NOTES ON SCHUR FUNCTIONS

ANDERS SKOVSTED BUCH

These notes are work in progress. The goal is to provide quick proofs of some of the main identities satisfied by Schur functions. Some alternative references are [Mac95, Ful97].

1. Definition of Schur functions

sec:symfcn

1.1. **Symmetric functions.** Let $X = (x_1, x_2, \dots)$ and $Y = (y_1, y_2, \dots)$ be two countably infinite sets of independent commuting variables. Define the *double complete symmetric function* $S_p = S_p(X;Y) \in \mathbb{Z}[\![X,Y]\!]$, for $p \in \mathbb{Z}$, by the generating series

$$\sum_p S_p\, t^p = \frac{\prod_{j=1}^{\infty}(1-y_j\, t)}{\prod_{i=1}^{\infty}(1-x_i\, t)}\,.$$

The power series S_p is homogeneous of total degree p. We have $S_0 = 1$ and $S_p = 0$ for $p < 0$, and the functions S_p for $p \geq 1$ are algebraically independent. The *ring of symmetric functions* Λ is the subring of $\mathbb{Z}[\![X,Y]\!]$ generated by the elements S_p,

$$\Lambda = \mathbb{Z}[S_1, S_2, S_3, \dots] \subset \mathbb{Z}[\![X,Y]\!]\,.$$

Let $f \in \Lambda$ be a symmetric function, let R be a commutative ring, and let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ be finite sets of elements of R. We then let

$$f(a;b) = f(a_1,\dots,a_n;b_1,\dots,b_m) = f(a_1,\dots,a_n,0,0,\dots\,;\, b_1,\dots,b_m,0,0,\dots) \in R$$

denote the result of substituting $x_i = a_i$ for $1 \leq i \leq n$, $x_i = 0$ for $i > n$, $y_j = b_j$ for $1 \leq j \leq m$, and $y_j = 0$ for $j > m$. We will always use a semicolon to separate the two sets of arguments.

The resulting functions $f : R^n \times R^m \to R$ are *super-symmetric* in the following sense. First, $f(a;b)$ is separately symmetric in each set of arguments a and b. In addition, $f(a;b)$ is unchanged if 0 is added to either set of arguments, or if the same element $c \in R$ is added to both sets of arguments:

$$f(a;b) = f(a,0\,;\,b) = f(a\,;\,b,0) = f(a,c\,;\,b,c)\,.$$

This follows from the definition of the generators S_p of Λ.

The functions S_p satisfy the following identities. If the second set of arguments is omitted, then

$$S_p(a) = S_p(a;0) = h_p(a_1,\dots,a_n)$$

is the *complete symmetric polynomial*, defined as the sum of all monomials of degree p in $a = (a_1, \dots, a_n)$. If the first set of arguments is omitted, then

$$S_p(0;b) = e_p(-b_1,\dots,-b_m) = (-1)^p\, e_p(b_1,\dots,b_m)\,,$$

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where $e_p(b_1,\dots,b_n)$ is the *elementary symmetric polynomial*, defined as the sum of all square-free monomials of degree p in $b=(b_1,\dots,b_m)$. In general, we have

$$S_p(a;b) = \sum_{i+j=p} h_i(a)\, e_j(-b) = \sum_{j=0}^{p} (-1)^j h_{p-j}(a)\, e_j(b)\,.$$

1.2. **Schur functions.** Given an integer sequence $\lambda = (\lambda_1,\dots,\lambda_\ell) \in \mathbb{Z}^\ell$, define the (double) *Schur function* $S_\lambda \in \Lambda$ by

$$S_\lambda \;=\; \det\left(S_{\lambda_i+j-i}\right)_{\ell\times\ell} \;=\; \begin{vmatrix} S_{\lambda_1} & S_{\lambda_1+1} & S_{\lambda_1+2} & \dots & S_{\lambda_1+\ell-1} \\ S_{\lambda_2-1} & S_{\lambda_2} & S_{\lambda_2+1} & \dots & S_{\lambda_2+\ell-2} \\ S_{\lambda_3-2} & S_{\lambda_3-1} & S_{\lambda_3} & \dots & S_{\lambda_3+\ell-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_\ell-\ell+1} & S_{\lambda_\ell-\ell+2} & S_{\lambda_\ell-\ell+3} & \dots & S_{\lambda_\ell} \end{vmatrix}\,.$$

