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ABSTRACT

Let X be a non-singular variety and $E_0 \to E_1 \to \cdots \to E_n$ a sequence of vector bundles over X with maps between them. A set of *rank conditions* for this sequence is a collection $r = (r_{ij})$ of non-negative integers, $0 \leq i < j \leq n$. The associated *quiver variety* is the locus

$$\Omega_r(E_\bullet) = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij} \ \forall i < j\}\,.$$

With Fulton we recently found a formula for the cohomology class of $\Omega_r(E_\bullet)$ in the cohomology ring of X, when this locus is irreducible and of maximal codimension in X. This formula extends the Thom-Porteous formula, and it is general enough to give new expressions for all types of Schubert polynomials.

Our formula writes the cohomology class of $\Omega_r(E_\bullet)$ as a linear combination with integer coefficients of products of Schur polynomials:

$$[\Omega_r(E_\bullet)] = \sum c_\mu(r)\, s_{\mu_1}(E_1 - E_0) \cdot \ldots \cdot s_{\mu_n}(E_n - E_{n-1})\,.$$

The sum is over all sequences of partitions $\mu = (\mu_1, \ldots, \mu_n)$. The coefficients $c_\mu(r)$ are given by an explicit algorithm. Surprisingly, these coefficients all seem to be non-negative. We have conjectured a generalized Littlewood-Richardson rule saying that each coefficient is equal to the number of sequences of semistandard Young tableaux satisfying certain properties.

In the first half of my thesis I will prove this conjecture in the special case where the sequence $E_\bullet$ contains four vector bundles. I will also pose a stronger but simpler conjecture, and prove that it implies the generalized Littlewood-Richardson rule. In contrast to the Littlewood-Richardson rule, this stronger conjecture is easy to verify on a computer, and this has been done in 500.000 randomly chosen examples with $n \leq 10$.

In the second half of the thesis I will derive some applications of the quiver formula to Stanley symmetric functions. In particular I will show that certain coefficients in Stanley symmetric functions are special cases of the coefficients $c_\mu(r)$. I will also give geometric proofs of some formulas describing Stanley's symmetric function for a product of two permutations as well as for a longest permutation.

ACKNOWLEDGEMENTS

It is a pleasure to thank my advisor William Fulton for introducing me to and guiding me within many exciting areas of algebraic geometry and combinatorics. The work described in this thesis can be viewed as a continuation of a joint geometric project with Fulton, which resulted in a general formula for quiver varieties [1]. I would like to thank him for a very pleasant collaboration as well as for numerous suggestions, ideas, comments, etc. regarding this continued work. It has been a luxury to work in an environment where encouragements as well as time for discussions have been unlimited resources.

I am indebted to Sergey Fomin for his interest in and contributions to this thesis. In particular I owe him thanks for constructing an involution on pairs of tableaux, which has been a key tool in my work. I also thank him for collaborating about finding a general proof of a conjectured version of the quiver formula.

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Not only mathematical favors are needed in order to complete graduate study. I would like to thank my fiancee Heather Marie Propes for endless amounts of support and encouragement. In particular she has moved three times and more than 6000 miles in order that I could have the best possibilities for studying.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

In this introductory chapter we introduce notation and recall the main results from [1].

1.1 Degeneracy loci

Let X be a non-singular complex variety and $E_0 \to E_1 \to E_2 \to \cdots \to E_n$ a sequence of vector bundles and bundle maps over X. A set of *rank conditions* for this sequence is a collection of non-negative integers $r = (r_{ij})$ for $0 \leq i < j \leq n$. The associated degeneracy locus or quiver variety is the subset

$$\Omega_r(E_\bullet) = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij} \ \forall i < j\}$$

of X. Equivalently this locus is an intersection of the zero sections of bundle maps:

$$\Omega_r(E_\bullet) = \bigcap_{i<j} Z(\wedge^{r_{ij}+1} E_i \to \wedge^{r_{ij}+1} E_j)\,.$$

The later definition shows that $\Omega_r(E_\bullet)$ has a natural structure of subscheme of X.

Let r_{ii} denote the rank of the bundle E_i. We will demand that the rank conditions can *occur*, i.e. that there exists a sequence of vector spaces and linear maps $V_0 \to V_1 \to \cdots \to V_n$ so that $\dim(V_i) = r_{ii}$ and $\operatorname{rank}(V_i \to V_j) = r_{ij}$. This is equivalent to the conditions $r_{ij} \leq \min(r_{i,j-1}, r_{i+1,j})$ for $i < j$, and $r_{ij} - r_{i,j-1} - r_{i+1,j} + r_{i+1,j-1} \geq 0$ for $j - i \geq 2$.

Let E and F be vector bundles of ranks e and f over X and let $I = (a_1, \ldots, a_p)$ be a sequence of integers. We define the double Schur polynomial $s_I(F - E)$ as follows.

Let h_k be the coefficient of the term of degree k in the formal power series expansion of the quotient of total Chern polynomials for the duals of E and F:

$$\sum_{k\geq 0} h_k t^k = \frac{c_t(E^\vee)}{c_t(F^\vee)} = \frac{1 - c_1(E)t + c_2(E)t^2 - \cdots + (-1)^e c_e(E)t^e}{1 - c_1(F)t + c_2(F)t^2 - \cdots + (-1)^f c_f(F)t^f} .$$

Then the polynomial $s_I(F - E)$ is the determinant of the $p \times p$ matrix whose $(i,j)^{\text{th}}$ entry is h_{a_i+j-i},

$$s_I(F - E) = \det(h_{a_i+j-i})_{1\leq i,j\leq p} .$$

The expected (and maximal) codimension of the locus $\Omega_r(E_\bullet)$ in X is

$$d(r) = \sum_{i<j} (r_{i,j-1} - r_{ij}) \cdot (r_{i+1,j} - r_{ij}) .$$

When the locus has this codimension it is a Cohen-Macaulay subscheme of X [10] (see also Lemma A.2 of [8]). The main result of [1] gives a formula for the cohomology class of $\Omega_r(E_\bullet)$ when it has its expected codimension:

$$[\Omega_r(E_\bullet)] = \sum_\mu c_\mu(r)\, s_{\mu_1}(E_1 - E_0) \cdots s_{\mu_n}(E_n - E_{n-1}) . \tag{1.1}$$

Here the sum is over sequences of partitions $\mu = (\mu_1, \ldots, \mu_n)$; the coefficients $c_\mu(r)$ are certain integers given by an explicit combinatorial algorithm which we shall describe in the next section.

There is no immediate geometric reason for the products of Schur polynomials appearing in the formula. However, it is even more surprising that the coefficients $c_\mu(r)$ all seem to be non-negative. Attempts to prove this has led to a conjecture saying that these coefficients count the number of different sequences of tableaux satisfying certain conditions [1]. This conjecture will be explained in Section 1.3, and the main goal of Chapter 2 is to prove it in certain special cases.

The coefficients $c_\mu(r)$ are known to generalize Littlewood-Richardson coefficients [1]. In Chapter 3 we shall prove that they also generalize the coefficients which are obtained when Stanley symmetric functions are expressed in the basis of Schur functions. We will use this to give new proofs of some results about Stanley symmetric functions and to obtain additional evidence for the conjectured formula for the coefficients $c_\mu(r)$.

1.2 Description of the algorithm

Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ be the ring of symmetric functions. The variable h_i may be identified with the complete symmetric function of degree i. If $I = (a_1, a_2, \ldots, a_p)$ is a sequence of integers, define the Schur function $s_I \in \Lambda$ to be the determinant of the $p \times p$ matrix whose $(i,j)^{\text{th}}$ entry is h_{a_i+j-i}:

$$s_I = \det(h_{a_i+j-i})_{1 \le i,j \le p} \,.$$

(Here one sets $h_0 = 1$ and $h_{-q} = 0$ for $q > 0$.) A Schur function is always equal to either zero or plus or minus a Schur function s_λ for a partition λ. This follows from interchanging the rows of the matrix defining s_I. Furthermore, the Schur functions given by partitions form a basis for the ring of symmetric functions [14], [6]. Note that if the coefficients of a formal power series $c_t(E^\vee)/c_t(F^\vee)$ are substituted for the variables h_i, then the Schur function s_I becomes the double Schur polynomial $s_I(F - E)$ defined above.

We will give the algorithm for computing the coefficients $c_\mu(r)$ by constructing an element P_r in the n^{th} tensor power of the ring of symmetric functions $\Lambda^{\otimes n}$, such that

$$P_r = \sum_\mu c_\mu(r)\, s_{\mu_1} \otimes \cdots \otimes s_{\mu_n} \,.$$

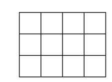
It is convenient to arrange the rank conditions in a *rank diagram*:

$$\begin{array}{ccccccccccc}
E_0 & \to & E_1 & \to & E_2 & \to & \cdots & & \to & & E_n \\
r_{00} & & r_{11} & & r_{22} & & \cdots & & & & r_{nn} \\
& r_{01} & & r_{12} & & \cdots & & & r_{n-1,n} & & \\
& & r_{02} & & \cdots & & & r_{n-2,n} & & & \\
& & & \ddots & & & & & & & \\
& & & & r_{0n} & & & & & &
\end{array}$$

In this diagram we replace each small triangle of numbers

$$\begin{array}{ccc}
r_{i,j-1} & & r_{i+1,j} \\
& r_{ij} &
\end{array}$$

by a rectangle R_{ij} with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns.

$R_{ij} =$  $r_{i+1,j} - r_{ij}$

$r_{i,j-1} - r_{ij}$

These rectangles are then arranged in a *rectangle diagram*:

$$\begin{array}{ccccccc}
R_{01} & & R_{12} & & \cdots & & R_{n-1,n} \\
& R_{02} & & \cdots & & R_{n-2,n} & \\
& & \ddots & & & & \\
& & & R_{0n} & & &
\end{array}$$

It turns out that the information carried by the rank conditions is very well represented in this diagram. First, the expected codimension $d(r)$ for the locus $\Omega_r(E_\bullet)$ is equal to the total number of boxes in the rectangle diagram. Furthermore, the condition that the rank conditions can occur is equivalent to saying that the rectangles get narrower when one travels south-west, while they get shorter when one travels south-east. Finally, the element P_r depends only on the rectangle diagram.

We will define $P_r \in \Lambda^{\otimes n}$ by induction on n. When $n = 1$ (corresponding to a sequence of two vector bundles), the rectangle diagram has only one rectangle $R = R_{01}$. In this case we set

$$P_r = s_R \in \Lambda^{\otimes 1}$$

where R is identified with the partition for which it is the Young diagram. This case recovers the Giambelli-Thom-Porteous formula.

If $n \geq 2$ we let $\bar{r}$ denote the bottom n rows of the rank diagram. Then $\bar{r}$ is a valid set of rank conditions, so by induction we can assume that

$$P_{\bar{r}} = \sum_{\mu} c_\mu(\bar{r})\, s_{\mu_1} \otimes \cdots \otimes s_{\mu_{n-1}} \tag{1.2}$$

is a well defined element of $\Lambda^{\otimes n-1}$. Now P_r is obtained from $P_{\bar{r}}$ by replacing each basis element $s_{\mu_1} \otimes \cdots \otimes s_{\mu_{n-1}}$ in (1.2) with the sum

$$\sum \left(\prod_{i=1}^{n-1} c^{\mu_i}_{\sigma_i \tau_i} \right) s_{\begin{array}{|c|c} \hline R_{01} & \sigma_1 \\ \hline \end{array}} \otimes \cdots \otimes s_{\begin{array}{|c|c} \hline R_{i-1,i} & \sigma_i \\ \hline \tau_{i-1} & \\ \end{array}} \otimes \cdots \otimes s_{\begin{array}{|c|} \hline R_{n-1,n} \\ \hline \tau_{n-1} \\ \end{array}} .$$

This sum is over all partitions $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1, \ldots, \tau_{n-1}$ such that σ_i has fewer rows than $R_{i-1,i}$ and the Littlewood-Richardson coefficient $c^{\mu_i}_{\sigma_i \tau_i}$ is non-zero. A diagram consisting of a rectangle $R_{i-1,i}$ with (the Young diagram of) a partition σ_i attached to its right side, and τ_{i-1} attached beneath should be interpreted as the sequence of integers giving the number of boxes in each row of this diagram.

