

A JOURNEY FROM THE OCTONIONIC \mathbb{P}^2 TO A FAKE \mathbb{P}^2

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ABSTRACT. We discover a family of surfaces of general type with $K^2 = 3$ and $p = q = 0$ as free C_{13} quotients of special linear cuts of the octonionic projective plane $\mathbb{O}\mathbb{P}^2$. A special member of the family has 3 singularities of type A_2 , and is a quotient of a fake projective plane. We use the techniques of [BF20] to define this fake projective plane by explicit equations in its bicanonical embedding.

1. INTRODUCTION

Fake projective planes are complex projective surfaces of general type with Hodge numbers equal to those of the usual projective plane $\mathbb{C}\mathbb{P}^2$. There are exactly 50 complex conjugate pairs, constructed as ball quotients in [CS11] and they are fascinating gemstones in the vast mine of algebraic surfaces of general type. The first explicit equations of a pair of fake projective planes were constructed in [BK19], and additional six pairs were given explicitly in [BF20]. We refer the reader to [BK19] for more background and history.

Many fake projective planes \mathbb{P}_{fake}^2 admit an action of the cyclic group C_3 . The quotient \mathbb{P}_{fake}^2/C_3 is then a singular surface with $K^2 = 3$ and three singular points of type A_2 . It can be deformed to construct interesting smooth surfaces with $K^2 = 3$, genus $p = 0$, and irregularity $q = 0$. In [BF20] the process was reversed and since the current paper is in many ways analogous, we describe [BF20] in some detail below.

The paper [BF20] first builds a family of special complete intersections of seven Plücker hyperplanes in the Grassmannian $\text{Gr}(3, \mathbb{C}^6)$ which admit a free action of the cyclic group C_{14} . This gives a family of surfaces W which has $K_W^2 = 3$, $p = q = 0$. Then the authors find an element of this family such that the quotient by C_{14} has an additional C_3 symmetry and three A_2 singularities. Its Galois cover (that was not at all easy to construct) is a fake projective plane with the automorphism group $(C_3)^2$, labeled by $(C2, p = 2, \emptyset, d_3D_3)$ in Cartwright-Steger classification [CS11+].

In the table [BCP11, Table 1] there are listed surfaces with fundamental group C_{13} instead of C_{14} . The current paper is the result of our efforts to replicate the approach of [BF20] and to construct their fundamental covers as complete intersections in some homogeneous space. We were not quite able to do it, instead we constructed them as *almost* complete intersections of the 16 dimensional octonionic projective plane $\mathbb{O}\mathbb{P}^2$ in \mathbb{P}^{26} by certain 15 linear equations, equivariant with respect to an order 13 element in the Cartan subgroup of the E_6 group of automorphisms of $\mathbb{O}\mathbb{P}^2$.

Afterwards, the process was rather similar to that of [BF20], although there were some technical complications due to lack of unramified double covers. In particular, we had more difficulty controlling the size of the coefficients and had to work with 60K decimal digit numbers at some intermediate steps.

The paper is organized as follows. In Section 2 we describe our motivation for using the octonionic projective plane and the special linear cuts that achieve our goal of constructing surfaces with $K^2 = 3$, $p = q = 0$. We also describe how we found a special element of this family with three A_2 singularities. In Section 3 we briefly explain the construction of the fake projective plane \mathbb{P}_{fake}^2 , labeled by $(C18, p = 3, \emptyset, d_3D_3)$ in [CS11+], and state some open problems.

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2. SPECIAL CUTS OF THE OCTONIONIC PROJECTIVE PLANE

2.1. Motivation. As was mentioned in the Introduction, we set out to find a family of surfaces of general type with $K^2 = 3$, $p = q = 0$ and fundamental group C_{13} . Here is how this search led us to consider cuts of the octonionic projective plane $\mathbb{O}\mathbb{P}^2$.