The diagonal entries in the determinant are $S_{\lambda_1}, S_{\lambda_2}, \dots, S_{\lambda_\ell}$, and the subscripts increase consecutively from left to right. For example,

$$S_{(3,-1,2)} = \begin{vmatrix} S_3 & S_4 & S_5 \\ S_{-2} & S_{-1} & S_0 \\ S_0 & S_1 & S_2 \end{vmatrix} = \begin{vmatrix} S_3 & S_4 & S_5 \\ 0 & 0 & 1 \\ 1 & S_1 & S_2 \end{vmatrix} = S_4 - S_1 S_3 = -S_{(3,1)}\,.$$

The element $S_\lambda \in \Lambda$ is homogeneous of total degree

$$|\lambda| = \sum_{i=1}^{\ell} \lambda_i\,.$$

Notice that S_λ is unchanged if λ is extended by zeros:

$$S_{(\lambda,0)} = S_\lambda\,.$$

The specialization $S_\lambda(x_1,\dots,x_n)$ to one finite set of variables is called a *Schur polynomial*, and $S_\lambda(x_1,\dots,x_n;y_1,\dots,y_m)$ is called a *double Schur polynomial*.

sec:straight

1.3. **Straightening law.** For $a,b\in\mathbb{Z}$ and arbitrary integer sequences λ' and λ'' we have

$$S_{(\lambda',a,b,\lambda'')} = S_{(\lambda',b-1,a+1,\lambda'')}\,.$$

In fact, the determinants defining these functions differ by interchanging two rows. In particular, we have

$$S_{(\lambda',a,a+1,\lambda'')} = 0\,.$$

A *partition* is a weakly decreasing sequence of non-negative integers, and we identify two partitions if they differ only by trailing zeros. For example, the sequences $(4,3,1)$ and $(4,3,1,0,0)$ define the same partition.

It follows from the straightening law that any Schur function is given by

$$S_\lambda = \begin{cases} 0 & \text{if } \lambda_i - i = \lambda_j - j \text{ for some } i \neq j; \\ \pm S_{\widetilde{\lambda}} & \text{otherwise, where } \widetilde{\lambda} \text{ is a partition.} \end{cases}$$

In the second case, $\widetilde{\lambda}$ is the unique partition for which the strictly decreasing sequence $(\widetilde{\lambda}_1 - 1,\dots,\widetilde{\lambda}_\ell-\ell)$ is a permutation of $(\lambda_1-1,\dots,\lambda_\ell-\ell)$, and the sign of $S_{\widetilde{\lambda}}$ is the sign of this permutation.

Exercise 1.1. The Schur functions S_λ indexed by partitions form a $\mathbb{Z}$-basis of Λ.

1.4. **Skew Schur functions.** Given two integer sequences $\lambda, \mu \in \mathbb{Z}^\ell$, define the *skew Schur function* $S_{\lambda/\mu}$ by

$$S_{\lambda/\mu} = \det\left(S_{\lambda_i - i - \mu_j + j}\right)_{\ell\times\ell}\,.$$

For example,

$$S_{(5,4,1)/(3,1,0)} = \begin{vmatrix} S_{0+5-3} & S_{1+5-1} & S_{2+5-0} \\ S_{-1+4-3} & S_{0+4-1} & S_{1+4-0} \\ S_{-2+1-3} & S_{-1+1-1} & S_{0+1-0} \end{vmatrix}\,.$$

The function $S_{\lambda/\mu}$ is homogeneous of total degree

$$|\lambda/\mu| = |\lambda| - |\mu| = \sum_{i=1}^{\ell} \lambda_i - \sum_{j=1}^{\ell} \mu_j\,.$$

Notice that $S_{(\lambda,0)/(\mu,0)} = S_{\lambda/\mu}$. We can therefore define skew Schur functions for integer sequences λ and μ of different lengths by adding an appropriate number of zeros:

$$S_{\lambda/\mu} = S_{(\lambda,0,\dots,0)/(\mu,0,\dots,0)}\,.$$

The usual Schur function is $S_\lambda = S_{\lambda/0}$.

