in the opposite order. Given a lace diagram $\mathbf{w} = (w_1,\dots,w_n)$ for the dimension vector $e$, we let $\mathfrak{S}(w_1,\dots,w_n)$ be the product of the Schubert polynomials $\mathfrak{S}_{w_a}(x^a;x^{a-1})$ for all rightward arrows $a$, as well as the polynomials $\mathfrak{S}_{w_a}(\widetilde{x}^{a-1};\widetilde{x}^a)$ for all leftward arrows $a$.

$$\mathfrak{S}(w_1,\dots,w_n) = \left(\prod_{a:\delta(a)=1} \mathfrak{S}_{w_a}(x^a;x^{a-1})\right) \cdot \left(\prod_{a:\delta(a)=-1} \mathfrak{S}_{w_a}(\widetilde{x}^{a-1};\widetilde{x}^a)\right) \tag{4}$$

Since each permutation $w_a$ is a partial permutation from $e_{h(a)}$ elements to $e_{t(a)}$ elements, it follows that the corresponding Schubert polynomial receives the required number of variables. Finally, for any $G$-orbit $\mu \subset V$ we define the polynomial

$$Q_\mu = \sum_{(w_1,\dots,w_n)} \mathfrak{S}(w_1,\dots,w_n)$$

where the sum is over all minimal lace diagrams for $\mu$. Our main result is the following theorem, which generalizes the equioriented component formula proved in [15] and [7].

**Theorem 1.** *The polynomial $Q_\mu$ represents the $G$-equivariant cohomology class of the orbit closure $\overline{\mu}$ in $H^*_G(V)$.*

M. Shimozono reports that he had speculated that this formula was true, but had not been able to prove it.

2.5. **Degeneracy loci.** Theorem 1 can be interpreted as a formula for degeneracy loci defined by a quiver $F_\bullet$ of vector bundle morphisms over a non-singular complex variety $X$. This quiver consists of a vector bundle $F_i$ of rank $e_i$ for each vertex $i \in Q_0$, and a bundle map $F_{t(a)} \to F_{h(a)}$ for each arrow $a \in Q_1$. These bundle maps define a section $s : X \to H$ to the bundle $\pi : H = \bigoplus_{a\in Q_1} \operatorname{Hom}(F_{t(a)}, F_{h(a)}) \to X$. Since each fiber $\pi^{-1}(x)$ of $H$ is identical to the representation space $V$, a $G$-orbit $\mu \subset V$ defines a Zariski closed subset $H_\mu$ in $H$ as the union of the orbit closures $\overline{\mu} \subset V = \pi^{-1}(x)$ for all $x \in X$. The corresponding degeneracy locus in $X$ is defined as the scheme theoretic inverse image $X_\mu = s^{-1}(H_\mu)$. We assume that the bundle maps of $F_\bullet$ are sufficiently generic, so that $X_\mu$ obtains its maximal possible codimension $d(\mu)$ in $X$.

It follows from the definition of equivariant cohomology that the cohomology class $[H_\mu] \in H^*(H)$ is given by the polynomial $Q_\mu$, when the Chern roots of $\pi^* F_i$ are substituted for the variables $x^i$. Using Bobiński and Zwara's result that the orbit closure $\overline{\mu}$ (and therefore $H_\mu$) is Cohen-Macaulay [4], it follows from [12, Prop. 7.1] that $[X_\mu] = s^*[H_\mu]$ in $H^*(X)$, so the cohomology class of $X_\mu$ is also given by $Q_\mu$ when the variables $x^i$ are identified with the Chern roots of $F_i$.

If $X$ admits an ample line bundle $L$, then this formula remains true in the Chow group of $X$. In fact, by twisting the bundles $F_i$ with a power of $L$, we may assume that these bundles are globally generated. In this case one can construct a bundle $Y = \bigoplus_{a\in Q_1} \operatorname{Hom}(B_{t(a)}, B_{h(a)})$ over a product of Grassmannians $\prod_{i\in Q_0} \operatorname{Gr}^{e_i}(\mathbb{C}^N)$ with tautological quotient bundles $B_i$ for $i \in Q_0$, such that the quiver $F_\bullet$ on $X$ is the pullback of the universal quiver $B_\bullet$ on $Y$ along a morphism of varieties $f : X \to Y$. Since the Chow cohomology of $Y$ agrees with singular cohomology, our formula for the Chow class of $X_\mu$ follows from the identity $[X_\mu] = f^*[Y_\mu]$, which again uses that $Y_\mu$ is Cohen-Macaulay.