It can happen that the rectangle $R_{i-1,i}$ is empty, since the number of rows or columns can be zero. If the number of rows is zero, then σ_i is required to be empty, and the diagram is the Young diagram of τ_{i-1}. If the number of columns is zero, then the algorithm requires that the length of σ_i is at most equal to the number of rows $r_{ii} - r_{i-1,i}$ of $R_{i-1,i}$, and the diagram consists of σ_i in the top $r_{ii} - r_{i-1,i}$ rows and τ_{i-1} below this, possibly with some zero-length rows in between.

1.3 A conjecture for the coefficients $c_\mu(r)$

Finally we will describe the conjectured formula for the coefficients $c_\mu(r)$. We will need the notions of (semistandard) Young tableaux and multiplication of tableaux, see for example [6].

A *tableau diagram* for a set of rank conditions is a filling of all the boxes in the corresponding rectangle diagram with integers, such that each rectangle R_{ij} becomes a tableau T_{ij}. Furthermore, it is required that the entries of each tableau T_{ij} are strictly larger than the entries in tableaux above T_{ij} in the diagram, within 45 degree angles. These are the tableaux T_{kl} with $i \leq k < l \leq j$ and $(k,l) \neq (i,j)$.

A *factor sequence* for a tableau diagram with n rows is a sequence of tableaux $(W_1, \ldots, W_n)$, which is obtained as follows: If $n = 1$ then the only factor sequence is the sequence (T_{01}) containing the only tableau in the diagram. When $n \geq 2$, a factor sequence is obtained by first constructing a factor sequence $(U_1, \ldots, U_{n-1})$ for the bottom $n-1$ rows of the tableau diagram, and choosing arbitrary factorizations of the tableaux in this sequence:

$$U_i = P_i \cdot Q_i \,.$$

Then the sequence

$$(W_1, \ldots, W_n) = (T_{01} \cdot P_1 \,,\, Q_1 \cdot T_{12} \cdot P_2 \,, \ldots ,\, Q_{n-1} \cdot T_{n-1,n})$$

is the factor sequence for the whole tableau diagram. The conjecture from [1], which is the theme of Chapter 2, can now be stated as follows:

Conjecture 1.1. *The coefficient $c_\mu(r)$ is equal to the number of different factor sequences $(W_1, \ldots, W_n)$ for any fixed tableau diagram for the rank conditions r, such that W_i has shape μ_i for each i.*

This conjecture first of all implies that the coefficients $c_\mu(r)$ are non-negative and that they are independent of the side lengths of empty rectangles in the rectangle diagram. In addition it implies that the number of factor sequences does only depend on the rectangle diagram and not on the choice of a filling of its boxes with integers.

Example 1.2. Suppose we are given a sequence of four vector bundles and the following rank conditions:

$$\begin{array}{ccccccc} E_0 & \to & E_1 & \to & E_2 & \to & E_3 \\ 1 & & 4 & & 3 & & 3 \\ & 1 & & 2 & & 2 & \\ & & 1 & & 1 & & \\ & & & 0 & & & \end{array}$$

These rank conditions then give the following rectangle diagram:

$$\begin{array}{ccccc} \Big| & & \ydiagram{2} & & \ydiagram{1} \\ & | & & \ydiagram{1} & \\ & & \ydiagram{1} & & \end{array}$$

From the bottom row of this diagram we get

$$P_{\bar{\bar{r}}} = s_{\ydiagram{1}}\,.$$

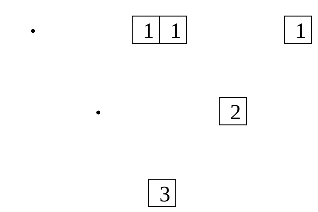
Then using the algorithm we obtain

$$P_{\bar{r}} = s_{\ydiagram{1}} \otimes s_{\ydiagram{1}} \;+\; 1 \otimes s_{\ydiagram{1,1}}$$

and

$$\begin{aligned} P_r = \;& s_{\ydiagram{1}} \otimes s_{\ydiagram{3}} \otimes s_{\ydiagram{1}} \;+\; s_{\ydiagram{1}} \otimes s_{\ydiagram{2}} \otimes s_{\ydiagram{1,1}} \;+\; 1 \otimes s_{\ydiagram{3,1}} \otimes s_{\ydiagram{1}} \;+ \\ & 1 \otimes s_{\ydiagram{2,1}} \otimes s_{\ydiagram{1,1}} \;+\; 1 \otimes s_{\ydiagram{3}} \otimes s_{\ydiagram{1,1}} \;+\; 1 \otimes s_{\ydiagram{2}} \otimes s_{\ydiagram{1,1,1}}\,. \end{aligned}$$

Thus the formula for the cohomology class of $\Omega_r(E_\bullet)$ has six terms. Now, one possible tableau diagram for the given rank conditions is the following:



This diagram has the following six factor sequences:

$$\left(\begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right), \left(\begin{array}{|c|}\hline 3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right), \left(\emptyset, \begin{array}{|c|c|c|}\hline 1&1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}\right),$$

$$\left(\emptyset, \begin{array}{|c|c|}\hline 1&1\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right), \left(\emptyset, \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right), \left(\emptyset, \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right).$$

Since only the rectangle diagram matters for the formula, we will often depict a rank diagram simply as a triangle of dots in place of a triangle of numbers. This is especially convenient when working with paths through the rank diagram, which we shall do in Section 2.1. Such a diagram will often be decorated with the rectangles from the rectangle diagram, or by the tableaux from a tableau diagram. When this is done, each rectangle or tableau is put in the middle of the triangle of dots representing the numbers that produced the rectangle. In this way the rank conditions used in the above example would be represented by the diagram:

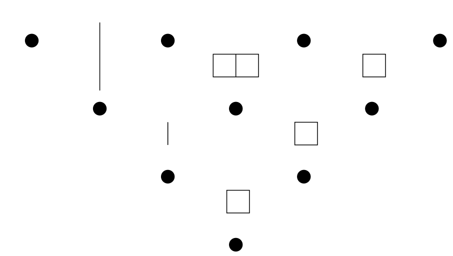


Figure 1.1: The rectangle diagram used in Example 1.2.

CHAPTER 2
RESULTS ABOUT THE CONJECTURED FORMULA

The goal of this chapter is to prove some combinatorial results about the formula for quiver varieties described in Section 1.2. The main result is that the conjectured formula for the coefficients $c_\mu(r)$ is true in some special cases which include all situations where the sequence $E_\bullet$ has up to four bundles. We will also show that the conjecture follows from a stronger but simpler conjecture, for which substantial computational verification has been obtained. For both of these results, a sign-reversing involution on pairs of tableaux constructed by S. Fomin plays a fundamental role.

In Section 2.1 we define a generalization of the formula P_r which for many purposes is easier to work with. We will also define corresponding generalizations of factor sequences and the conjectured formula. In Section 2.2 we will prove a useful criterion for recognizing factor sequences. Section 2.3 gives an account of Fomin's involution, which in Section 2.4 is used to formulate the stronger conjecture mentioned above. Finally, Section 2.5 contains a proof of this stronger conjecture in special cases. Throughout the chapter we shall make use of the row and column bumping algorithms for tableau multiplication as well as the reverse versions of these algorithms. This and more is explained in [6].

2.1 Paths through the rank diagram

In this section we will introduce a generalization of the formula P_r. Define a *path* through the rank diagram to be a union of line segments between neighboring rank conditions, which form a continuous path from r_{00} to r_{nn} such that any vertical line intersects this path at most once.

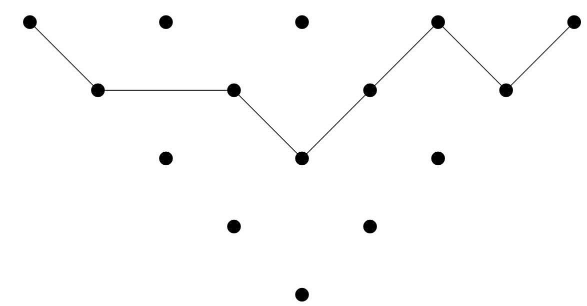


Figure 2.1: Path through the rank diagram.

The length of a path is the number of contained line segments (which is between n and $2n$). Given a path γ of length ℓ, we will define an element $P_\gamma \in \Lambda^{\otimes \ell}$. It is convenient to identify the natural basis elements of $\Lambda^{\otimes \ell}$ with labelings of the line segments of γ with partitions. More generally, if $I_1, \ldots, I_\ell$ are sequences of integers, we will identify the labeling of the line segments in γ by these sequences, left to right, with the element $s_{I_1} \otimes \cdots \otimes s_{I_\ell} \in \Lambda^{\otimes \ell}$. All basis elements occurring in P_γ will label line segments on the side of the rank diagram with the empty partition. If γ is the highest path, going horizontally from r_{00} to r_{nn}, then P_γ is equal to P_r.

We define P_γ inductively as follows. If γ is the lowest possible path, going from r_{00} to r_{0n} to r_{nn}, then we set $P_\gamma = 1 \otimes 1 \otimes \cdots \otimes 1 \in \Lambda^{\otimes 2n}$. In other words P_γ is equal to the single basis element which assigns the empty partition to each line segment. If γ is any other path, then we can find a path γ' which is equal to γ, except it goes lower at one place, in one of the following ways:

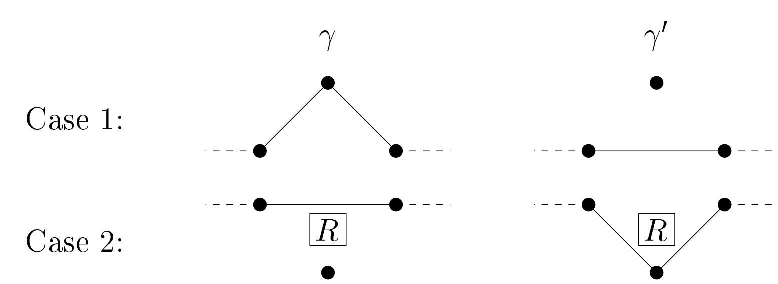
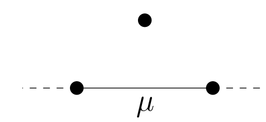


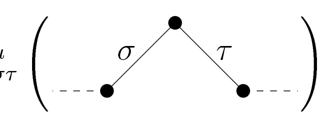
Figure 2.2: The path γ' goes lower than γ at the indicated place.

By induction we may assume that $P_{\gamma'}$ is well defined.

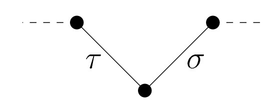
If we are in Case 1 we now obtain P_γ from $P_{\gamma'}$ by replacing each basis element



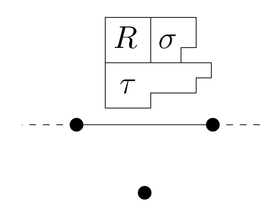
occurring in $P_{\gamma'}$ with the sum

$$\sum_{\sigma,\tau} c^{\mu}_{\sigma\tau}$$


For Case 2, let R be the rectangle associated to the triangle where γ and γ' differ. Then P_γ is obtained from $P_{\gamma'}$ by replacing each basis element



occurring in $P_{\gamma'}$ with zero if σ has more rows than R, and otherwise with the element:



An easy induction shows that this definition is independent of the choice of γ'. The element P_γ has geometric meaning similar to that of P_r. It describes the cohomology class of a degeneracy locus $\Omega_r(\gamma)$ defined in [1].

If we are given a tableau diagram, the notion of a factor sequence can also be extended to paths. Any factor sequence for a path γ will contain one tableau for each line segment in γ. As with basis elements of $\Lambda^{\otimes \ell}$, we will often regard such a sequence as a labeling of the line segments in γ with tableaux.

If γ is the lowest path from r_{00} to r_{0n} to r_{nn} then the only factor sequence is the sequence $(\emptyset, \ldots, \emptyset)$ which assigns the empty tableau to each line segment. Otherwise we can find a lower path γ' as in Case 1 or Case 2 of Figure 2.2. In order to obtain a factor sequence for γ we must first construct one for γ'.

If we are in Case 1, let $(\dots, W, \dots)$ be a factor sequence for γ' such that W is the label of the displayed line segment, and let $W = P \cdot Q$ be an arbitrary factorization of W. Then the sequence $(\dots, P, Q, \dots)$ is a factor sequence for γ. For Case 2, let T be the tableau corresponding to the rectangle R. If $(\dots, Q, P, \dots)$ is a factor sequence for γ' with Q and P the tableaux assigned to the displayed line segments, then $(\dots, Q \cdot T \cdot P, \dots)$ is a factor sequence for γ.