Assume that λ and μ are partitions. We say that μ is *contained in* λ, written $\mu \subset \lambda$, if $\mu_i \le \lambda_i$ for all i. Notice that $S_{\lambda/\mu}$ is non-zero only if $\mu \subset \lambda$:

eqn:skew_zero (1) $$S_{\lambda/\mu} \neq 0 \;\Rightarrow\; \mu \subset \lambda\,.$$

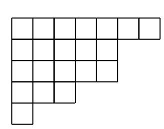
In fact, if $\mu_i > \lambda_i$, then the i-th diagonal entry of the determinant defining $S_{\lambda/\mu}$ is $S_{\lambda_i - \mu_i} = 0$, and all entries south-west of this entry are also zero. It follows from Theorem 1.4 below that the converse of (1) is also true.

The straightening law from Section 1.3 applies to both λ and μ. As a consequence we have

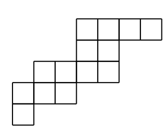
$$S_{\lambda/\mu} = \begin{cases} 0 & \text{if } S_\lambda = 0 \text{ or } S_\mu = 0 \text{ or } \widetilde{\mu} \not\subset \widetilde{\lambda}; \\ \pm S_{\widetilde{\lambda}/\widetilde{\mu}} & \text{otherwise,} \end{cases}$$

where $\widetilde{\lambda}$ and $\widetilde{\mu}$ denote the partitions obtained from the straightening law applied to S_λ and S_μ.

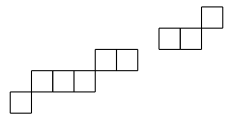
1.5. **Young diagrams.** A partition λ can be identified with its *Young diagram* of boxes, which has λ_i boxes in row i. The row number i increases from top to bottom, and the rows of boxes are left-justified. For example:

$(7, 5, 5, 3, 1) \;=$ 

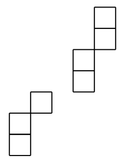
The inclusion relation $\mu \subset \lambda$ means that the Young diagram of μ is contained in the Young diagram of λ. When $\mu \subset \lambda$, we let λ/μ denote the *skew diagram* of boxes in the Young diagram of λ that are outside the Young diagram of μ. For example:

$(7, 5, 5, 3, 1)/(3, 3, 1) \;=$ 

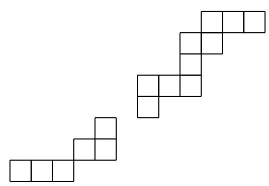
A *horizontal strip* is a skew diagram with at most one box in each column:



A *vertical strip* is a skew diagram with at most one box in each row:



A skew diagram is called a *rim* if it is a union of a horizontal strip and a vertical strip. Equivalently, the diagram contains no 2×2 squares. For example:



1.6. **Expansions of Schur polynomials.** In this section we let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_m)$ be finite sets of elements of a commutative ring R, and $c \in R$ denotes a single element. We first prove a basic formula for the expansion of a double skew Schur polynomial defined by arbitrary integer sequences.

lemma:expand_seq

Lemma 1.2. *For any integer sequences $\lambda, \mu \in \mathbb{Z}^\ell$, we have*

$$S_{\lambda/\mu}(a\,;\,b,c) = \sum_{\varepsilon \in \{0,1\}^\ell} (-c)^{|\varepsilon|}\, S_{\lambda/\mu+\varepsilon}(a,b)\,,$$

where the sum is over all sequences $\varepsilon = (\varepsilon_1, \dots, \epsilon_\ell)$ with $\epsilon_i \in \{0,1\}$.