Let X be the universal cover of a surface in question, with a free action of an order 13 automorphism g . Then $K_X^2 = 39$ and $\chi(K_X) = 13$. It is reasonable to expect that $h^1(X, K_X) = 0$ and $h^0(X, K_X) = 12$. Then the pluricanonical ring $\bigoplus_{n \geq 0} H^0(X, nK_X)$ of X must have the graded dimension

$$\sum_{n \geq 0} \dim H^0(X, nK_X) t^n = 1 + 12t + 52t^2 + 130t^3 + \dots = \frac{1 + 9t + 19t^2 + 9t^3 + t^4}{(1-t)^3}.$$

It is also reasonable to assume that the pluricanonical ring is generated in degree one, so X is embedded into \mathbb{P}^{11} . It is also plausible that its image is cut out by $12(12+1)/2 - 52 = 26$ quadrics. By the Holomorphic Lefschetz formula, as in [H11, Theorem 2.1], the trace of the action of g on $H^0(X, K_X)$ is (-1) , which means that $H^0(X, K_X)$ has a basis of eigenvectors of g with eigenvalues ζ_{13}^i for $i = 1, \dots, 12$. Similarly, the action of g on the space of quadrics splits it into 13 two-dimensional eigenspaces.

Inspired by [BF20], we undertook a rather exhaustive computer search for homogeneous varieties of degree 39 and other relevant invariants, but were not successful. However, the octonionic projective plane $\mathbb{O}\mathbb{P}^2$ has degree 78, and we observed the following remarkable coincidence: the homogeneous coordinate ring of $\mathbb{O}\mathbb{P}^2$ has graded dimension

$$\frac{(1 + 9t + 19t^2 + 9t^3 + t^4)(1 + t)}{(1-t)^{17}}.$$

More specifically, $\mathbb{O}\mathbb{P}^2$ is the dim 16 singular locus of the E_6 -invariant cubic in \mathbb{P}^{26} cut out by the 27 quadratic equations which are the partial derivatives of the cubic. So our idea was to take a linear cut of $\mathbb{O}\mathbb{P}^2$ by 15 equations (so that we are in \mathbb{P}^{11}) which only drop the dimension by 14. We also want one of the quadratic equations to reduce to zero on the linear cut. The best analogy would be cutting a quadric $xy = zw$ with Hilbert series $\frac{1+t}{(1-t)^3}$ by two linear equations $x = 0$ and $z = 0$ to get a line with the Hilbert series $\frac{1}{(1-t)^2}$ but, ultimately, it was a lucky guess.

2.2. Octonionic projective plane. There are several incarnations of the E_6 -invariant cubic found in the literature. We used the one in Jacob Lurie's undergraduate thesis [L11], namely

$$\begin{aligned} & -P_{10}P_{13}P_{16} - P_{11}P_{14}P_{17} - P_{12}P_{15}P_{18} - P_{16}P_{17}P_{18} + P_1P_{10}P_{19} - P_1P_{18}P_2 + P_{11}P_2P_{20} - P_1P_{14}P_{21} \\ & - P_{16}P_{20}P_{21} - P_{18}P_{22}P_{23} - P_{17}P_{19}P_{24} - P_{13}P_2P_{24} - P_{14}P_{15}P_{25} - P_{19}P_{22}P_{25} - P_{12}P_{13}P_{26} - P_{20}P_{23}P_{26} \\ & - P_{10}P_{11}P_{27} - P_{21}P_{24}P_{27} - P_{25}P_{26}P_{27} + P_{12}P_{21}P_3 - P_{10}P_{23}P_3 - P_2P_{25}P_3 - P_{15}P_{20}P_4 + P_{13}P_{22}P_4 \\ & - P_{17}P_3P_4 - P_{12}P_{19}P_5 + P_{14}P_{23}P_5 - P_{27}P_4P_5 - P_{11}P_{22}P_6 + P_{15}P_{24}P_6 - P_1P_{26}P_6 - P_{16}P_5P_6 \\ & - P_{11}P_{12}P_7 - P_{23}P_{24}P_7 + P_{16}P_{25}P_7 - P_1P_4P_7 - P_{10}P_{15}P_8 - P_{21}P_{22}P_8 + P_{17}P_{26}P_8 - P_2P_5P_8 \\ & - P_{13}P_{14}P_9 - P_{19}P_{20}P_9 + P_{18}P_{27}P_9 - P_3P_6P_9 - P_7P_8P_9 \end{aligned}$$