Finally we define coefficients $c_\mu(\gamma) \in \mathbb{Z}$ by the expression

$$P_\gamma = \sum_\mu c_\mu(\gamma)\, s_{\mu_1} \otimes \cdots \otimes s_{\mu_\ell} \in \Lambda^{\otimes \ell}$$

where ℓ is the length of γ. Conjecture 1.1 then has the following generalization:

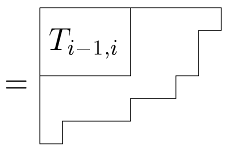
Conjecture 2.1. *The coefficient $c_\mu(\gamma)$ is equal to the number of different factor sequence $(W_1, \dots, W_\ell)$ for the path γ, such that W_i has shape μ_i for each i.*

2.2 A criterion for factor sequences

In this section we will prove a simple criterion for recognizing factor sequences. We will start by discussing this criterion for ordinary factor sequences.

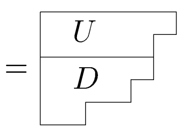
Let $\{T_{ij}\}$ be a tableau diagram and let $(W_1, \dots, W_n)$ be a sequence of tableaux. At first glance it would appear that to check if this sequence is a factor sequence, we would have to find all factor sequences $(U_1, \dots, U_{n-1})$ for the bottom $n-1$ rows of the tableau diagram, as well as all factorizations $U_i = P_i \cdot Q_i$, to see if our sequence $(W_1, \dots, W_n)$ is obtained from any of these, i.e. $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ for all i. Equivalently we could find all factorizations of each W_i into three factors $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ (with $Q_0 = P_n = \emptyset$), and check if $(P_1 \cdot Q_1, \dots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for any of these choices. The criterion for factor sequences allows us to check this for just one factorization of each W_i.

Notice that if the sequence $(W_1, \dots, W_n)$ is a factor sequence, obtained from an inductive factor sequence $(U_1, \dots, U_{n-1})$ as above, then the conditions on the filling of a tableau diagram imply that the entries of each tableau $T_{i-1,i}$ are strictly smaller than the entries of Q_{i-1} and P_i. This implies that $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ contains the rectangular tableau $T_{i-1,i}$ in its upper-left corner.

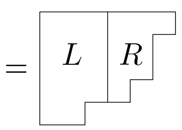
$W_i =$ 

We shall therefore investigate ways to factor a tableau into three pieces, one of which is a contained rectangular tableau.

A quick way to factor any tableau is by cutting it along a horizontal or vertical line. Let T be a tableau and $a \geq 0$ an integer. Let U the top a rows of T, and D the rest of T. Then $T = D \cdot U$. We will call this factorization the *horizontal cut* through T after the a^{th} row. Vertical cuts are defined similarly.

$T =$ 

$= D \cdot U$

$T =$ 

$= L \cdot R$

Lemma 2.2. *Let $T = P \cdot Q$ be any factorization of T and let a be the number of rows in Q. The following are equivalent:*

(i) $T = P \cdot Q$ is a horizontal cut.

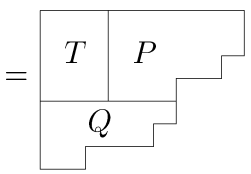
(ii) The i^{th} row of T has the same number of boxes as the i^{th} row of Q for $1 \leq i \leq a$.

(iii) Whenever the top row of P has a box in column $j \geq 1$, the a^{th} row of Q has a strictly smaller box in this column (unless $a = 0$).

Similarly, if P has b columns, then $T = P \cdot Q$ is a vertical cut iff the first b columns of T and P have the same heights, iff the boxes in the last column of P are smaller than or equal to the boxes in similar positions in the first column of Q.

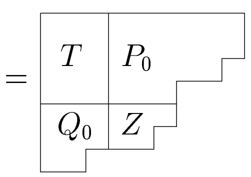
Proof. It is clear that (i) implies (ii) and (iii). If (iii) is true then P and Q fit together to form a tableau with Q in the top a rows and P below. By taking a horizontal cut through this tableau, we see that it must be the product of P and Q. But then it is equal to T and (i) follows. Finally, suppose (ii) is true. When the boxes of P are column bumped into Q to form the product T, all of these boxes must then stay below the a^{th} row. This process therefore reconstructs P below Q and (i) follows. The statements about vertical cuts are proved similarly. $\square$

Now let W be any tableau whose shape contains a rectangle $(b)^a$ with a rows and b columns. We define the *canonical factorization* of W with respect to the rectangle $(b)^a$ to be the one obtained by first taking a horizontal cut through W after the a^{th} row, and then a vertical cut through the top part of W after the b^{th} column.

$W =$  $= Q \cdot T \cdot P$

Note that this definition depends on a, even when b is zero and the rectangle $(b)^a$ is empty. When the product of three tableau Q, T, and P looks like in this picture, we shall say that the pair of tableaux (Q, P) *fits around* the rectangular tableau T.

More generally, let Q_0 be the part of W below T, P_0 the part of W to the right of T, and let Z be the remaining part between Q_0 and P_0.

$W =$ 

We define a *simple factorization* of W with respect to the rectangle $(b)^a$ to be any factorization $W = Q \cdot T \cdot P$, such that $Q = Q_0 \cdot \tilde{Q}$ and $P = \tilde{P} \cdot P_0$ for some factorization $Z = \tilde{Q} \cdot \tilde{P}$.

Note that if $Z = \tilde{Q} \cdot \tilde{P}$ is any factorization of Z and if we put $Q = Q_0 \cdot \tilde{Q}$ and $P = \tilde{P} \cdot P_0$, then $Q \cdot T \cdot P = W$. This follows because $P = \tilde{P} \cdot P_0$ must be a horizontal cut through P, and therefore $T \cdot P = \tilde{P} \cdot T \cdot P_0$. In fact, given arbitrary tableaux $\tilde{Q}$ and $\tilde{P}$ one can show that $Q \cdot T \cdot P = W$ if and only if $\tilde{Q} \cdot \tilde{P} = Z$, but we shall not need this here.

We are now ready to formulate the criterion for factor sequences. Let $\{R_{ij}\}$ be the rectangles corresponding to the tableau diagram $\{T_{ij}\}$. If $(W_1, \ldots, W_n)$ is a factor sequence, a simple factorization of any W_i will always be with respect to the relevant rectangle $R_{i-1,i}$ from the rectangle diagram.

Theorem 2.3. *Let $(W_1, \ldots, W_n)$ be a sequence of tableaux such that each W_i contains $T_{i-1,i}$ in its upper-left corner. Let $W_i = Q_{i-1} \cdot T_{i-1,i} \cdot P_i$ be any simple factorization of W_i with respect to the rectangle $R_{i-1,i}$. Then $(W_1, \ldots, W_n)$ is a factor sequence if and only if Q_0 and P_n are empty tableaux and $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the bottom $n-1$ rows of the tableau diagram $\{T_{ij}\}$.*

We shall derive this result from Proposition 2.7 below. Since this criterion can be applied recursively to the sequence $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$, it gives an easy algorithm to determine if a sequence $(W_1, \ldots, W_n)$ is a factor sequence. Note that the easiest way to produce the simple factorizations is to take the canonical factorization of each W_i. When this choice is made, the work required in the algorithm essentially consists of $n(n-1)/2$ tableau multiplications. Note also that this criterion makes use of the height of any empty rectangles in the rectangle diagram.

For proving this criterion we need some definitions. Let T be a tableau whose shape is the rectangle $(b)^a$ with a rows and b columns. We will consider pairs of tableaux (X, Y) such that all entries in X and Y are strictly larger than the entries of T. For such a pair, let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after the b^{th} column, and let $Y = \tilde{Y} \cdot Y_0$ be the horizontal cut after row a.

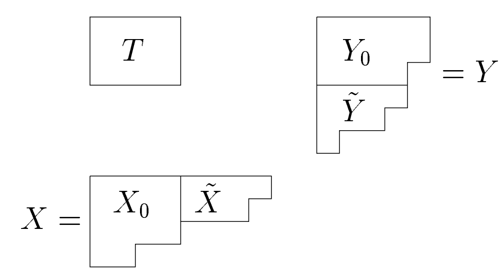


Figure 2.3: Ingredients for the relation $\models$.

If (X', Y') is another pair of tableaux, we will write $(X, Y) \models (X', Y')$ if either

1. for some factorization $\tilde{X} = M \cdot N$ we have $X' = X_0 \cdot M$ and $Y' = N \cdot Y$, or
2. for some factorization $\tilde{Y} = M \cdot N$ we have $X' = X \cdot M$ and $Y' = N \cdot Y_0$.

Note that this implies that $X' \cdot T \cdot Y' = X \cdot T \cdot Y$. In in the first case this follows because $X \cdot T = X_0 \cdot T \cdot \tilde{X}$ and $X' \cdot T = X_0 \cdot T \cdot M$, and the second case is similar. We will let $\rightarrow$ denote the transitive closure of the relation $\models$. This notation depends on the choice of T, as well as the numbers a and b if T is empty.

Lemma 2.4. *Let W be a tableau containing T in its upper-left corner. Suppose that the entries of T are smaller than all other entries in W. If $W = Q \cdot T \cdot P$ is a simple factorization of W with respect to the rectangle $(b)^a$, and if $W = X \cdot T \cdot Y$ is any factorization, then $(X, Y) \rightarrow (Q, P)$.*

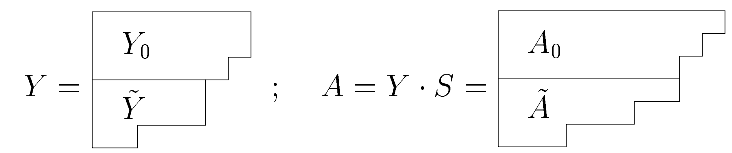
Proof. Let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after column b, and put $Y' = \tilde{X} \cdot Y$. Then let $Y' = \tilde{Y}' \cdot Y_0'$ be the horizontal cut through Y' after row a, and put $X'' = X_0 \cdot \tilde{Y}'$.

We claim that the pair (X'', Y_0') fits around T. Using Lemma 2.2 and that the entries of T are smaller than all other entries, it is enough to prove that the $b + j^{\text{th}}$ entry in the top row of X'' is strictly larger than the j^{th} entry in the bottom row of Y_0'. This will follow if the $b + j^{\text{th}}$ entry in the top row of X'' is larger than or equal to the j^{th} entry in the top row of $\tilde{Y}'$. Since $X'' = X_0 \cdot \tilde{Y}'$ and X_0 has at most b columns, this follows from an easy induction on the number of rows of $\tilde{Y}'$.

It follows from the claim that $W = X'' \cdot T \cdot Y_0'$ is the canonical factorization of W, and therefore we have $(X, Y) \models (X_0, Y') \models (X'', Y_0') \models (Q, P)$ as required. $\square$

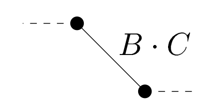
Notice that if $W = X \cdot T \cdot Y$ is a simple factorization and $(X, Y) \models (X', Y')$, then $W = X' \cdot T \cdot Y'$ must also be a simple factorization. It follows that Lemma 2.4 would be false without the requirement that $W = Q \cdot T \cdot P$ is simple.

Lemma 2.5. *Let $a \geq 0$ be an integer, and let Y and S be tableaux with product $A = Y \cdot S$. Let $A = \tilde{A} \cdot A_0$ and $Y = \tilde{Y} \cdot Y_0$ be the horizontal cuts through A and Y after row a, and let $\tilde{Y} = M \cdot N$ be any factorization. Then $N \cdot Y_0 \cdot S = \tilde{A}' \cdot A_0$ for some tableau $\tilde{A}'$, and $M \cdot \tilde{A}' = \tilde{A}$.*

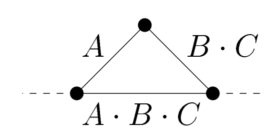


Proof. The first statement follows from the observation that the bottom rows of Y can't influence the top part of $Y \cdot S$, which is a consequence of the row bumping algorithm. Lemma 2.2 then shows that the factorization $A = (M \cdot \tilde{A}') \cdot A_0$ is a horizontal cut, so $M \cdot \tilde{A}' = \tilde{A}$ as required. □

Lemma 2.6. *Let γ be a path through the rank diagram, and let $(\dots, A, B \cdot C, \dots)$ be a factor sequence for γ such that the product $B \cdot C$ is the label of a down-going line segment. Then $(\dots, A \cdot B, C, \dots)$ is also a factor sequence for γ.*



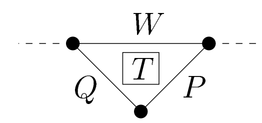
Proof. We will first consider the case where the line segment corresponding to A goes up. Let γ' be the path under γ that cuts short this line segment and its successor.