Proof. Set $h'_p = S_p(a\,;\,b,c)$ and $h_p = S_p(a;b)$ for $p \in \mathbb{Z}$. Since the definition of double complete symmetric functions implies that

$$\sum_p h'_p\, t^p = (1 - c\,t) \sum_p h_p\, t^p\,,$$

we obtain

$$h'_p = h_p - c\, h_{p-1}\,.$$

The j-th column of the determinant defining $S_{\lambda/\mu}(a;b,c)$ is therefore given by

$$\begin{bmatrix} h'_{\lambda_1-1-\mu_j+j} \\ h'_{\lambda_2-2-\mu_j+j} \\ \vdots \\ h'_{\lambda_\ell-\ell-\mu_j+j} \end{bmatrix} = \begin{bmatrix} h_{\lambda_1-1-\mu_j+j} \\ h_{\lambda_2-2-\mu_j+j} \\ \vdots \\ h_{\lambda_\ell-\ell-\mu_j+j} \end{bmatrix} - c \begin{bmatrix} h_{\lambda_1-1-\mu_j-1+j} \\ h_{\lambda_2-2-\mu_j-1+j} \\ \vdots \\ h_{\lambda_\ell-\ell-\mu_j-1+j} \end{bmatrix}.$$

The first vector on the right hand side is the j-th column of the determinant defining $S_{\lambda/\mu+\varepsilon}(a;b)$ when $\varepsilon_j = 0$, and the vector multiplied to c is the j-th column in the determinant defining $S_{\lambda/\mu+\varepsilon}(a;b)$ when $\varepsilon_j = 1$. The lemma now follows because determinants are multilinear functions of column vectors. $\square$

When the sequence μ is a partition, the expansion of Lemma 1.2 can be interpreted in terms of adding vertical strips to μ.

`prop:expand_part` **Proposition 1.3.** *Let $\lambda, \mu \in \mathbb{Z}^\ell$ and assume that μ is a partition. Then,*

`eqn:expand_vert`
$$S_{\lambda/\mu}(a\,;\,b,c) = \sum_{\nu/\mu \text{ vertical strip}} (-c)^{|\nu/\mu|}\, S_{\lambda/\nu}(a;b)\,, \tag{2}$$

where the sum is over all partitions ν containing μ, such that ν/μ is a vertical strip. In addition,

`eqn:expand_horiz`
$$S_{\lambda/\mu}(a,c\,;\,b) = \sum_{\nu/\mu \text{ horizontal strip}} c^{|\nu/\mu|}\, S_{\lambda/\nu}(a;b)\,, \tag{3}$$

where the sum is over all partitions ν containing μ, such that ν/μ is a horizontal strip.

Proof. If $\mu \in \mathbb{Z}^\ell$ is a partition and $\kappa \in \{0,1\}^\ell$, then it follows from the straightening law that $S_{\lambda/\mu+\kappa}$ is non-zero only if $\nu = \mu + \kappa$ is a partition, and in this case ν/μ is a vertical strip. Identity (2) therefore follows from Lemma 1.2.

Using (2), the right hand side of (3) is equal to

$$\begin{aligned}
\sum_{\nu/\mu \text{ horiz.}} c^{|\nu/\mu|}\, S_{\lambda/\nu}(a,c\,;\,b,c) \;&=\; \sum_{\substack{\mu\subset\nu\subset\pi \\ \nu/\mu \text{ horiz.} \\ \pi/\nu \text{ vert.}}} c^{|\nu/\mu|}\,(-c)^{|\pi/\nu|}\, S_{\lambda/\pi}(a,c\,;\,b) \\
&=\; \sum_{\pi} c^{|\pi/\mu|}\, S_{\lambda/\pi}(a,b\,;\,y) \left(\sum_{\substack{\nu:\ \mu\subset\nu\subset\pi \\ \nu/\mu \text{ horiz.} \\ \pi/\nu \text{ vert.}}} (-1)^{|\nu/\mu|} \right).
\end{aligned}$$

The last two expressions are sums over partitions ν and π for which $\mu \subset \nu \subset \pi$, ν/μ is a horizontal strip, and π/ν is a vertical strip.