The 27 variables P_1, \dots, P_{27} are indexed by the lines on the Fermat cubic surface in $\mathbb{C}\mathbb{P}^3$ and the terms correspond to triples of coplanar lines. The sign prescription is more intricate, given in terms of the C_3 action on the cubic, see [L11]. The octonionic projective plane $\mathbb{O}\mathbb{P}^2$ is cut out by the 27 partial derivatives of the above cubic.

There is a Cartan subgroup $(\mathbb{C}^*)^6$ of E_6 that acts diagonally on the variables P_i . We picked an element g of order 13 of it which acts by $P_i \mapsto \zeta_{13}^{a_i} P_i$ with the weights a_i given by

$$(6, 7, 7, 10, 3, 6, 10, 3, 0, 5, 8, 8, 4, 9, 5, 4, 9, 0, 2, 11, 11, 12, 1, 2, 12, 1, 0).$$

As the reader can see, the action of g on the variables has a three-dimensional eigenspace of weight zero and 12 two-dimensional eigenspace of other weights. For the three invariant variables P_9, P_{18}, P_{27} , the corresponding partial derivatives of the cubic

$$(1) \quad \begin{aligned} & -P_{13}P_{14} - P_{19}P_{20} + P_{18}P_{27} - P_3P_6 - P_7P_8, \\ & -P_{12}P_{15} - P_{16}P_{17} - P_1P_2 - P_{22}P_{23} + P_{27}P_9, \\ & -P_{10}P_{11} - P_{21}P_{24} - P_{25}P_{26} - P_4P_5 + P_{18}P_9 \end{aligned}$$

involve all of the variables P_i .

At this point, our expectations of the g -action on $H^0(X, K_X)$ indicate that we need to take a linear cut by

$$(P_9, P_{18}, P_{27}, P_{23} + d_1P_{26}, P_{19} + d_2P_{24}, P_5 + d_3P_8, P_{13} + d_4P_{16}, P_{10} + d_5P_{15}, P_1 + d_6P_6, \\ P_2 + d_7P_3, P_{11} + d_8P_{12}, P_{14} + d_9P_{17}, P_4 + d_{10}P_7, P_{20} + d_{11}P_{21}, P_{22} + d_{12}P_{25})$$

for some constants d_1, \dots, d_{12} . Moreover, we want a linear combination of the g -invariant quadrics (1) to vanish on the linear subspace of the cut. In view of the Cartan subgroup symmetry, it is reasonable to pick this linear combination to be the sum of the above quadrics. This gives 6 simple equations on d_i , namely $d_i d_{13-i} = -1$. We have been able to verify by computer at a specific point that the resulting scheme is a smooth surface of degree 13 and is thus a good candidate for our X .