2.6. **Symmetry of the component formula.** In order to apply the interpolation method from [9] to prove Theorem 1, we first need to show that the polynomial $Q_\mu$ belongs to the subring $H^*_G(V)$ of symmetric polynomials in $H^*_T(V)$ (of course, this is *implied* by Theorem 1). We prove this as in [7], except that there are more cases to consider.

**Lemma 2.** *The polynomial $Q_\mu$ is separately symmetric in each set of variables $x^i$, $0 \le i \le n$.*

*Proof.* We must show that for any $0 \le i \le n$ and $1 \le j < e_i$, the divided difference operator $\partial^i_j = \partial_{x^i_j, x^i_{j+1}}$ maps $Q_\mu$ to zero. We verify this using the identities (1) and (2) of Schubert polynomials. Let $\mathbf{w} = (w_1, \dots, w_n)$ be a minimal lace diagram for $\mu$. For convenience, we identify each variable $x^i_k$ with dot $k$ from the top of column $i$. Notice that if two line segments connected to $x^i_j$ and $x^i_{j+1}$ cross each other, then the minimality of the lace diagram implies that $0 < i < n$, and only the connections on one side of these dots are allowed to cross.

Assume first that the line segments connecting $x^i_j$ and $x^i_{j+1}$ to dots of column $i-1$ cross each other. Let $\mathbf{u} = (u_1, \dots, u_n)$ be the lace diagram obtained from $\mathbf{w}$ by removing this crossing. In other words, we set $u_p = w_p$ for $p \neq i$, while $u_i = w_i s_j$ if arrow $i$ points right and $u_i = s_{e_i - j} w_i$ if arrow $i$ points left. We claim that

$$\partial_j^i(\mathfrak{S}(w_1, \dots, w_n)) = \mathfrak{S}(u_1, \dots, u_n)\,.$$

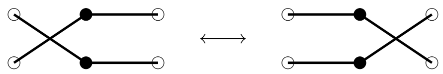
By using the identity $\partial_j^i(fg) = \partial_j^i(f)g$, which holds for polynomials $f$ and $g$ such that $g$ is symmetric in $\{x_j^i, x_{j+1}^i\}$, we need only check that $\partial_j^i$ maps the $i$th factor of $\mathfrak{S}(\mathbf{w})$ to the $i$th factor of $\mathfrak{S}(\mathbf{u})$. This follows from (1) when arrow $i$ points right and from (2) when arrow $i$ points left.

One checks similarly that, if the line segments connecting $x_j^i$ and $x_{j+1}^i$ to dots of column $i+1$ cross each other, then $\partial_j^i(\mathfrak{S}(\mathbf{w})) = -\mathfrak{S}(\mathbf{u})$, where the lace diagram $\mathbf{u}$ is obtained from $\mathbf{w}$ by removing this crossing. Furthermore, if none of the lines connected to $x_j^i$ and $x_{j+1}^i$ cross each other, then $\partial_j^i(\mathfrak{S}(\mathbf{w})) = 0$.

For each minimal lace diagram $\mathbf{w}$ for $\mu$ in which the connections to $x_j^i$ and $x_{j+1}^i$ from one side cross each other, one can construct another minimal lace diagram $\mathbf{w}'$ for $\mu$ by moving the crossing to the opposite side of these dots. The lemma follows from this because $\partial_j^i(\mathfrak{S}(\mathbf{w}) + \mathfrak{S}(\mathbf{w}')) = 0$. □

2.7. **Existence of minimal lace diagrams.** The orbit-preserving transformation of lace diagrams exploited in the proof of Lemma 2 is illustrated by the following picture (of parts of the extended lace diagrams):

(5)



These transformations played a similar role in [7]. Notice that the transformation (5) can be applied to any lace diagram, as long as the middle dots and at least one from each column of outer dots are not in the extended part of the diagram.

**Proposition 1.** *Let $\mu \subset V$ be any $G$-orbit. Then there exists at least one minimal lace diagram representing $\mu$, and every minimal lace diagram for $\mu$ can be obtained from any other such diagram by using the transformations (5).*

*Proof.* Given any minimal lace diagram for $\mu$, we can use the transformations (5) repeatedly, in left to right direction, until all crossings of the lace diagram involve the right hand side extension of one of the crossing strands. It is therefore enough to prove that each orbit $\mu$ has a unique minimal lace diagram with this property.