It suffices to show that, if $\mu \subset \pi$ are partitions, then

`eqn:cancel`
$$\sum_{\substack{\nu:\ \mu\subset\nu\subset\pi \\ \nu/\mu \text{ horiz.} \\ \pi/\nu \text{ vert.}}} (-1)^{|\nu/\mu|} \;=\; \delta_{\mu,\pi}\,. \tag{4}$$

If the sum is not empty, then π/μ must be a rim. Further, if ν satisfies the condition of the sum, then any box of π/μ located immediately left of another box in π/μ must be contained in ν, while any box of π/μ located immediately below another box must be outside ν. If $\pi/\mu \neq \emptyset$, then the North-East box of π/μ can be freely added to or removed from ν. Since such a change to ν switches the sign of $(-1)^{|\nu/\mu|}$, we deduce that (4) vanishes, as required.

$$\pi/\mu \quad = \quad \begin{array}{cccccc}
 & & & & & \boxed{?} \\
 & & & & & \boxed{\pi} \\
 & & & \boxed{\nu} & \boxed{\nu} & \boxed{\pi} \\
\boxed{\nu} & \boxed{\nu} & \boxed{\nu} & \boxed{\pi} & & \\
\boxed{\pi} & & & & & \\
\boxed{\pi} & & & & &
\end{array}$$

$\square$

1.7. **Tableaux.** Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be two sets of distinct commuting variables, and choose a total order on the union

$$x \cup y = \{x_1, \dots, x_n, y_1, \dots, y_m\}\,.$$

Let λ/μ be a skew diagram. Define a *bitableau* of shape λ/μ labeled by $(x; y)$ to be a labeling T of the boxes in λ/μ with variables from $x \cup y$, such that the following conditions are satisfied:

- The labels of the boxes in each row of λ/μ are weakly increasing from left to right with respect to the total order on $x \cup y$.
- The labels of the boxes in each column of λ/μ are weakly increasing from top to bottom with respect to the total order on $x \cup y$.
- Given any variable x_i from x, the set of boxes of λ/μ labeled by x_i is a horizontal strip.
- Given any variable y_j from y, the set of boxes of λ/μ labeled by y_j is a vertical strip.

Any box of λ/μ will also be considered a box of T, and the label of a box will be called the variable contained in the box. Let weight(T) be the product of the variables in all boxes of T, and set $(-1)^T = (-1)^k$, where k is the number of boxes in T containing variables from y.

thm:bitableau

Theorem 1.4. *For any partitions $\mu \subset \lambda$ we have*

$$S_{\lambda/\mu}(x; y) \;=\; \sum_T (-1)^T \operatorname{weight}(T)\,,$$

where the sum is over all bitableaux T of shape λ/μ labeled by $(x; y)$, relative to any chosen total order on $x \cup y$.

Proof. This follows from Proposition 1.3 by induction on the number of variables. For the induction step we use (2) if the smallest variable c in $x \cup y$ is from y, while we use (3) if c is from x. $\square$

Example 1.5. Let $x = (x_1, x_2)$ and $y = (y_1)$, and order these variables by $x_1 < y_1 < x_2$. The bitableaux of shape $(2, 1)$ labeled by $(x; y)$ are:

$$\begin{array}{|c|c|}\hline x_1 & x_1 \\ \hline y_1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline x_1 & x_1 \\ \hline x_2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline x_1 & y_1 \\ \hline y_1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline x_1 & y_1 \\ \hline x_2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline y_1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline x_1 & x_2 \\ \hline x_2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline y_1 & x_2 \\ \hline y_1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline y_1 & x_2 \\ \hline x_2 \\ \cline{1-1}\end{array}$$

We obtain

$$S_{(2,1)}(x_1, x_2\,;\, y_1) = -x_1^2 y_1 + x_1^2 x_2 + x_1 y_1^2 - 2\, x_1 y_1 x_2 + x_1 x_2^2 + y_1^2 x_2 - y_1 x_2^2\,.$$

1.8. **Consequences of the tableau formula.**

Corollary 1.6. *Let $\mu \subset \lambda$ be partitions, and let x, y, and z be three sets of variables. Then,*

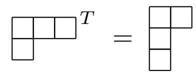
$$S_{\lambda/\mu}(x; y) \;=\; \sum_{\nu :\, \mu \subset \nu \subset \lambda} S_{\nu/\mu}(x; z)\, S_{\lambda/\nu}(z; y)\,.$$