Specifically, the 26 quadrics that cut out X are given by

$$(2) \quad \begin{aligned} & -t_{10}^2 + d_2t_2t_5 - t_1t_6 - d_2t_{11}t_9, \quad t_3^2 + t_2t_4 - d_1t_{12}t_7 + t_{11}t_8, \quad -d_1t_1t_5 - d_1t_{12}t_7 + d_2t_{11}t_8 - t_{10}t_9, \\ & -t_{12}t_4 - t_{11}t_5 + t_{10}t_6 - t_7t_9, \quad -t_4t_6 - t_3t_7 + d_2t_2t_8 - d_1t_1t_9, \quad t_3t_4 + t_2t_5 + t_1t_6 - t_{12}t_8, \\ & d_1t_1t_2 + d_1t_{12}t_4 + t_{10}t_6 - t_8^2, \quad -d_2t_{11}t_{12} + t_5^2 + t_3t_7 + t_1t_9, \quad d_2t_{11}t_2 - t_{10}t_3 - t_6t_7 + t_4t_9, \\ & t_4^2 - t_3t_5 + d_2t_2t_6 + d_1t_1t_7, \quad -t_{12}t_6 + t_{11}t_7 - t_{10}t_8 - t_9^2, \quad -d_2t_2t_3 + t_1t_4 + d_2t_{11}t_7 - t_{10}t_8, \\ & t_{10}t_{12} + t_4t_5 - t_2t_7 - t_1t_8, \quad d_1t_1t_3 - d_1t_{12}t_5 + d_2t_{11}t_6 - t_8t_9, \quad -t_{10}t_{11} + t_3t_5 + t_2t_6 - d_1t_{12}t_9, \\ & -d_2t_{11}^2 + d_1t_{10}t_{12} - t_4t_5 + t_3t_6, \quad d_2t_2^2 + t_1t_3 - t_{10}t_7 - t_8t_9, \quad d_1t_1t_{12} + t_6t_7 - t_5t_8 - t_4t_9, \\ & -d_1t_{12}^2 + t_5t_6 + t_3t_8 - t_2t_9, \quad d_1t_1^2 - d_2t_{11}t_4 - t_{10}t_5 + t_7t_8, \quad -t_{12}t_3 - t_{11}t_4 + t_7t_8 + t_6t_9, \\ & d_1d_2t_{12}t_2 - d_2t_{11}t_3 - t_{10}t_4 - t_6t_8, \quad -t_1t_{11} - t_{10}t_2 + t_5t_7 - t_3t_9, \quad d_1t_1t_{10} + t_5t_6 + t_4t_7 + d_2t_2t_9, \\ & d_2t_{12}t_2 + t_{10}t_4 - t_7^2 - t_5t_9, \quad d_1t_1t_{11} + t_6^2 + t_4t_8 + t_3t_9 \end{aligned}$$

in the homogeneous coordinates $(t_1 : \dots : t_{12})$ of \mathbb{P}^{11} . The action of g is $t_i \mapsto \zeta_{13}^i t_i$.

Remark 2.1. *The action of the Cartan subgroup of E_6 reduces the dimension of the space of parameters d from six to two (taking into account the need to preserve the invariant quadric that has to vanish on the cut reduces E_6 to F_4). We expect the total family to have dimension four, but it is not clear how one can build it. What makes the elements above special is that these surfaces X admit an additional C_3 symmetry that extends the C_{13} action to the semidirect product of these two groups. Namely, by scaling the variables (but still calling them t_i) we could rewrite the equations (2) as*