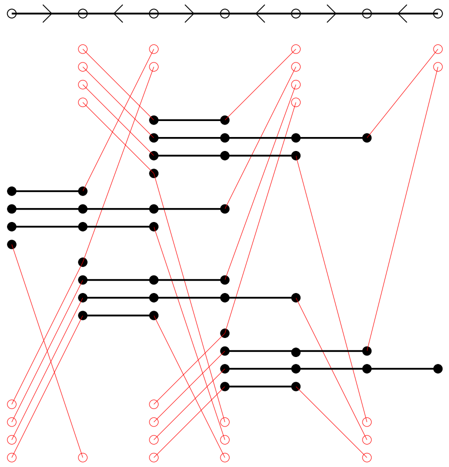
We will say that two (non-extended) strands *overlap* if both contain a dot in the same column. Notice that if all crossings of a lace diagram occur in the extended part of the diagram, then the lace diagram is uniquely determined by specifying, for each pair of overlapping strands, which strand is placed above the other. The uniqueness therefore follows from the observation that, if all crossings between two overlapping strands involve the right side extension of one of them, then this condition dictates which strand is over the other.

Finally, to prove that a minimal lace diagram exists, it is sufficient to give a total order on the set of all pairs of integers $(i,j)$ with $0 \leq i \leq j \leq n$, such that if $(i,j) < (p,q)$ and a strand from column $i$ to column $j$ is placed above a strand from column $p$ to column $q$, then these strands cross at most once, and if they do, the crossing must occur at the right side extension of one of them. Such an ordering can be defined explicitly by writing $(i,j) < (p,q)$ if and only if one of the following conditions hold:

(1) $\delta(i) = -1$ and $\delta(p) = 1$.

(2) $\delta(i) = \delta(p) = -1$ and $i > p$.
(3) $\delta(i) = \delta(p) = 1$ and $i < p$.
(4) $i = p$ and $\delta(j+1) = -1$ and $\delta(q+1) = 1$.
(5) $i = p$ and $\delta(j+1) = \delta(q+1) = -1$ and $j < q$.
(6) $i = p$ and $\delta(j+1) = \delta(q+1) = 1$ and $j > q$.

The following is an example of a minimal lace diagram where the strands are arranged according to this order.



$\square$

## 3. Proof of the main theorem

3.1. **The interpolation method.** For each $G$-orbit $\mu \subset V$ we let $G_\mu$ denote the stabilizer subgroup of a point $p_\mu$ in $\mu$. The inclusion $G_\mu \subset G$ induces a map $BG_\mu \to BG$, which gives an equivariant restriction map $\phi_\mu : H^*_G(V) = H^*(BG) \to H^*(BG_\mu) = H^*_G(\mu)$. The *Euler class* $\mathcal{E}(\mu) \in H^*_G(\mu)$ is the top equivariant Chern class of the normal bundle to $\mu$ in $V$. We will prove our formula for the class of $\overline{\mu}$ as an application of the *interpolation method* of Fehér and Rimányi. This method works more generally when $G$ is an arbitrary complex Lie group acting on a vector space $V$ with finitely many orbits, such that $\mathcal{E}(\mu)$ is not a zero-divisor in $H^*_G(\mu)$ for each orbit $\mu$. We need the following statement [9, Thm. 3.5].

**Theorem 2.** *Let $\mu \subset V$ be a $G$-orbit. The $G$-equivariant cohomology class of the closure of $\mu$ is the unique class $[\overline{\mu}] \in H^*_G(V)$ satisfying $\phi_\mu([\overline{\mu}]) = \mathcal{E}(\mu) \in H^*_G(\mu)$ and $\phi_\eta([\overline{\mu}]) = 0$ for every $G$-orbit $\eta \subset V$ for which $\eta \neq \mu$ and* $\operatorname{codim} \eta \leq \operatorname{codim} \mu$.