Proof. We may assume that x and y are disjoint. Let z' and z'' be arbitrary (disjoint) sets of variables, and choose a total order on $x \cup y \cup z' \cup z''$ such that all variables from $x \cup z'$ are smaller than all variables from $y \cup z''$. Then Theorem 1.4 implies that

$$S_{\lambda/\mu}(x, z''; z', y) \;=\; \sum_{\nu :\, \mu \subset \nu \subset \lambda} S_{\nu/\mu}(x; z')\, S_{\lambda/\nu}(z''; y)\,.$$

The result follows by setting $z' = z'' = z$. $\square$

Given a partition λ, the *conjugate* partition λ^T is obtained by interchanging rows and columns in the Young diagram of λ. For example:



Corollary 1.7. *For partitions $\mu \subset \lambda$ and sets of variables x and y, we have*

$$S_{\lambda^T/\mu^T}(x;y) = (-1)^{|\lambda/\mu|}\, S_{\lambda/\mu}(y;x)\,.$$

Proof. This follows from Theorem 1.4 because the transpose of a bitableau of shape λ^T/μ^T labeled by $(x;y)$ is a bitableau of shape λ/μ labeled by $(y;x)$. $\square$

Example 1.8. For any partition λ we have

$$S_\lambda(x) = \det\left(h_{\lambda_i+j-i}(x)\right) = \det\left(e_{\lambda_i^T+j-i}(x)\right).$$

Special cases include

$$h_p = \det\left(e_{1+j-i}\right)_{p\times p} \qquad \text{and} \qquad e_p = \det\left(h_{1+j-i}\right)_{p\times p}.$$

cor:vanish **Corollary 1.9.** *Let λ be a partition. If $\lambda_{n+1} \geq m+1$, then*

$$S_\lambda(x_1,\dots,x_n;y_1,\dots,y_m) = 0\,.$$

Proof. This holds because there are no bitableaux of shape λ labeled by $(x;y)$ when $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_m)$. For example, suppose T is such a bitableau, subject to the ordering

$$x_1 < x_2 < \dots < x_n < y_1 < y_2 < \dots y_m\,.$$

Since row $n+1$ of T contains at least $m+1$ boxes, and each variable from y must occupy a vertical strip in T, the leftmost box of row $n+1$ contains a variable from x. Since each variable from x occupies a horizontal strip, this implies that the first column of T must contain at least $n+1$ distinct variables from x, which is impossible. $\square$

Let $(m)^n = (m,m,\dots,m)$ be the partition containing n copies of m.

Corollary 1.10. *We have*

$$S_{(m)^n}(x_1,\dots,x_n;y_1,\dots,y_m) \;=\; \prod_{i=1}^n\prod_{j=1}^m (x_i - y_j)\,.$$

Proof. It follows from Corollary 1.9 and the super-symmetry property that the Schur polynomial vanishes if we substitute $y_m = x_n$:

$$S_{(m)^n}(x_1,\dots,x_n;y_1,\dots,y_{m-1},x_n) = S_{(m)^n}(x_1,\dots,x_{n-1};y_1,\dots,y_{m-1}) = 0\,.$$

It follows that $S_{(m)^n}(x_1,\dots,x_n;y_1,\dots,y_m)$ is divisible by $x_n - y_m$, hence divisible by $\prod_{i,j}(x_i - y_j)$ by symmetry. Since this product has the same degree as $S_{(m)^n}$, we deduce that

$$S_{(m)^n}(x_1,\dots,x_n;y_1,\dots,y_m) \;=\; c\cdot\prod_{i=1}^n\prod_{j=1}^m (x_i - y_j)$$

for some constant $c \in \mathbb{Z}$. Finally, the easy identity $S_{(m)^n}(x_1,\dots,x_n) = \prod_{i=1}^n x_i^m$ reveals that $c = 1$. $\square$

Exercise 1.11 (Factorization formula)**.** Let $\lambda \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^\ell$, and assume that $\lambda_i \geq 0$ for all i. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ we have

$$S_{(m)^n+\lambda,\mu}(x;y) \;=\; S_\mu(0;y)\, S_{(m)^n}(x;y)\, S_\lambda(x)\,,$$

where $(m)^n + \lambda, \mu = (m + \lambda_1, \dots, m + \lambda_n, \mu_1, , \dots, \mu_\ell)$.

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