Then by definition $(\dots, A \cdot B \cdot C, \dots)$ is a factor sequence for γ', which means that $(\dots, A \cdot B, C, \dots)$ is a factor sequence for γ. In general γ lies over a path like the one above, and the general case follows from this. □

Similarly one can prove that if $(\dots, A \cdot B, C, \dots)$ is a factor sequence for a path, such that $A \cdot B$ is the label of an up-going line segment, then $(\dots, A, B \cdot C, \dots)$ is also a factor sequence for this path.

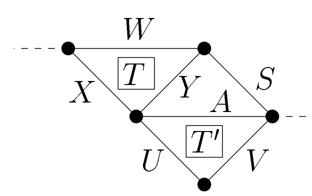
Proposition 2.7. *Let γ and γ' be paths related as in Case 2 of Figure 2.2, and let $(\dots, W, \dots)$ be a factor sequence for γ such that W is the label of the displayed horizontal line segment.*



If $W = Q \cdot T \cdot P$ is any simple factorization of W, then $(\dots, Q, P, \dots)$ is a factor sequence for γ'.

Proof. Since $(\dots, W, \dots)$ is a factor sequence for γ, there exists a factorization $W = X \cdot T \cdot Y$ such that $(\dots, X, Y, \dots)$ is a factor sequence for γ'. By Lemma 2.4 we have $(X, Y) \to (Q, P)$. It is therefore enough to show that if $(X, Y) \models (X', Y')$ then $(\dots, X', Y', \dots)$ is a factor sequence for γ'.

Let a be the number of rows in (the rectangle corresponding to) T, and let $Y = \tilde{Y} \cdot Y_0$ be the horizontal cut through Y after the a^{th} row. We will do the case where a factor of $\tilde{Y}$ is moved to X, the other case is proved using a symmetric argument. We then have a factorization $\tilde{Y} = M \cdot N$ such that $X' = X \cdot M$ and $Y' = N \cdot Y_0$. We can assume that the paths γ and γ' go down after they meet, and that the original factor sequence for γ is $(\dots, W, S, \dots)$.

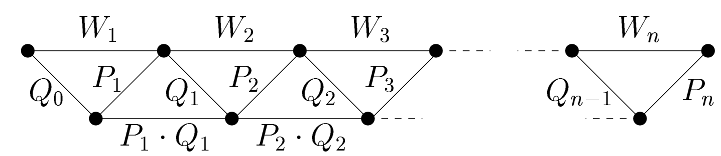


Put $A = Y \cdot S$. Then $(\dots, X, A, \dots)$ is a factor sequence for the path with these labels in the picture. Now let T' be the rectangular tableau associated to the lower triangle, and let $A = U \cdot T' \cdot V$ be the canonical factorization of A. Since this is a simple factorization we may assume by induction that $(\dots, X, U, V, \dots)$ is a factor sequence. Using Lemma 2.5 we deduce that $N \cdot Y_0 \cdot S = U' \cdot T' \cdot V$ for some tableau U', such that $M \cdot U' = U$. Since $(\dots, X, M \cdot U', V, \dots)$ is a factor sequence, so is $(\dots, X \cdot M, U', V, \dots)$ by Lemma 2.6. This means that $(\dots, X \cdot M, U' \cdot T' \cdot V, \dots) = (\dots, X', Y' \cdot S, \dots)$ is a factor sequence, which in turn implies that $(\dots, X', Y', S, \dots)$ is a factor sequence for γ' as required. $\square$

The proof of Proposition 2.7 also gives the following:

Corollary 2.8. *Let* $(\dots, X, Y, \dots)$ *be a factor sequence for the path* γ' *in the proposition. If* $(X, Y) \to (X', Y')$ *then* $(\dots, X', Y', \dots)$ *is also a factor sequence for* γ'.

Proof of Theorem 2.3. The "if" implication follows from the definition. If the sequence $(W_1, \dots, W_n)$ is a factor sequence, then n applications of Proposition 2.7 shows that $(Q_0, P_1, Q_1, P_2, \dots, Q_{n-1}, P_n)$ is a factor sequence for the path with these labels.



It follows that Q_0 and P_n are empty, and $(P_1 \cdot Q_1, \dots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the bottom $n-1$ rows. This proves "only if". $\square$

2.3 An involution of Fomin

In this section we will describe a sign-reversing involution on pairs of tableaux constructed by Sergey Fomin. The purpose of this involution is to cancel out the difference between the coefficients $c_\mu(r)$ produced by the algorithm in Section 1.2, and their conjectured values.

Fix a positive integer a. If P and Q are tableaux of shapes σ and τ such that P has at most a rows, we let $S(\frac{P}{Q})$ denote the symmetric function $s_I \in \Lambda$ where I is the sequence of integers $I = (\sigma_1, \dots, \sigma_a, \tau_1, \tau_2, \dots)$. Let $\mathcal{P}_a$ be the set of all pairs (Q, P) such that $S(\frac{P}{Q}) \neq 0$ and such that P and Q do not fit together as a tableau with P in the top a rows and Q below. This means that the a^{th} row of P must be shorter than the top row of Q, or some box in the top row of Q must be smaller than or equal to the box in the same position of the a^{th} row of P. For example, if $a = 2$ the following pairs are in P_a:

$$\left(\begin{array}{|c|c|c|c|}\hline 3&5&6&7\\\hline 4\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1&3&7&8\\\hline 2&4\\\cline{1-2}\end{array}\right) \text{ and } \left(\begin{array}{|c|c|c|}\hline 4&4&7\\\hline 5&6\\\cline{1-2} 8\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|c|c|}\hline 2&3&3&4&6\\\hline 3&4&5&5\\\cline{1-4}\end{array}\right).$$

Lemma 2.9 (Fomin's involution). *There exists an involution of $\mathcal{P}_a$ with the property that if (Q, P) is mapped to (Q', P') then*

(i) $Q' \cdot P' = Q \cdot P$,

(ii) $S(\frac{P'}{Q'}) = -S(\frac{P}{Q})$, *and*

(iii) the first column of Q' is equal to the first column of Q.

Fomin supplied the proof of this lemma in the form of the beautiful algorithm described below. While Fomin's original description uses path representations of tableaux, we have translated the algorithm into notation that is closer to the rest of this thesis.

We will work with *diagrams with weakly increasing rows*. These will be "Young diagrams" for finite sequences of non-negative integers, where all boxes are filled with integers so that the rows are weakly increasing. Empty rows are allowed as in the following example:

$$\begin{array}{|c|c|c|c|c|c|}\hline 2&3&6&6\\\hline 3&4&5&6&7&7\\\hline \\\cline{1-2} 4&6\\\cline{1-2}\end{array}$$

Figure 2.4: Diagram with weakly increasing rows.

A *violation* for such a diagram to be a tableau is a box in the second row or below, such that there is no box directly above it, or the box directly above it is not strictly smaller. The above diagram has 4 violations in its second row and 2 in row four.

If D is a diagram with weakly increasing rows, and if I is the sequence of row lengths, we put $S(D) = s_I \in \Lambda$. Let $\operatorname{rect}(D)$ denote the tableau obtained by multiplying the rows of D together as tableaux, from bottom to top. We will identify a pair $(Q,P) \in \mathcal{P}_a$ with the diagram D consisting of P in the top a rows and Q below. For this diagram we then have $Q \cdot P = \mathrm{rect}(D)$ and $S(\frac{P}{Q}) = S(D)$.

We will start by taking care of the special case where $a = 1$ and both P and Q have at most one row. In this case Lemma 2.9 without property (iii) is equivalent to the identity $s_{\ell,k} = h_\ell h_k - h_{\ell+1} h_{k-1}$ in the plactic monoid, which is a special case of a result by Lascoux and Schützenberger [16], [11]. The simple proof of this result given in [3] develops techniques which Fomin used to establish Lemma 2.9 in full generality.

Lemma 2.10. *Let D be a diagram with two rows and at least one violation in the second row. Then there exists a unique diagram D' such that* $\mathrm{rect}(D') = \mathrm{rect}(D)$ *and* $S(D') = -S(D)$. *Furthermore, D' also has two rows and at least one violation in the second row. The leftmost violations of D and D' appear in the same column and contain the same number. The parts of D and D' to the left of this column agree.*

Proof. Let p and q be the lengths of the top and bottom rows of D. The requirement $S(D') = -S(D)$ then implies that D' must have two rows with $q-1$ boxes in the top row and $p+1$ in the bottom row. Now it follows from the Pieri formula [6, §2.2] that the product $\mathrm{rect}(D)$ of the rows in D has at most two rows. Furthermore, since D contains a violation, the second row of $\mathrm{rect}(D)$ has at most $q-1$ boxes. Using the Pieri formula again, this implies that there is exactly one way to factorize $\mathrm{rect}(D)$ into a row of length $p+1$ times another of length $q-1$. This establishes the existence and uniqueness of D'.

Explicitly, one may use the inverse row bumping algorithm to obtain this factorization of $\mathrm{rect}(D)$. This is done by bumping out a horizontal strip of $q-1$ boxes which includes all boxes in the second row, working from right to left.

Let x be the leftmost violation of D, where D has the form:

$$D = \begin{array}{|c|c|c|} \hline A & \multicolumn{2}{c|}{E} \\ \hline B & x & F \\ \hline \end{array}\,.$$

Suppose the parts A and B each contain t boxes. Now form the product $F \cdot E$ and let c_j and d_j be the boxes of this product as in the picture:

$$F \cdot E = \begin{array}{|c|c|c|c|ccc|c|}\hline c_1 & c_2 & c_3 & c_4 & & \cdots & & c_k \\ \hline d_1 & d_2 & & \cdots & & d_l & & \\ \cline{1-6} \end{array} .$$

Since x is a violation in D, it must be smaller than all boxes in E and F. Therefore we have

$$x \cdot F \cdot E = \begin{array}{|c|c|c|c|c|ccc|c|}\hline x & c_1 & c_2 & c_3 & c_4 & & \cdots & & c_k \\ \hline d_1 & d_2 & & \cdots & & d_l & & & \\ \cline{1-6} \end{array} .$$

Now since each $d_j > c_j$ it follows that if a horizontal strip of length $q-t-1$ is bumped off this tableau, x will remain where it is. In other words we can factor $x \cdot F \cdot E$ into $x \cdot F' \cdot E'$ such that $x \cdot F'$ and E' are rows of lengths $p-t+1$ and $q-t-1$ respectively. Since the entries of A and B are no larger than x, the products $B \cdot x \cdot F'$ and $A \cdot E'$ are rows of lengths $p+1$ and $q-1$. But the product of these rows is $\operatorname{rect}(D)$, so they must be the rows of D' by the uniqueness. This proves that D' has the stated properties. $\square$

Notice that the uniqueness also implies that the transformation of diagrams described in the lemma is inverse to itself, i.e. an involution.

Now suppose D is any diagram with weakly increasing rows. Then Lemma 2.10 can be applied to any subdiagram of two consecutive rows, such that the second of these rows contains a violation. If this subdiagram is replaced by the new two-row diagram given by the lemma, we arrive at a diagram D' satisfying $S(D') = -S(D)$ and $\operatorname{rect}(D') = \operatorname{rect}(D)$. We will call this an *exchange operation* between the two rows of D.

We shall need an ordering on the violations in a diagram. Here the smallest of two violations is the south-west most one. If the two violations are equally far south-west, then the north-west most one is smaller. In other words, a violation in row i and column j is smaller than another in row i' and column j' iff $j - i < j' - i'$, or $j - i = j' - i'$ and $i < i'$.

Notice that when an exchange operation between two rows is carried out, violations may appear or disappear in these two rows as well as in the row below them. However, the properties given in Lemma 2.10 imply that all of the changed violations will be larger than the left-most violation in the second of the rows exchanged. It follows that the minimal violation in a diagram will remain constant if any (sequence of) exchange operations is carried out. Similarly, all boxes south-west of the minimal violation will remain fixed.