$$(3) \quad \begin{aligned} & -t_{10}^2 - d_1d_2^2(t_2t_5 + t_1t_6 - t_{11}t_9), \quad d_1d_2^2t_3^2 + t_2t_4 + t_{12}t_7 - t_{11}t_8, \quad d_1d_2^2t_1t_5 + t_{12}t_7 - d_2(t_{11}t_8 + t_{10}t_9), \\ & -t_{12}t_4 + d_1d_2(-t_{11}t_5 + t_{10}t_6 + d_2t_7t_9), \quad t_4t_6 + d_2(-t_3t_7 - t_2t_8 + d_1d_2t_1t_9), \quad d_1d_2(t_3t_4 - t_2t_5 + d_2t_1t_6) - t_{12}t_8, \\ & d_1d_2^2t_1t_2 + t_{12}t_4 + d_2t_{10}t_6 - d_2t_8^2, \quad t_{11}t_{12} + d_1d_2(t_5^2 - t_3t_7 + d_2t_1t_9), \quad -t_{11}t_2 - t_{10}t_3 + t_6t_7 + t_4t_9, \\ & t_4^2 + d_1d_2^2(t_3t_5 + t_2t_6 - t_1t_7), \quad -t_{12}t_6 + t_{11}t_7 - t_{10}t_8 - d_1d_2^2t_9^2, \quad -d_1d_2^2t_2t_3 + d_2t_1t_4 + d_2t_{11}t_7 - t_{10}t_8, \\ & t_{10}t_{12} - d_1d_2(t_4t_5 - t_2t_7 + d_2t_1t_8), \quad d_1d_2^2t_1t_3 + t_{12}t_5 - d_2(t_{11}t_6 + t_8t_9), \quad t_{10}t_{11} - d_1d_2(d_2t_3t_5 - t_2t_6 + t_{12}t_9), \\ & -d_2t_{11}^2 + t_{10}t_{12} + d_2t_4t_5 + d_1d_2^2t_3t_6, \quad t_{10}t_7 + d_1d_2(t_2^2 + d_2t_1t_3 - t_8t_9), \quad t_1t_{12} - t_6t_7 + t_5t_8 - t_4t_9, \\ & -t_{12}^2 - d_1d_2^2(t_5t_6 - t_3t_8 + t_2t_9), \quad d_1d_2^2t_1^2 + t_{11}t_4 + t_{10}t_5 - t_7t_8, \quad t_{11}t_4 - d_2(t_{12}t_3 + t_7t_8) + d_1d_2^2t_6t_9, \\ & -t_{10}t_4 + d_1d_2(t_{12}t_2 + d_2t_{11}t_3 - t_6t_8), \quad d_2t_1t_{11} - t_{10}t_2 + d_2t_5t_7 - d_1d_2^2t_3t_9, \quad -t_4t_7 + d_1d_2(t_1t_{10} - t_5t_6 + d_2t_2t_9), \\ & d_2t_{12}t_2 + t_{10}t_4 - d_2t_7^2 + d_1d_2^2t_5t_9, \quad t_4t_8 + d_1d_2(-t_1t_{11} + t_6^2 + d_2t_3t_9) \end{aligned}$$

with the additional symmetry $t_i \mapsto t_{3i \bmod 13}$. The details are in [BBF20+, Section2.nb].

2.3. Constructing a cut with A_2 singularities. Our method of constructing a fake projective plane largely followed the blueprint of [BF20].

We set $d_3 = d_4 = d_5 = d_6 = 1$ and tried to find out which (d_1, d_2) give singular cuts. In order to achieve this, we worked on an affine coordinate chart of $\mathbb{O}\mathbb{P}^2$ which can be obtained by solving the equations of $\mathbb{O}\mathbb{P}^2$ for eleven of the variables as follows.

$$\begin{aligned} P_4 &= P_{10}P_{16} + P_2P_{24} + P_{12}P_{26} + P_{14}P_9, P_6 = -P_{14}P_{17} + P_2P_{20} - P_{10}P_{27} - P_{12}P_7, \\ P_8 &= -P_1P_{14} - P_{16}P_{20} - P_{24}P_{27} + P_{12}P_3, P_{11} = P_{15}P_{24} - P_1P_{26} - P_{16}P_5 - P_3P_9, \\ P_{13} &= P_{15}P_{20} + P_{17}P_3 + P_{27}P_5 + P_1P_7, P_{18} = -P_{20}P_{26} - P_{10}P_3 + P_{14}P_5 - P_{24}P_7, \\ P_{19} &= -P_{14}P_{15} - P_{26}P_{27} - P_2P_3 + P_{16}P_7, P_{21} = -P_{10}P_{15} + P_{17}P_{26} - P_2P_5 - P_7P_9, \\ P_{22} &= 1, P_{23} = -P_{12}P_{15} - P_{16}P_{17} - P_1P_2 + P_{27}P_9, P_{25} = P_1P_{10} - P_{17}P_{24} - P_{12}P_5 - P_{20}P_9 \end{aligned}$$