3.2. **Description of the Euler class.** We also need a description of the restriction maps $\phi_\mu : H^*_G(V) \to H^*_G(\mu)$ and Euler classes $\mathcal{E}(\mu) \in H^*_G(\mu)$ which was proved in [10] for any quiver of Dynkin type. Fix a lace diagram $\mathbf{w}$ representing the orbit $\mu$, and choose variables $b_1, \dots, b_k$ corresponding to the strands of $\mathbf{w}$. Then $H^*_G(\mu)$ can be identified with a subring of the polynomial ring $\mathbb{Z}[b_1, \dots, b_k]$; this was done in [10, §3] by showing that $G_\mu$ has a maximal torus of dimension equal to the number of strands. By [10, Prop. 3.10], the restriction map $\phi_\mu : H^*_G(V) \to H^*_G(\mu)$ extends to a ring homomorphism $\phi_{\mathbf{w}} : H^*_T(V) \to \mathbb{Z}[b_1, \dots, b_k]$, which maps $x^i_j$ to the variable of the strand passing through dot $j$ from the top of column $i$ of the lace diagram. This map $\phi_{\mathbf{w}}$ depends on the chosen lace diagram $\mathbf{w}$ for $\mu$.

To describe the Euler class $\mathcal{E}(\mu) \in H^*_G(\mu)$, we need some definitions for quiver representations with arbitrary dimension vectors. Let $\phi = (\phi_1, \dots, \phi_n)$ and $\phi' = (\phi'_1, \dots, \phi'_n)$ be representations of $Q$ with dimension vectors $e = (e_0, \dots, e_n)$ and $e' = (e'_0, \dots, e'_n)$. A homomorphism $\alpha : \phi \to \phi'$ is a tuple $\alpha = (\alpha_0, \dots, \alpha_n)$ of linear maps $\alpha_i : \mathbb{C}^{e_i} \to \mathbb{C}^{e'_i}$ such that $\alpha_{h(a)}\phi_a = \phi'_a \alpha_{t(a)}$ for all arrows $a$. The set $\operatorname{Hom}(\phi, \phi')$ of all such homomorphisms is a complex vector space. By using an injective resolution of the representation $\phi'$, one can also define the extension module $\operatorname{Ext}(\phi, \phi') = \operatorname{Ext}^1(\phi, \phi')$. Let $E_Q$ be the Euler form defined by $E_Q(\phi, \phi') = E_Q(e, e') = \sum_{i \in Q_0} e_i e'_i - \sum_{a \in Q_1} e_{t(a)} e'_{h(a)}$. The homomorphism and extension modules are related by the identity [19]

$$E_Q(\phi, \phi') = \dim \operatorname{Hom}(\phi, \phi') - \dim \operatorname{Ext}(\phi, \phi') \,. \tag{6}$$

A quiver representation is *indecomposable* if it cannot be written as a direct sum of other quiver representations. For a quiver of Dynkin type, the indecomposable representations correspond to the positive roots of the corresponding root system [13] (see also [3]). For our quiver of type $A$, there is one indecomposable representation $X^{ij}$ for each pair of integers $(i, j)$ with $0 \leq i \leq j \leq n$. The dimension vector of $X^{ij}$ assigns the dimension 1 to all vertices $k \in Q_0$ with $i \leq k \leq j$, and assigns dimension zero to all other vertices. For each arrow $i < a \leq j$, the map $X^{ij}_a : \mathbb{C} \to \mathbb{C}$ is the identity. Given a $G$-orbit $\mu \subset V$, the indecomposable summands in the decomposition of a representation $\phi \in \mu$ correspond to the strands in the lace diagram for $\mu$. More canonically, the multiplicity of $X^{ij}$ in $\phi$ is equal to the number of strands $s_{ij}(\mu)$ from column $i$ to column $j$.

We can now state the formula for the Euler class $\mathcal{E}(\mu)$, using the above described embedding $H^*_G(\mu) \subset \mathbb{Z}[b_1, \dots, b_k]$. For each pair of variables $b_p, b_q$ we let $\operatorname{Ext}(b_p, b_q)$ denote the extension module of the indecomposable representations corresponding to the strands of $b_p$ and $b_q$. The following was proved in [10, Cor. 3.13].

**Proposition 2.** *The Euler class of the $G$-orbit $\mu \subset V$ is given by*

$$\mathcal{E}(\mu) = \prod_{1 \leq p,q \leq k} (b_p - b_q)^{\dim \operatorname{Ext}(b_q, b_p)} \,.$$

3.3. **Proof of the formula.** We need to compute the dimension of an extension module $\operatorname{Ext}(X^{ij}, X^{pq})$. Let $N(X^{ij}, X^{pq})$ denote the number of arrows $a \in Q_1$ such that $t(a) \in [i, j]$, $h(a) \in [p, q]$, and such that $h(a) \notin [i, j]$ or $t(a) \notin [p, q]$.