Proof of Lemma 2.9. Given a pair $(Q, P) \in \mathcal{P}_a$, let $\mathcal{D}_{Q,P}$ be the finite set of all non-tableau diagrams D with weakly increasing rows, such that $\mathrm{rect}(D) = Q \cdot P$ and $S(D) = \pm S(\frac{P}{Q})$, and so that the minimal violation in D is in row $a+1$. The pair (Q, P) is then identified with one of the diagrams in this set. We will describe an involution of the set $\mathcal{D}_{Q,P}$ and another of the complement of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ in $\mathcal{D}_{Q,P}$. The restriction of Fomin's involution to $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ is then obtained by applying the involution principle of Garsia and Milne [9] to these involutions.

The involution of $\mathcal{D}_{Q,P}$ simply consists of doing an exchange operation between the rows a and $a+1$ of a diagram. This is possible because all diagrams are required to have a violation in row $a+1$.

Now note that a diagram $D \in \mathcal{D}_{Q,P}$ is in the complement of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ if and only if D has a violation outside the $a+1^{\mathrm{st}}$ row. We take the involution of $\mathcal{D}_{Q,P} \setminus \mathcal{P}_a$ to be an exchange operation between the row of the minimal violation outside row $a+1$, and the row above this violation. This is indeed an involution since the minimal violation outside row $a+1$ will be the same after the exchange operation.

These involutions now combine to give an involution of $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ by the involution principle. To carry it out, start by forming the diagram with P in the top a rows and Q below it. Then do an exchange operation between row a and row $a+1$. If all violations in the resulting diagram are in row $a+1$ we are done. P' is then the top a rows of this diagram and Q' is the rest. Otherwise we continue by doing an exchange operation between the row of the minimal violation outside row $a+1$ and the row above it, followed by another exchange operation between row a and row $a+1$. We continue in this way until all violations are in row $a+1$.

Finally, the properties of P' and Q' follow from the properties of exchange operations. In particular, the requirement $S(\frac{P'}{Q'}) = -S(\frac{P}{Q})$ follows because we always carry out an odd number of exchange operations. $\square$

Example 2.11. The pair $(P, Q) = \left(\begin{array}{|c|c|c|c|}\hline 3&5&6&7\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&3&7&8\\\hline 2&4\\\cline{1-2}\end{array}\right)$ in $\mathcal{P}_2$ gives the following sequence of exchange operations:

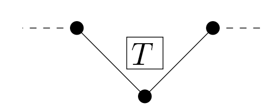
$$\begin{array}{llll}1&3&7&8\\2&4&&\\3&5&6&7\\4&&&\end{array} \rightsquigarrow \begin{array}{llll}1&3&7&8\\2&4&7&\\3&5&6&\\4&&&\end{array} \rightsquigarrow \begin{array}{lllll}1&3&&&\\2&4&7&7&8\\3&5&6&&\\4&&&&\end{array} \rightsquigarrow \begin{array}{llllll}1&3&&&&\\2&4&&&&\\3&5&6&7&7&8\\4&&&&&\end{array}$$

This pair therefore corresponds to $(P', Q') = \left(\begin{array}{|c|c|c|c|c|c|}\hline 3&5&6&7&7&8\\\hline 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right)$ by Fomin's involution.

There are examples of pairs (Q, P) for which the set $\mathcal{P}_a \cap \mathcal{D}_{Q,P}$ has more than two diagrams, all with the same first column. This means that the involution constructed above is not the only one that satisfies the conditions of Lemma 2.9. One way to produce different involutions is to use another ordering among violations. The only property of the order that we have used is that when an exchange operation is performed, any appearing and disappearing violations must be larger than the leftmost violation in the second of the rows being exchanged. For example, given any irrational parameter $\xi \in (0, 1)$, we obtain a new order by letting a violation in position (i, j) be smaller than another in position (i', j') if and only if $j - \xi i < j' - \xi i'$.

2.4 The stronger conjecture

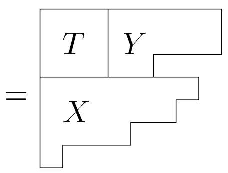
In this section we will present a simple conjecture which implies Conjecture 2.1. Let γ be a path through the rank diagram which at some triangle has an angle pointing down:



Let T be the rectangular tableau associated to this triangle, and suppose the corresponding rectangle has a rows and b columns.

If X and Y are tableaux whose entries are strictly larger than the entries of T, and if Y has at most a rows, we will let

$$\frac{T|Y}{X} =$$



denote the diagram with weakly increasing rows consisting of $T \cdot Y$ in the top a rows and X below. The sequence of row lengths of this diagram then gives an element $S(\frac{T|Y}{X})$ in the ring of symmetric functions Λ. Note that (X, Y) fits around T if and only if the diagram $\frac{T|Y}{X}$ is a tableau.

Suppose that (X, Y) does not fit around T and $S(\frac{T|Y}{X})$ is non-zero. Let $X = X_0 \cdot \tilde{X}$ be the vertical cut through X after the b^{th} column. Then $(\tilde{X}, Y)$ is an element of the set $\mathcal{P}_a$ defined in the previous section. Let $(\tilde{X}', Y')$ be the result of applying Fomin's involution to this pair, and set $X' = X_0 \cdot \tilde{X}'$. Since the first columns of $\tilde{X}$ and $\tilde{X}'$ agree, X' consists of X_0 with $\tilde{X}'$ attached to its right side by Lemma 2.2. It follows that $S(\frac{T|Y'}{X'}) = -S(\frac{T|Y}{X})$. (Note that one could also get from (X, Y) to (X', Y') by applying Fomin's involution to the pair $(X, T \cdot Y)$.)

Conjecture 2.12. *Let $(\dots, X, Y, \dots)$ be a factor sequence for γ with X and Y the labels of the displayed line segments, such that Y has at most a rows. Suppose (X, Y) does not fit around T and $S(\frac{T|Y}{X}) \neq 0$. If X' and Y' are obtained from X and Y by applying Fomin's involution as described above, then $(\dots, X', Y', \dots)$ is also a factor sequence for γ.*

If we fix the location of the down-pointing angle of γ (i.e. the location of T in the tableau diagram), then the strongest case of this conjecture is when the rest of γ goes as low as possible. If Conjecture 2.12 is true for all locations of the down-pointing angle, then the conjectured formula for the coefficients $c_\mu(\gamma)$ is correct.

Theorem 2.13. *Conjecture 2.1 follows from Conjecture 2.12.*

Proof. If $W_1, \dots, W_\ell$ are diagrams with weakly increasing rows, e.g. tableaux, we will write $S(W_1,\dots,W_\ell) = S(W_1)\otimes\cdots\otimes S(W_\ell) \in \Lambda^{\otimes\ell}$. With this notation we must prove that if γ is a path of length ℓ, then

$$P_\gamma = \sum_{(W_i)} S(W_1,\dots,W_\ell) \tag{2.1}$$

where the sum is over all factor sequences (W_i) for γ.

Let γ' be a path under γ as in Case 1 or Case 2 of Figure 2.2. By induction we can assume that Conjecture 2.1 is true for γ', i.e.

$$P_{\gamma'} = \sum_{(U_i)} S(U_1,\dots,U_{\ell'}) \tag{2.2}$$

where this sum is over the factor sequences for γ'. We must prove that the right hand side of (2.1) is obtained by replacing each basis element of (2.2) in the way prescribed by the definition of P_γ. If we are in Case 1 then this follows from the Littlewood-Richardson rule [6, §5.1]: If U is a tableau of shape μ and σ and τ are partitions, then there are $c^\mu_{\sigma\tau}$ ways to factor U into a product $U = P\cdot Q$ such that P has shape σ and Q has shape τ.

Assume we are in Case 2. By induction we then have $P_{\gamma'} = \sum S(\dots,X,Y,\dots)$ where the sum is over all factor sequences $(\dots,X,Y,\dots)$ for γ'; X and Y are the labels of the two line segments where γ' is lower than γ. Let T be the rectangular tableau of the corresponding triangle, and let a be the number of rows in its rectangle. Then by definition we get

$$P_\gamma = \sum_{(\dots,X,Y,\dots)} S(\dots,\frac{T|Y}{X},\dots) \tag{2.3}$$

where the sum is over all factor sequences $(\dots,X,Y,\dots)$ for γ' such that Y has at most a rows.

Now suppose we have a factor sequence $(\dots,X,Y,\dots)$ such that the diagram $\frac{T|Y}{X}$ is a tableau. Then this tableau must be the product $X\cdot T\cdot Y$, and so $(\dots,\frac{T|Y}{X},\dots)$ is a factor sequence for γ. Thus the term $S(\dots,\frac{T|Y}{X},\dots)$ matches one of the terms of (2.1). On the other hand it follows from Proposition 2.7 that every term of (2.1) is matched in this way.

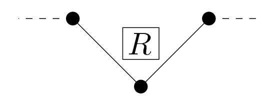
We conclude from this that the terms in (2.1) is the subset of the terms in (2.3) which come from factor sequences such that (X, Y) fits around T. We claim that the sum of the remaining terms in (2.3) is zero. In fact, if $(\dots, X, Y, \dots)$ is a factor sequence for γ' such that (X, Y) doesn't fit around T and $S(\frac{T|Y}{X}) \neq 0$, then we may apply Fomin's involution in the way described above to get tableaux X' and Y'. If Conjecture 2.12 is true, then the sequence $(\dots, X', Y', \dots)$ is also a factor sequence, and since $S(\frac{T|Y'}{X'}) = -S(\frac{T|Y}{X})$, the terms of (2.3) given by these two factor sequences cancel each other out. $\square$

The number of factor sequences for a tableau diagram can be extremely large. For this reason it is almost impossible to verify Conjecture 1.1 or Conjecture 2.1 by computing both sides of their equations. In contrast, instances of Conjecture 2.12 can be tested easily even on large examples. Given a tableau diagram and a path, one can generate a factor sequence for this path by choosing factorizations of tableaux by random. Then one can apply Fomin's involution to the sequence, and use the criterion of Proposition 2.7 to check that the result is still a factor sequence. Such checks have been carried out repeatedly for each of 500,000 randomly chosen tableau diagrams with up to 10 rows of tableaux, without finding any violations of Conjecture 2.12. Together with the results in the next section, we consider this to be convincing evidence for the conjectures.

2.5 Proof in a special case

In this final section we will show that Conjecture 2.12 is true in certain special cases. These cases will be sufficient to prove the conjectured formula for $c_\mu(r)$ when all rectangles in the fourth row of the rectangle diagram are empty, and when no two non-empty rectangles in the third row are neighbors. This covers all situations with at most four vector bundles.

Let γ be a path through the rank diagram with a down-pointing angle as in the previous section. Let R be the rectangle of the corresponding triangle.



We will describe two cases where Conjecture 2.12 can be proved. Both cases require a special configuration of the rectangles surrounding R. Suppose R is the rectangle R_{ij} in the rectangle diagram. We will say that a different rectangle $R' = R_{kl}$ is *below* R if $k \leq i < j \leq l$. R' is *strictly below* R if $k < i < j < l$.

Proposition 2.14. *Conjecture 2.12 is true for γ if all rectangles strictly below R are empty.*

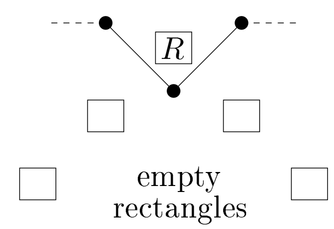


Figure 2.5: Rectangles strictly below R are empty.

Note that this covers all rectangles on the side of the rectangle diagram.

Proof. Let T be the tableau corresponding to R, and suppose $(\ldots, X, Y, \ldots)$ is a factor sequence for γ. Since all tableau on the line going south-west from T in the tableau diagram are narrower than T, it follows that also X has fewer columns than T. Similarly Y has fewer rows than T. But this means that (X, Y) fits around T and the statement of Conjecture 2.12 is trivially true. $\square$

In the other situation we shall describe, we allow three non-empty tableaux below T as shown in Figure 2.6. All other tableaux below T are required to be empty. Let γ be the higher and γ' the lower of the two paths in the diagram.

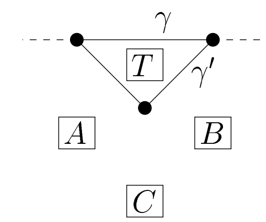


Figure 2.6: All but three of the tableaux below T are empty.