We obtained this chart by connecting the formulas for the Cartan cubic from [GE96] and [L11]. We then further solved for five of the variables to reduce their number while still keeping the equations relatively short. Then we looked for tangent vectors for the surfaces with $d_1 = 1$ that lie in a codimension three subspace, by a multivariable Newton method starting at random points. The idea is that some of these would happen at values of d_2 where the surface $X = X_{1,d_2}$ acquires a node. After some trial and error we saw that solutions to

$$-27 - 34d_2 - 397d_2^2 - 172d_2^3 - 821d_2^4 + 190d_2^5 - 83d_2^6 + 16d_2^7 = 0$$

give singular surfaces. As in [BF20], we then perturbed d_1 slightly to $1 + 10^{-20}$ to find a nearby point on the locus of singular surfaces. This lead us to conjecture that generic points (d_1, d_2) on the curve

$$\begin{aligned} 0 &= -4d_1^3 + 8d_1^4 - 4d_1^5 - 12d_1^2d_2 - 16d_1^3d_2 + 28d_1^4d_2 - 39d_1^5d_2 + 12d_1^6d_2 - 12d_1d_2^2 - 28d_1^2d_2^2 - 54d_1^3d_2^2 \\ &+ 78d_1^4d_2^2 - 34d_1^5d_2^2 + 28d_1^6d_2^2 - 12d_1^7d_2^2 - 4d_2^3 - 39d_1d_2^3 - 34d_1^2d_2^3 - 277d_1^3d_2^3 + 192d_1^4d_2^3 - 277d_1^5d_2^3 \\ &+ 54d_1^6d_2^3 - 16d_1^7d_2^3 + 4d_1^8d_2^3 - 8d_2^4 - 28d_1d_2^4 - 78d_1^2d_2^4 - 192d_1^3d_2^4 + 192d_1^5d_2^4 - 78d_1^6d_2^4 + 28d_1^7d_2^4 \\ &- 8d_1^8d_2^4 - 4d_2^5 - 16d_1d_2^5 - 54d_1^2d_2^5 - 277d_1^3d_2^5 - 192d_1^4d_2^5 - 277d_1^5d_2^5 + 34d_1^6d_2^5 - 39d_1^7d_2^5 + 4d_1^8d_2^5 + 12d_1d_2^6 \\ &+ 28d_1^2d_2^6 + 34d_1^3d_2^6 + 78d_1^4d_2^6 + 54d_1^5d_2^6 - 28d_1^6d_2^6 + 12d_1^7d_2^6 - 12d_1^2d_2^7 - 39d_1^3d_2^7 - 28d_1^4d_2^7 - 16d_1^5d_2^7 \\ &+ 12d_1^6d_2^7 + 4d_1^3d_2^8 + 8d_1^4d_2^8 + 4d_1^5d_2^8 \end{aligned}$$

give nodal X_{d_1,d_2} .

We then looked for singular points of this curve. There were several such points, one of which was a cusp of the curve. We focused our attention on it and discovered a surface X_{d_1,d_2} with 39 A_2 singularities. Specifically, both d_1 and d_2 can be given given as roots of

$$0 = 2187 + 7290d + 23433d^2 + 21640d^3 + 66393d^4 - 21640d^5 + 23433d^6 - 7290d^7 + 2187d^8$$

approximately given by $(d_1, d_2) \approx (1.93 + 2.30i, 0.0125 - 0.515i)$. Of course, the same is true for all of the Galois conjugates of this pair. From now on we will call this surface X_0 .

We observed that four of the Galois conjugate pairs of (d_1, d_2) give isomorphic surfaces. To see that, we noticed that scheme X_{d_1,d_2} cut out by (3) is isomorphic to $X_{-1/d_2,d_1}$ under the coordinate change

$$(4) \quad (t_1, \dots, t_{12}) \mapsto (d_2t_5, t_{10}, d_2t_2, -d_1t_7, t_{12}, t_4, -d_1d_2t_9, -d_1d_2t_1, d_2t_6, -d_1t_{11}, -d_1d_2t_3, -d_1t_8)$$

The idea behind it was to use $i \rightarrow 5i \pmod{13}$, and we heavily relied on Mathematica computations, see [BBF20+, Section2.nb].

We then used the symmetry (4) to average the C_{13} -invariants of the coordinate ring of the surface X_0 suspected to have A_2 singularities, to get X_0/C_{13} defined in the 13-dimensional weighted projective space $W\mathbb{P}(2^4, 3^{10})$ by 9 equations of degree five and 29 equations of degree six. Due to the above symmetrization, the coefficients were in the field $\mathbb{Q}(\sqrt{-2})$.