**Lemma 3.** *The dimension of the extension module of the indecomposable representations $X^{ij}$ and $X^{pq}$ is given by*

$$\dim \operatorname{Ext}(X^{ij}, X^{pq}) = \begin{cases} 1 & \text{if } [i,j] \cap [p,q] \neq \emptyset \text{ and } N(X^{ij}, X^{pq}) = 2 \\ 1 & \text{if } [i,j] \cap [p,q] = \emptyset \text{ and } N(X^{ij}, X^{pq}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If $[i, j] \cap [p, q] = \emptyset$ then $\operatorname{Hom}(X^{ij}, X^{pq}) = 0$ and $N(X^{ij}, X^{pq}) \leq 1$. The lemma follows from (6) because $E_Q(X^{ij}, X^{pq}) = -N(X^{ij}, X^{pq})$.

Otherwise $[i, j] \cap [p, q] \neq \emptyset$, in which case we have $N(X^{ij}, X^{pq}) \leq 2$. It follows from the definition that $\operatorname{Hom}(X^{ij}, X^{pq}) = \mathbb{C}$ if $N(X^{ij}, X^{pq}) = 0$, while $\operatorname{Hom}(X^{ij}, X^{pq}) = 0$ otherwise. The lemma now follows because $E_Q(X^{ij}, X^{pq}) = 1 - N(X^{ij}, X^{pq})$. ∎

Another way to state this lemma is that $\mathrm{Ext}(X^{ij}, X^{pq})$ is non-zero (with dimension one) exactly when two strands corresponding to $X^{ij}$ and $X^{pq}$ are forced to cross each other (see the proof of Lemma 1), and when $(i,j) < (p,q)$ in the order used in the proof of Proposition 1. Notice also that if two such strands have a single crossing point, then the slope at the crossing point of the strand corresponding to $X^{pq}$ is larger than the slope of the strand corresponding to $X^{ij}$. As a consequence we obtain the following description of the Euler class $\mathcal{E}(\mu)$.

**Corollary 1.** *Let $\mathbf{w}$ be a minimal lace diagram for the $G$-orbit $\mu \subset V$, and let $H^*_G(\mu) \subset \mathbb{Z}[b_1, \dots, b_k]$ be the corresponding inclusion of rings. Then the Euler class $\mathcal{E}(\mu) \in H^*_G(\mu)$ is the product of all factors $(b_p - b_q)$ for which the strands of $b_p$ and $b_q$ cross each other and the strand of $b_p$ has the highest slope at the crossing point.*

**Corollary 2.** *The codimension of the $G$-orbit $\mu \subset V$ is equal to the length $d(\mu)$ of any minimal lace diagram for $\mu$.*

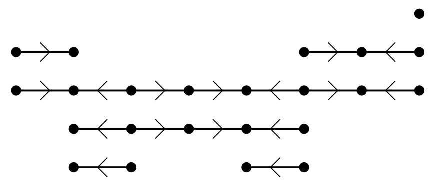
*Proof of Theorem 1.* It follows from Lemma 2 that $Q_\mu$ is an element of $H^*_G(V)$. According to Theorem 2, we need to prove that $\phi_\mu(Q_\mu) = \mathcal{E}(\mu)$ and that $\phi_\eta(Q_\mu) = 0$ for any $G$-orbit $\eta \subset V$ such that $\eta \neq \mu$ and $\operatorname{codim} \eta \leq \operatorname{codim} \mu$. It is enough to show that if $\mathbf{u}$ is a minimal lace diagram for $\mu$ and $\mathbf{w}$ is any lace diagram for the same dimension vector $e$ such that $\ell(\mathbf{w}) \leq \ell(\mathbf{u})$, then we have

$$\phi_{\mathbf{w}}(\mathfrak{S}(\mathbf{u})) = \begin{cases} \mathcal{E}(\mu) & \text{if } \mathbf{w} = \mathbf{u} \\ 0 & \text{if } \mathbf{w} \neq \mathbf{u}. \end{cases}$$