Lemma 2.15. *Let* $(\ldots, X, Y, \ldots)$ *be a labeling of the line segments of* γ' *with tableaux. The following are equivalent:*

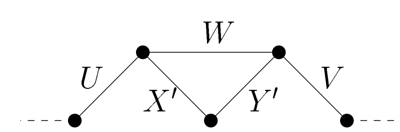
(1) $(\ldots, X, Y, \ldots)$ *is a factor sequence for* γ'*.*

(2) $(\ldots, X \cdot T \cdot Y, \ldots)$ *is a factor sequence for* γ *and the part of* X *that is wider than* T *and the part of* Y *that is taller than* T *have entries only from* C*.*

Proof. It is clear that (1) implies (2). For the other implication, put $W = X \cdot T \cdot Y$ and let $W = X' \cdot T \cdot Y'$ be the canonical factorization of W. Then it follows from Proposition 2.7 that $(\ldots, X', Y', \ldots)$ is a factor sequence for γ'. Since $(X, Y) \to (X', Y')$ by Lemma 2.4, we may assume that $(X, Y) \models (X', Y')$.

We will handle the case where a factor of the bottom part of Y is moved to X, the other case being symmetric. This means that for some tableau M we have $X' = X \cdot M$ and $Y = M \cdot Y'$. Since the bottom part of Y has entries only from C, this is also true for M.

We may assume that γ and γ' go down outside the displayed angle and that our factor sequence is $(\ldots, U, X', Y', V, \ldots)$.



Then by definition there exists a factorization $C = C_1' \cdot C_2'$ such that $A \cdot C_1' = U \cdot X'$ and $C_2' \cdot B = Y' \cdot V$. Since $U \cdot X \cdot M$ consists of A with C_1' attached on its right side, and since all entries of M are strictly larger than the entries of A, it follows that

$U \cdot X$ consists of A with some tableau C_1 attached on the right side. Furthermore $C_1 \cdot M = C_1'$ by Lemma 2.2.

Put $C_2 = M \cdot C_2'$. Then we have $C_1 \cdot C_2 = C$, $A \cdot C_1 = U \cdot X$, and $C_2 \cdot B = Y \cdot V$. It follows that $(\dots, U, X, Y, V, \dots)$ is a factor sequence as required. □

Proposition 2.16. *Conjecture 2.12 is true for the path γ' in Figure 2.6.*

Proof. Let $(\dots, X, Y, \dots)$ be a factor sequence for γ' which satisfies the conditions in Conjecture 2.12, and let X' and Y' be the tableaux obtained from X and Y using Fomin's involution. Since the part of X that is wider than T has entries only from C, the same will be true for X' by Lemma 2.9 (iii). Since Y' has fewer rows than T and since $(\dots, X' \cdot T \cdot Y', \dots) = (\dots, X \cdot T \cdot Y, \cdots)$ is a factor sequence for γ, it follows from Lemma 2.15 that $(\dots, X', Y', \dots)$ is a factor sequence for γ'. □

Corollary 2.17. *Conjecture 1.1 is true if all rectangles in the fourth row of the rectangle diagram are empty, and if no two non-empty rectangles in the third row are neighbors.*

Proof. When the rectangle diagram satisfy these properties, then all instances of Conjecture 2.12 follow from either Proposition 2.14 or Proposition 2.16. The corollary therefore follows from Theorem 2.13. □

In Section 1.2 we defined a rectangle diagram to be any diagram which can be obtained by replacing the small triangles of numbers in a rank diagram with rectangles. However, everything we have done is still true if one defines a rectangle diagram to be any diagram of rectangles, each given by a number of rows and columns, such that the number of rows decreases when one moves south-east while the number of columns decreases when one moves south-west. This definition is slightly more general because the side lengths of the rectangles in a rectangle diagram obtained from rank conditions satisfy certain relations. Although we don't know any geometric interpretation of the more general rectangle diagrams, they seem to be the natural definition for combinatorial purposes.

CHAPTER 3
STANLEY SYMMETRIC FUNCTIONS

3.1 Introduction

The purpose of this chapter is to show a connection between Stanley symmetric functions and the formula for quiver varieties given in [1]. Recall that a simple reflection in the symmetric group S_m is a transposition that interchanges two consecutive integers. A *reduced word* for a permutation $w \in S_m$ is a tuple of simple reflections $(\tau_1, \tau_2, \dots, \tau_\ell)$ with $\ell = \ell(w)$ the length of w, such that $w = \tau_1 \tau_2 \cdots \tau_\ell$. Stanley asked how many reduced words does a permutation w have.

To answer this question, Stanley [17] defined a power series $F_w(x)$ in infinitely many variables $x_1, x_2, \dots$; it is homogeneous of degree $\ell = \ell(w)$, has non-negative integer coefficients, and the number of reduced words for w is the coefficient in $F_w(x)$ of the monomial $x_1 x_2 \cdots x_\ell$. Stanley then proved that this power series is symmetric. This implies that it can be written in the basis of Schur functions:

$$F_w(x) = \sum_{\lambda \vdash \ell} \alpha_{w\lambda}\, s_\lambda(x) \tag{3.1}$$

where the sum is over all partitions λ of ℓ and the coefficients $\alpha_{w\lambda}$ are integers. Since the coefficient of $x_1 x_2 \cdots x_\ell$ in a Schur function $s_\lambda(x)$ is equal to the number f^λ of standard Young tableaux of shape λ (see e.g. [14] or [6]), it follows that the number of reduced words for w is given as

$$\sum_{\lambda \vdash \ell} \alpha_{w\lambda}\, f^\lambda .$$

The constants f^λ are considered well understood, so only a description of the coefficients $\alpha_{w\lambda}$ remained to be found. Stanley credits Edelman and Greene for proving that these coefficients are non-negative [2] (see also [13]). Fomin and Greene have shown that $\alpha_{w\lambda}$ is equal to the number of semistandard Young tableaux T of shape λ, such that the column word of T is a reduced word for w [3]. Another useful fact is that $\alpha_{w^{-1}\lambda} = \alpha_{w\lambda'}$ where λ' is the conjugate of λ [13], [15, (7.22)].

Stanley's symmetric function is known to be a limit of Schubert polynomials $\mathfrak{S}_w(x)$ defined by Lascoux and Schützenberger [12], [15]. For $n \in \mathbb{N}$, let $1^n \times w \in S_{n+m}$ denote the shifted permutation which acts as the identity on $1, \ldots, n$ and maps i to $w(i-n)+n$ for $n+1 \leq i \leq n+m$. If one specializes to finitely many variables $x_1, x_2, \ldots, x_N$, then

$$F_w(x_1, \ldots, x_N, 0, 0, \ldots) = \mathfrak{S}_{1^n \times w^{-1}}(x_1, \ldots, x_N, 0, 0, \ldots) \tag{3.2}$$

for all $n \geq N$ [15, (7.18)].

The formula (1.1) for quiver varieties specializes to a formula for the double Schubert polynomial $\mathfrak{S}_w(x;y)$ for the permutation $w \in S_m$:

$$\mathfrak{S}_w(x;y) = \sum c_w(a,b,\lambda)\, y_2^{a_2} \cdots y_{m-1}^{a_{m-1}}\, (-x_2)^{b_2} \cdots (-x_{m-1})^{b_{m-1}}\, s_\lambda(x/y)\,. \tag{3.3}$$

The sum is over exponents $a_2, \ldots, a_{m-1}$ and $b_2, \ldots, b_{m-1}$ and a partition λ. The coefficients $c_w(a,b,\lambda)$ are special cases of the coefficients $c_\mu(r)$.

The main result in this chapter is that the coefficient $c_w(0,0,\lambda)$ (corresponding to zero exponents) is equal to Stanley's coefficient $\alpha_{w^{-1}\lambda}$. In this way, (3.3) writes a Schubert polynomial $\mathfrak{S}_w(x)$ as a symmetric polynomial equal to Stanley's symmetric function for w^{-1} plus a non-symmetric polynomial.

In Section 3.2 we recall from [1] how to apply the quiver formula (1.1) to calculate Schubert polynomials. In Section 3.3 we prove the identity $\alpha_{w^{-1}\lambda} = c_w(0,0,\lambda)$ and use this to give a new proof of Stanley's result [17] that the symmetric function $F_{w_0}(x)$ for the longest permutation w_0 in S_m is equal to the Schur function $s_\lambda(x)$ for the staircase partition $\lambda = (m-1, m-2, \ldots, 1)$. In Section 3.4 we use geometry of degeneracy loci to prove the well known formula for a Schubert polynomial of a product of two permutations. Finally, in Section 3.5, we discuss the relations to the conjectured Littlewood-Richardson rule stated in Conjecture 1.1.

3.2 Schubert polynomials

Let $w \in S_{m+1}$ be a permutation, and let $E_\bullet$ be a sequence of bundles over X

$$F_1 \subset F_2 \subset \cdots \subset F_m \to G_m \twoheadrightarrow G_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow G_1$$

consisting of a full flag with a general map to a dual full flag. Define the locus

$$\Omega_w = \{x \in X \mid \operatorname{rank}(F_q(x) \to G_p(x)) \le r_w(p,q)\ \forall p,q\}$$

where $r_w(p,q) = \#\{i \le p \mid w(i) \le q\}$. Fulton has proved [5] that the cohomology class of this locus is given by the double Schubert polynomial defined by Lascoux and Schützenberger [12]:

$$[\Omega_w] = \mathfrak{S}_w(x_1,\ldots,x_m;y_1,\ldots,y_m)$$

where $x_i = c_1(\ker(G_i \to G_{i-1}))$ and $y_i = c_1(F_i/F_{i-1})$. Now $\Omega_w = \Omega_r(E_\bullet)$ where $r = (r_{ij})$ are the obvious rank conditions. This means that the double Schubert polynomial becomes a special case of the quiver formula:

$$\begin{aligned}\mathfrak{S}_w(x;y) &= [\Omega_r(E_\bullet)] \\ &= \sum c_\mu(r)\, s_{\mu_1}(F_2 - F_1)\cdots s_{\mu_{m-1}}(F_m - F_{m-1}) \cdot s_{\mu_m}(G_m - F_m) \cdot \\ &\qquad s_{\mu_{m+1}}(G_{m-1} - G_m)\cdots s_{\mu_{2m-1}}(G_1 - G_2)\end{aligned}$$

As noted in [1], significant simplifications can be made by using the equalities

$$s_\lambda(F_{i+1} - F_i) = s_\lambda(y_{i+1}) = \begin{cases} y_{i+1}^a & \text{if } \lambda = (a) \text{ is a row with } a \text{ boxes} \\ 0 & \text{otherwise} \end{cases}$$

and

$$s_\lambda(G_i - G_{i+1}) = s_\lambda(0/x_{i+1}) = \begin{cases} (-x_{i+1})^b & \text{if } \lambda = (1^b) \text{ is a column with } b \text{ boxes} \\ 0 & \text{otherwise.} \end{cases}$$

Using this and the fact that $s_\lambda(G_m - F_m)$ is the super-symmetric Schur polynomial $s_\lambda(x/y)$ in the variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$, we obtain a formula

$$\mathfrak{S}_w(x;y) = \sum c_w(a,b,\lambda)\, y_2^{a_2} \cdots y_m^{a_m}\, (-x_2)^{b_2} \cdots (-x_m)^{b_m}\, s_\lambda(x/y)\,. \tag{3.4}$$

The sum is over exponents $a_2, \ldots, a_m$ and $b_2, \ldots, b_m$, and a single partition λ, and $c_w(a,b,\lambda)$ is the coefficient $c_\mu(r)$ for the sequence of partitions

$$\mu = ((a_2), \ldots, (a_m), \lambda, (1^{b_m}), \ldots, (1^{b_2}))\,.$$

Example 3.1. For the permutation $w = 2\,4\,3\,1$ we get the rank diagram

$$\begin{array}{ccccccccccc}
F_1 & \subset & F_2 & \subset & F_3 & \to & G_3 & \twoheadrightarrow & G_2 & \twoheadrightarrow & G_1 \\
1 & & 2 & & 3 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & & 1 & & 1 & & 1 & & 1 & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & & & 0 & & & & &
\end{array}$$

which in turn gives the rectangle diagram:

$$\begin{array}{ccccccccc}
\vert & & \vert & & \square & & - & & - \\
 & \vert & & \square & & \square & & - & \\
 & & \square & & \cdot & & \cdot & & \\
 & & & \vert & & \cdot & & & \\
 & & & & \vert & & & &
\end{array}$$