2.4. Finding singular points. It was not entirely trivial to find the singular points of X_0/C_{13} . We did it by calculating a degree 12 equation in the first four variables which gives a (non-normal) image of X_0/C_{13} in \mathbb{P}^3 . Then we looked for its curves of singularities by finding multiple singular points on random hyperplane cuts. Then we have looked for singular points outside of the curve of singularities, and indeed hit upon A_2 singularities. We were then able to verify that these were the only singularities by computing the degree of the singular locus over a finite field. As in the case of [BF20], the A_2 singularities were not defined over the quadratic extension of \mathbb{Q} , but a coordinate change gave us a model of $X_0/C_{13} \subseteq W\mathbb{P}^{13}$ still defined over $\mathbb{Q}(\sqrt{-2})$ and with three singular points defined over \mathbb{Q} .

3. CONSTRUCTING THE FAKE PROJECTIVE PLANE

3.1. Constructing the triple cover. By the work of Keum [Ke12], the surface X_0/C_{13} admits a Galois triple cover which is a fake projective plane. In this, it is very similar to the situation in [BF20] and we employed the same general method. It was useful that in both cases there was an additional order three automorphism σ because the FPP had a $C_3 \times C_3$ group of automorphisms. Specifically, we looked for sections f and d of $4K_{X_0/C_{13}}$ which satisfy

$$f \sigma(f) \sigma^2(f) = d^3.$$

Moreover f and d should have certain vanishing on the exceptional lines at the blowup of A_2 singularities. We refer the reader to [BF20] for details.

The nature of X_0/C_{13} made the computations more challenging. In particular, at some point we had to work with random points on the surface computed with 6×10^4 digits of accuracy. The equations for f and d had coefficients in $\mathbb{Q}(\sqrt{-2})$ which were about 1.5×10^4 digits long. As in [BF20], we solved it over a finite field of 19 elements, but now we used a p-adic version of the Newton's method to quickly gain the needed accuracy.

Once the triple cover was constructed, we used the fixed points of the automorphisms of \mathbb{P}_{fake}^2 to get a basis with nicer equations, only about 100 digits long coefficients, see [BBF20+, Section3.nb] for details. This surface is labeled by $(C18, p = 3, \emptyset, d_3 D_3)$ in the classification of [CS11+], since it is the only one with an automorphism group that contains $(C_3)^2$ and Picard group that contains C_{13} .

The details of the above process are in [BBF20+, Section3.nb].

3.2. Open questions. Let us now discuss open problems related to this construction.

The first question is how to verify that the special cuts X_{d_1, d_2} of $\mathbb{O}\mathbb{P}^2$ are simply connected. Since these are not complete intersections, the Lefschetz Hyperplane theorem can not be applied, so other methods are needed. It might perhaps follow from our construction and [CS11+], but a more direct argument is desirable.

A related question is how to construct non- C_3 -invariant deformations of X_{d_1, d_2} . It looks like they will no longer be cuts of $\mathbb{O}\mathbb{P}^2$ but perhaps one can get them by carefully examining the equations (2).

The quotient of the fake projective plane $(C18, p = 3, \emptyset, d_3 D_3)$ by $(C_3)^2$ is also covered by $(C18, p = 3, \{2I\})$. This fake projective plane in turn covers a surface $(C18, p = 3, \{2\})$, which is covered by three other fake projective planes. The method of [BF20] is, unfortunately, not quite applicable here, so how do we find (the equations of) these other surfaces?

It is known from [CS11+] that $(C18, p = 3, \emptyset, d_3 D_3)$ has Picard group $C_2 \times C_2 \times C_{13}$. While the C_{13} part can be inferred from our construction (even though it may not be entirely trivial to follow), the other two factors are mysterious. It would be interesting to see them explicitly, and they may be useful in answering both the previous and the next questions.

A perennial question is how one can reduce the size of the coefficients in the equations. There are currently only ad hoc tools that are not very successful, except in [BK19] case.

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