For any arrow $a \in Q_1$, $\phi_{\mathbf{w}}$ maps the $a$th factor of $\mathfrak{S}(\mathbf{u})$ to the specialized Schubert polynomial $\mathfrak{S}_{u_a}(b_{w_a(1)}, \dots, b_{w_a(m)}; b_1, \dots, b_m)$, where $b_1, \dots, b_m$ denote the variables of strands of $\mathbf{w}$ connecting column $a-1$ to column $a$. If $\delta(a) = 1$ then $b_1, \dots, b_m$ correspond to the strands passing through the dots of column $a-1$ ordered from top to bottom, and starting with the first non-extended dot. If $\delta(a) = -1$, then we use the strands passing through the dots of column $a$ in bottom to top order, starting with the lowest non-extended dot. Now the specialization $\mathfrak{S}_{u_a}(b_{w_a}; b)$ is zero unless $u_a \leq w_a$ in the Bruhat order. Since $\ell(\mathbf{w}) \leq \ell(\mathbf{u})$, it follows that $\phi_{\mathbf{w}}(\mathfrak{S}(\mathbf{u}))$ is zero unless $\mathbf{u} = \mathbf{w}$. Furthermore, it follows from (3) that $\mathfrak{S}_{u_a}(b_{u_a}; b)$ is equal to the product of the factors $(b_p - b_q)$ of Corollary 1 for which the strands of $b_p$ and $b_q$ cross between column $a-1$ and column $a$. This shows that $\phi_{\mathbf{u}}(\mathfrak{S}(\mathbf{u})) = \mathcal{E}(\mu)$, and finishes the proof. □

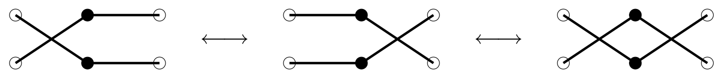
**Remark 1.** In most of our pictures of lace diagrams, the columns of (non-extended) dots have been aligned at the top. However, the definition of extended lace diagrams makes it natural to bottom-align two consecutive columns if they are connected by a leftward arrow, while columns connected with a rightward arrow are top-aligned as usual. When this convention is used, a minimal lace diagram for the open orbit in the representation space $V$ can be obtained by simply drawing all possible horizontal lines between dots of consecutive columns. For example, the open orbit for the quiver $(\circ \to \circ \leftarrow \circ \to \circ \to \circ \leftarrow \circ \to \circ \leftarrow \circ)$ with dimension vector

$e = (2, 4, 3, 2, 3, 4, 2, 3)$ is represented by the following minimal lace diagram.



## 4. A conjectural $K$-theoretic component formula

For an equioriented quiver of type $A$, Buch has proved a $K$-theoretic generalization of the component formula, which expresses the equivariant Grothendieck class of an orbit closure as an alternating sum of products of Grothendieck polynomials [6]. This sum is over all *KMS-factorizations* for the orbit, which generalize the equioriented minimal lace diagrams from [15]. A limit of the $K$-theoretic component formula in terms of stable Grothendieck polynomials was also obtained by Miller [18]. It was proved in [7] that all KMS-factorizations for an orbit can be obtained from a minimal lace diagram for the orbit by applying a series of transformations:



In these transformations, the middle dots and at least one of the outer dots from each side must be outside the extended part of the diagram, and furthermore the two middle dots must be consecutive in their column.

Given an orbit $\mu \subset V$ of representations of an arbitrary quiver of type $A$, define a *$K$-theoretic lace diagram* for this orbit to be any lace diagram that can be obtained from a minimal lace diagram representing $\mu$ by using these transformations. For such a diagram $\mathbf{w}$, we define a Laurent polynomial $\mathfrak{G}(\mathbf{w})$ by the expression (4), except that each Schubert polynomial $\mathfrak{S}_w(x; y)$ is replaced with the Grothendieck (Laurent) polynomial $\mathfrak{G}_w(x; y)$ from [17]. This polynomial is defined by the recursive identities $(x_i - x_{i+1})\mathfrak{G}_w(x; y) = x_i \mathfrak{G}_{ws_i}(x; y) - x_{i+1}\mathfrak{G}_{ws_i}(x_{s_i}; y)$ when $w(i) < w(i+1)$, as well as the expression $\mathfrak{G}_{w_0}(x; y) = \prod_{i+j\leq m}(1 - y_i/x_j)$ when $w_0$ is the longest permutation in $S_m$. Given a maximal torus $T \subset G$, we identify the variable $x^i_j$ of $\mathfrak{G}(\mathbf{w})$ with the $T$-equivariant class of a line bundle $V \times \mathbb{C} \to V$ with the action of $T$ given by $t.(\phi, z) = (t.\phi, t^i_j z)$, where $(t^i_1, \dots, t^i_{e_i})$ are chosen coordinates on $T \cap \mathrm{GL}(E_i)$. We finish this paper by posing the following conjecture, which generalizes Theorem 1 as well as [6, Thm. 6.3].