The bottom three rows of this rectangle diagram gives

$$P_{\tilde{r}} = s_{\square} \otimes 1 \otimes 1\,;$$

using the algorithm we then get

$$P_{\bar{r}} = s_{\square} \otimes s_{\square} \otimes s_{\square} \otimes 1 \;+\; 1 \otimes s_{(1,1)} \otimes s_{\square} \otimes 1$$

and

$$\begin{aligned} P_r = \;& s_{\square} \otimes s_{\square} \otimes s_{(2)} \otimes 1 \otimes 1 \;+\; s_{\square} \otimes s_{\square} \otimes s_{\square} \otimes s_{\square} \otimes 1 \;+ \\ & s_{\square} \otimes 1 \otimes s_{(2,1)} \otimes 1 \otimes 1 \;+\; s_{\square} \otimes 1 \otimes s_{(1,1)} \otimes s_{\square} \otimes 1 \;+ \\ & 1 \otimes s_{(1,1)} \otimes s_{(2)} \otimes 1 \otimes 1 \;+\; 1 \otimes s_{(1,1)} \otimes s_{\square} \otimes s_{\square} \otimes 1 \;+ \\ & 1 \otimes s_{\square} \otimes s_{(2,1)} \otimes 1 \otimes 1 \;+\; 1 \otimes s_{\square} \otimes s_{(1,1)} \otimes s_{\square} \otimes 1 \;+ \\ & 1 \otimes 1 \otimes s_{(2,1,1)} \otimes 1 \otimes 1 \;+\; 1 \otimes 1 \otimes s_{(1,1,1)} \otimes s_{\square} \otimes 1 \,. \end{aligned}$$

This gives the formula

$$\begin{aligned} \mathfrak{S}_w(x;y) = y_2\, y_3\, s_{(2)}(x/y) - x_3\, y_2\, y_3\, s_{\square}(x/y) + y_2\, s_{(2,1)}(x/y) - x_3\, y_2\, s_{(1,1)}(x/y) \\ + y_3\, s_{(2,1)}(x/y) - x_3\, y_3\, s_{(1,1)}(x/y) + s_{(2,1,1)}(x/y) - x_3\, s_{(1,1,1)}(x/y) \,. \end{aligned}$$

In general, the rectangle diagram associated to a permutation $w \in S_{m+1}$ contains only empty rectangles and 1×1 rectangles, and all of the non-empty ones are located in a diamond below the rectangle $R_{m-1,m}$. In other words, if R_{ij} is not empty then $i \leq m-1$ and $j \geq m$. In fact, R_{ij} is non-empty if and only if the diagram $D'(w)$ from [5] has a box in position $(2m-j, i+1)$, and this happens exactly when $w(2m+1-j) \leq i+1$ and $w^{-1}(i+2) \leq 2m-j$ [1].

3.3 Stable Schubert polynomials

In this section we will apply the quiver formula for Schubert polynomials to calculate Stanley symmetric functions. Let $w \in S_{m+1}$ be a permutation and $r = (r_{ij})$ the corresponding rank conditions. Notice at first that the rank diagram for the one step shifted permutation $1 \times w$ is obtained by adding one to each number r_{ij} in the rank diagram for w, and putting an extra row of ones on the sides of this diagram. For example, if $w = 3\,1\,2$, this looks like:

$$\begin{array}{ccccccc} 1 & & 2 & & 2 & & 1 \\ & 1 & & 1 & & 1 & \\ & & 1 & & 0 & & \\ & & & 0 & & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 3 & & 2 & & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ & & 1 & & 2 & & 1 & & 1 & & \\ & & & 1 & & 1 & & 1 & & & \\ & & & & 1 & & 1 & & & & \\ & & & & & 1 & & & & & \end{array}$$

Figure 3.1: Rank diagram for a shifted permutation.

This means that the rectangle diagram for $1 \times w$ is obtained by adding a rim of empty rectangles to the sides of the rectangle diagram for w.

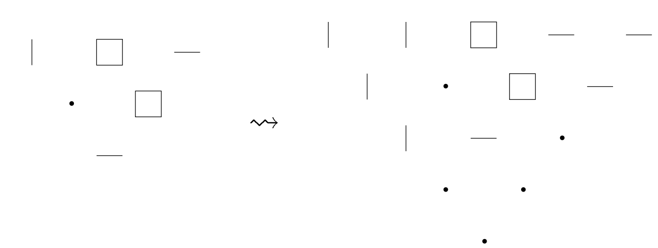


Figure 3.2: Rectangle diagram for a shifted permutation.

Similarly, one obtains the rectangle diagram for $1^n \times w$ by adding n rims of empty rectangles to the rectangle diagram for w.

Let $P_r \in \Lambda^{\otimes 2m-1}$ be the element associated to the rank conditions $r = (r_{ij})$ for w. The above comparison of rectangle diagrams then shows that $1^n \times w$ corresponds to the element

$$\underbrace{1 \otimes \cdots \otimes 1}_{n} \otimes P_r \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n} \in \Lambda^{\otimes 2m+2n-1}\,.$$

By (3.4) this gives us

$$\mathfrak{S}_{1^n \times w}(x;y) = \sum c_w(a,b,\lambda)\, y_{2+n}^{a_2} \cdots y_{m+n}^{a_m}\, (-x_{2+n})^{b_2} \cdots (-x_{m+n})^{b_m}\, s_\lambda(x/y)$$

where $s_\lambda(x/y)$ is in variables $x_1, \dots, x_{m+n}$ and $y_1, \dots, y_{m+n}$.

Now restrict to two fixed sets of variables $x_1, \dots, x_N$ and $y_1, \dots, y_M$, setting $x_i = y_j = 0$ for $i > N$ and $j > M$. When $n \geq \max(N-1, M-1)$, the only non-zero terms in the above expression for $\mathfrak{S}_{1^n \times w}$ are those with all exponents a_i and b_i equal to zero. Since this Schubert polynomial is homogeneous of degree equal to the length of w, the partitions λ occurring in these terms all have weight $\ell(w)$. This proves:

Theorem 3.2. *Let $w \in S_{m+1}$ and fix two sets of variables $x_1, \dots, x_N$ and $y_1, \dots, y_M$. When $n \geq \max(N-1, M-1)$, the double Schubert polynomial $\mathfrak{S}_{1^n \times w}$ in these variables is given by*

$$\mathfrak{S}_{1^n \times w}(x_1, \dots, x_N, 0, \dots, 0; y_1, \dots, y_M, 0, \dots, 0) = \sum_{\lambda \vdash \ell(w)} c_w(0,0,\lambda)\, s_\lambda(x/y)\,.$$

Comparing with equations (3.1) and (3.2) we obtain (since $\alpha_{w\lambda} = \alpha_{w^{-1}\lambda'}$):

Corollary 3.3. *Stanley's coefficient $\alpha_{w\lambda}$ is equal to $c_w(0,0,\lambda')$.*

Thus the formula (3.3) writes a Schubert polynomial as a symmetric part equal to Stanley's symmetric function plus additional non-symmetric terms. For example, if $w = 2\,4\,3\,1$ as in the above example, we have $F_w(x) = s_{\yng(3,1)}(x)$.

The identity $\alpha_{w\lambda'} = \alpha_{w^{-1}\lambda}$ becomes a special case of the identity $c_{w^{-1}}(b,a,\lambda') = c_w(a,b,\lambda)$, which in turn follows from the formula $c_{\mu^\vee}(r^\vee) = c_\mu(r)$ of [1]. Here $r^\vee$ are the rank conditions obtained by mirroring the rank diagram for $r = (r_{ij})$ in a vertical line, so $r^{\vee}_{ij} = r_{n-j,n-i}$, and $\mu^{\vee}$ is the sequence $(\mu'_n, \ldots, \mu'_1)$ of conjugate partitions in the opposite order.

Example 3.4. Let $w_0 = m \cdots 2\,1$ be the longest permutation in S_m. Then we have $r_{w_0}(p, q) = \max(p + q - m, 0)$. The rectangle diagram associated to w_0 therefore has exactly i non-empty rectangles in the i^{th} row for $1 \leq i \leq m-1$, and these are centered around the middle. All other rectangles are empty.

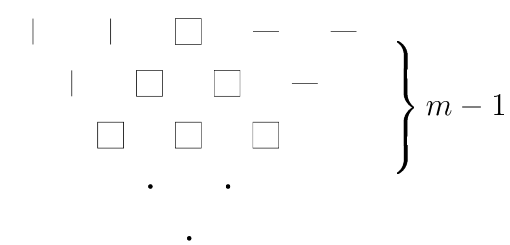


Figure 3.3: Rectangle diagram for a longest permutation.

We will use this diagram to compute Stanley's symmetric function for w_0. The idea is that in order for the algorithm to produce a contribution to F_{w_0}, all boxes must travel north-west until they meet the last non-empty rectangle in this direction, and from that point they must travel north-east.

Let $P_r^{(k)}$ denote the element in $\Lambda^{\otimes 2m-3-k}$ given by this rectangle diagram with the top k rows removed. In particular $P_r^{(0)} = P_r$ is the element associated to the whole diagram. The terms s_λ in Stanley's symmetric function F_{w_0} are in 1-1 correspondence with terms in P_r of the form

$$\underbrace{1 \otimes \cdots \otimes 1}_{m-2} \otimes s_\lambda \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-2}.$$

One may check that, in order for a term

$$s_{\mu_1} \otimes \cdots \otimes s_{\mu_{2m-3-k}}$$

in $P_r^{(k)}$ to contribute to F_{w_0}, the partition μ_i must be empty if the i^{th} rectangle in the

$k+1^{\text{th}}$ row of the rectangle diagram is empty, while it must have length at most one unless the i^{th} rectangle is the leftmost non-empty rectangle in the $k+1^{\text{th}}$ row. To be precise, the term $s_{\mu_1} \otimes \cdots \otimes s_{\mu_{2m-3-k}}$ contributes to F_{w_0} only if μ_i has length at most one for $i \neq m-1-k$, and is empty when $i \leq m-2-k$ and when $i \geq m$. The reason is that all rectangles in the diagram have height at most one, which means that any σ_i in the algorithm can have length at most one. So if any μ_i has two or more rows, boxes are forced to the right, creating a new partition with too many rows if $i \neq m-1-k$.

An examination of the algorithm then shows that $P_r^{(k)}$ contains only one such term, with coefficient 1. This term has μ_{m-1-k} equal to the staircase partition with $m-1-k$ rows, $\mu_{m-1-k} = (m-1-k, \ldots, 2, 1)$, while any other non-empty partitions μ_i is a single row with $m-1-k$ boxes, $\mu_i = (m-1-k)$.

Taking $k=0$, we see that $F_{w_0} = s_{(m-1,m-2,\ldots,2,1)}$. This was first proved by Stanley [17], and implies that the number of reduced words for w_0 is equal to the number of standard tableaux on the staircase partition with $m-1$ rows.

3.4 Redundant rank conditions and products of permutations

Suppose we are given a sequence of bundles $E_0 \to E_1 \to \cdots \to E_n$ and a set of rank conditions $r = \{r_{ij}\}$ for this sequence. The degeneracy locus $\Omega_r(E_\bullet)$ is then the subset of points $x \in X$ over which the maps on fibers satisfy all of the inequalities

$$\operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij}$$

for $i < j$. Some of these inequalities may be redundant in the sense that they follow from other inequalities. It is easy to see that the inequality involving the number r_{ij} is redundant if and only if this number is equal to one of $r_{i,j-1}$ or $r_{i+1,j}$. In other words, the inequality involving r_{ij} is necessary if and only the rectangle R_{ij} is not empty.

Now suppose there are integers $0 \leq p \leq q \leq n$ such that the rectangle R_{ij} is empty whenever exactly one of i and j is in the interval $[p,q]$. In this case the degeneracy locus $\Omega_r(E_\bullet)$ is the (scheme-theoretic) intersection of two larger loci $\Omega_{r'}(E'_\bullet)$ and $\Omega_{r''}(E''_\bullet)$ for the sequences $E'_\bullet : E_p \to E_{p+1} \to \cdots \to E_q$ and $E''_\bullet : E_1 \to \cdots \to E_{p-1} \to E_{q+1} \to \cdots \to E_n$, where r' and r'' are the restrictions of the rank conditions $r = \{r_{ij}\}$ to these sequences. We will say that $E'_\bullet$ is an *independent subsequence*. Note that if $p = q$, the bundle E_p is redundant and can be removed from the sequence $E_\bullet$ without changing $\Omega_r(E_\bullet)$. This special case was described in [1].