**Conjecture 1.** *The $T$-equivariant Grothendieck class of $\overline{\mu}$ is given by*

$$[\mathcal{O}_{\overline{\mu}}] = \sum_{\mathbf{w}} (-1)^{\ell(\mathbf{w})-d(\mu)} \mathfrak{G}(\mathbf{w})$$

*where the sum is over all $K$-theoretic lace diagrams for $\mu$.*

Aside from the analogies with known identities, this conjecture is motivated by the fact that one is naturally led to the $K$-theoretic transformations when attempting to prove that a linear combination of the products $\mathfrak{G}(\mathbf{w})$ of Grothendieck polynomials is symmetric in each set of variables $x^i$. We note that it is possible to compute the $T$-equivariant Grothendieck class of $\overline{\mu}$ by using a resolution of singularities of this locus like in [8, 5], but there is no known way to derive the conjectured formula from this approach, even for equioriented quivers.

## References


[1] S. Abeasis and A. Del Fra, *Degenerations for the representations of an equioriented quiver of type $A_m$*, Boll. Un. Mat. Ital. Suppl. **1980**, 157–171. MR 84e:16019

[2] ———, *Degenerations for the representations of a quiver of type $A_m$*, J. Algebra **93** (1985), 376–412. MR 86j:16028

[3] I. N. Bernstein, I. M. Gel'fand, and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspehi Mat. Nauk **28** (1973), 19–33. MR 0393065 (52 #13876)

[4] G. Bobiński and G. Zwara, *Normality of orbit closures for Dynkin quivers of type $\mathbb{A}_n$*, Manuscripta Math. **105** (2001), no. 1, 103–109. MR 1885816 (2002k:14077)

[5] A. S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103. MR 1 932 326

[6] ———, *Alternating signs of quiver coefficients*, J. Amer. Math. Soc. **18** (2005), no. 1, 217–237.

[7] A. S. Buch, L. M. Fehér, and R. Rimányi, *Positivity of quiver coefficients through Thom polynomials*, to appear in Adv. Math., 2004.

[8] A. S. Buch and W. Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), 665–687.

[9] L. M. Fehér and R. Rimányi, *Calculation of Thom polynomials and other cohomological obstructions for group actions*, Real and Complex Singularities (Sao Carlos, 2002), Ed. T. Gaffney and M. Ruas, Contemp. Math., #354, Amer. Math. Soc., Providence, RI, 2004, pp. 69–93.

[10] ———, *Classes of degeneracy loci for quivers: the Thom polynomial point of view*, Duke Math. J. **114** (August 2002), no. 2, 193–213.

[11] W. Fulton, *Notes from a course on equivariant cohomology*, Winter, 2003.

[12] ———, *Intersection theory*, Springer-Verlag, 1984, 1998.

[13] P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309. MR 0332887 (48 #11212)

[14] M. É. Kazarian, *Characteristic classes of singularity theory*, The Arnold-Gelfand mathematical seminars: geometry and singularity theory (V. I. Arnold et al., eds.), 1997, pp. 325–340.

[15] A. Knutson, E. Miller, and M. Shimozono, *Four positive formulas for type A quiver polynomials*, preprint, 2003.

[16] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450. MR 83e:14039

[17] ———, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 629–633. MR 84b:14030

[18] E. Miller, *Alternating formulae for $K$-theoretic quiver polynomials*, Duke Math. J. **128** (2005), no. 1, 1–17.

[19] C. M. Ringel, *Representations of $K$-species and bimodules*, J. Algebra **41** (1976), 269–302. MR 0422350 (54 #10340)



Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

*E-mail address*: asbuch@math.rutgers.edu

Department of Mathematics, The University of North Carolina at Chapel Hill, CB #3250, Phillips Hall, Chapel Hill, NC 27599, USA

*E-mail address*: rimanyi@email.unc.edu