When $E'_\bullet$ is an independent subsequence, the rectangle diagram for the rank conditions r' simply consists of the rectangles R_{ij} for $p \le i < j \le q$, while the rectangle diagram for r'' contains the remaining non-empty rectangles.

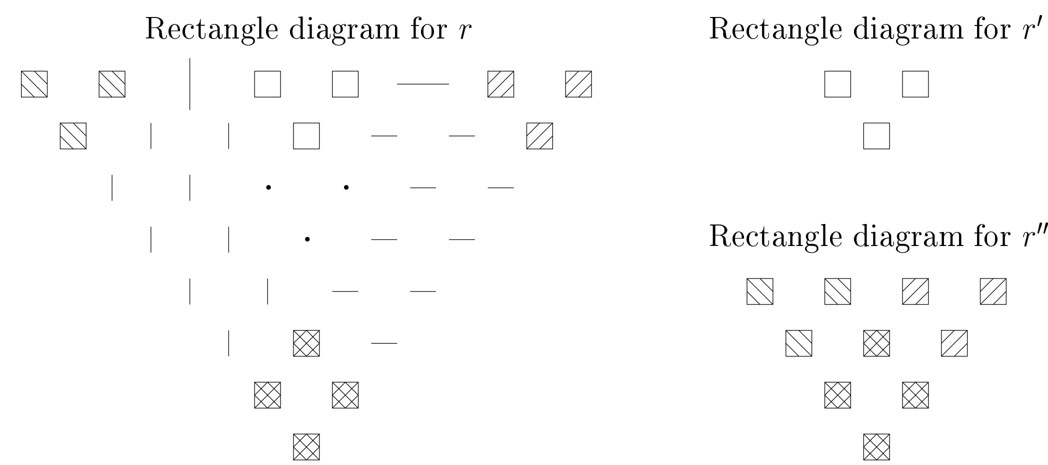


Figure 3.4: Rectangle diagrams for an independent subsequence.

This in particular means that the expected codimensions of $\Omega_{r'}(E'_\bullet)$ and $\Omega_{r''}(E''_\bullet)$ add up to that of $\Omega_r(E_\bullet)$. If all of these loci have their expected codimensions, then we get the equality $[\Omega_r(E_\bullet)] = [\Omega_{r'}(E'_\bullet)] \cdot [\Omega_{r''}(E''_\bullet)]$ in the cohomology ring of X. To see this, note at first that both $\Omega_{r'}(E'_\bullet)$ and $\Omega_{r''}(E''_\bullet)$ are Cohen-Macaulay [10] (see also [8, Lemma A.2]). If $f : \Omega_{r'}(E'_\bullet) \hookrightarrow X$ is the inclusion, we therefore get

$$[\Omega_r(E_\bullet)] = f_*[f^{-1}(\Omega_{r''}(E''_\bullet))] = f_* f^*[\Omega_{r''}(E''_\bullet)] = [\Omega_{r'}(E'_\bullet)] \cdot [\Omega_{r''}(E''_\bullet)]\,.$$

This means that the formula P_r satisfies

$$P_r = (\underbrace{1 \otimes \cdots \otimes 1}_{p} \otimes P_{r'} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-q}) \cdot \Phi_p^{q-p+2}(P_{r''})$$

where multiplication is performed factor-wise, and Φ_p^k denotes the k-fold coproduct expansion of the p^{th} factor of its arguments, i.e.

$$\Phi_p^k(s_{\mu_1} \otimes \cdots \otimes s_{\mu_p} \otimes \cdots \otimes s_{\mu_\ell}) = \sum_{\sigma_1,\ldots,\sigma_k} c^{\mu_p}_{\sigma_1,\ldots,\sigma_k}\, s_{\mu_1} \otimes \cdots \otimes s_{\mu_{p-1}} \otimes s_{\sigma_1} \otimes \cdots \otimes s_{\sigma_k} \otimes s_{\mu_{p+1}} \otimes \cdots \otimes s_{\mu_\ell}\,.$$

We will apply this to study the Schubert polynomial of a product of two permutations. If $w \in S_m$ and $u \in S_n$ are permutations, define the product $w \times u \in S_{m+n}$ to be the permutation which maps i to $w(i)$ if $1 \leq i \leq m$, while $m+i$ is mapped to $m + u(i)$ for $1 \leq i \leq n$. The rank diagram for this permutation is equal to that of $1^m \times u$, except the bottom $2m-2$ rows are replaced by the rank diagram for w. The diamond of non-empty rectangles in the rectangle diagram for $w \times u$ is therefore split into a top part containing the diamond of rectangles for u and a bottom part with the diamond of rectangles for w.

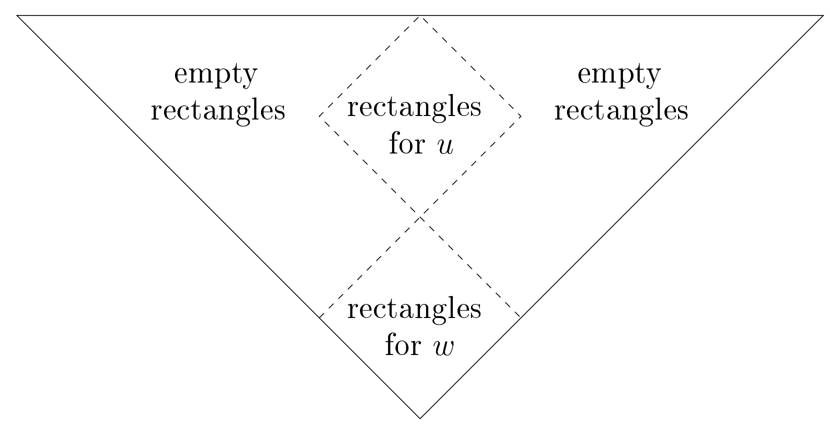


Figure 3.5: Rectangle diagram for a product of permutations

Given a sequence of bundles consisting of a full flag of length $m+n-1$ followed by a full dual flag of the same length as in Section 3.2, we deduce that the locus $\Omega_{w\times u}$ is the intersection of the loci Ω_w and $\Omega_{1^m\times u}$. We therefore recover the well known formula [15, (4.6)]

$$\mathfrak{S}_{w\times u} = \mathfrak{S}_w \cdot \mathfrak{S}_{1^m\times u}$$

for the Schubert polynomial of a product of two permutations. This immediately implies Stanley's identity $F_{w\times u} = F_w \cdot F_u$ [17]. Note that the same method also gives a formula for Fulton's universal Schubert polynomial [7] for a product of permutations.

3.5 Relations to a conjectured Littlewood-Richardson rule

In this final section we will discuss relations with Stanley symmetric functions of the generalized Littlewood-Richardson rule for the coefficients $c_\mu(r)$.

Let $r = (r_{ij})$ be the rank conditions given by a permutation w, and fix a tableau diagram for these rank conditions. Then Conjecture 1.1 implies that Stanley's coefficient $\alpha_{w\lambda}$ is equal to the number of different tableaux W of shape λ', for which $(\emptyset, \dots, \emptyset, W, \emptyset, \dots, \emptyset)$ is a factor sequence. Thus a proof of the general conjecture will give a new proof that Stanley's coefficients are non-negative, as well as an interesting way to compute them.

Example 3.5. Let $w = 2\,1\,4\,3 \dots (2p)\,(2p-1) \in S_{2p}$ for some $p > 0$. Then the rectangle diagram for w has a 1×1 rectangle in the middle of row $4i+1$ for $0 \leq i \leq p-1$. All other rectangles are empty.

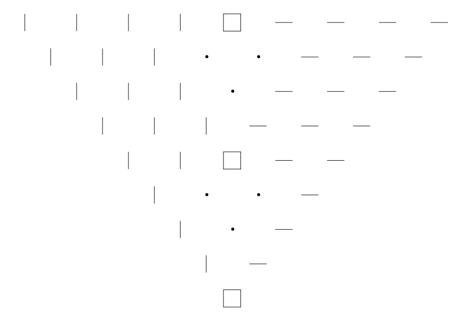


Figure 3.6: Rectangle diagram for the permutation $w = 2\,1\,4\,3\ldots(2p)\,(2p-1)$

A tableau diagram is obtained by filling the numbers $1, 2, \ldots, p$ in these boxes. It is easy to see that a sequence $(\emptyset, \ldots, \emptyset, W, \emptyset, \ldots, \emptyset)$ is a factor sequence for this diagram if and only if W is a standard tableau with p boxes. Therefore the conjecture predicts that Stanley's symmetric function is given by

$$F_w = \sum_{\lambda \vdash p} f^\lambda \, s_\lambda \, .$$

This can be confirmed using Stanley's formula $F_{w\times u} = F_w \cdot F_u$ [17]. Let $\sigma = 2\,1 \in S_2$. Then $w = \sigma \times \cdots \times \sigma$ (p times), which implies that

$$F_w = (F_\sigma)^p = (s_{\square})^p = \sum_{\lambda \vdash p} f^\lambda \, s_\lambda \, .$$

We thank F. Sottile for showing us a different proof of this fact.

Using the criterion for factor sequences of Theorem 2.3, one may also prove that the conjectured Littlewood-Richardson rule gives the correct prediction for Stanley's symmetric function of a longest permutation w_0.

In general, Stanley's symmetric function F_w is known to have a minimal term $s_{\lambda(w)}$ and a maximal term $s_{\mu(w)}$, both occurring with coefficient one. If $w \in S_{m+1}$, define

$$r_p(w) = \#\{q \mid q < p \text{ and } w(q) > w(p)\}$$

for $1 \leq p \leq m+1$, and let $\lambda(w)$ be the partition obtained by arranging the numbers $r_1(w), \ldots, r_{m+1}(w)$ in decreasing order. Let $\mu(w)$ be the conjugate of the partition $\lambda(w^{-1})$. Then $\alpha_{w,\lambda(w)} = \alpha_{w,\mu(w)} = 1$, and any partition λ with $\alpha_{w\lambda} \neq 0$ is between $\lambda(w)$ and $\mu(w)$ in the dominance order [17].

Let $\{T_{ij}\}_{1 \leq i < j \leq 2m}$ be a tableau diagram for (the rank conditions given by) w. There are two extremal ways to form a factor sequence $(\emptyset, \ldots, \emptyset, W, \emptyset, \ldots, \emptyset)$ for this diagram. The first is to make all factorizations of inductive factor sequences $(U_1, \ldots, U_k)$ be "rightward" whenever possible. This means that when factoring U_i into $U_i = P_i \cdot Q_i$, we take $P_i = \emptyset$ and $Q_i = U_i$ for $i \neq m$ while we take $P_m = U_m$ and $Q_m = \emptyset$ (if $k \geq m$). The middle tableau in the final factor sequence then is

$$W_{\text{right}} = T_m \cdot T_{m+1} \cdot \ldots \cdot T_{2m-1}$$

where

$$T_j = T_{0j} \cdot T_{1j} \cdot \ldots \cdot T_{m-1,j} \, .$$

Note that each tableau T_j has only one column. If we set $p = 2m + 1 - j$ and $q = w^{-1}(i+2)$ then T_{ij} is non-empty if and only if $q < p$ and $w(q) > w(p)$. It follows that T_j has exactly $r_p(w)$ boxes.

We claim that W_{right} has shape $\lambda(w)'$, corresponding to the maximal term of $F_{w^{-1}}$. It is enough to show that the if T_l and T_j both have a box in row t and $l < j$, then the box in T_l is smaller than the one in T_j. To prove this, let the t^{th} box in T_l come from T_{kl} and the t^{th} box in T_j come from T_{ij}. If the box in T_{kl} is not smaller than the box in T_{ij} then $k < i$. Now since the tableau T_{il} must be as wide as T_{kl} and as tall as T_{ij}, this tableau T_{il} can't be empty. Similarly, if T_{hj} corresponds to a box over T_{ij} in T_j, then T_{hl} gives a corresponding box in T_l. This shows that the boxes corresponding to T_{kl} and T_{ij} in T_l and T_j was not in the same row, a contradiction.

Similarly one can show that the tableau obtained by “leftward” factorizations,

$$W_{\text{left}} = (T_{0,m} \cdot T_{0,m+1} \cdots T_{0,2m-1}) \cdot (T_{1,m} \cdots T_{1,2m-1}) \cdots (T_{m-1,m} \cdots T_{m-1,2m-1})\,,$$

has shape $\mu(w)'$.

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