NOTES ON GRASSMANNIANS

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This is class notes under construction. We have not attempted to account for the history of the results covered here.

1. Construction of Grassmannians

1.1. **The set of points.** Let $k = \overline{k}$ be an algebraically closed field, and let k^n be the vector space of column vectors with n coordinates. Given a non-negative integer $m \leq n$, the Grassmann variety $\mathrm{Gr}(m,n)$ is defined as a set by

$$\mathrm{Gr}(m,n) = \{\Sigma \subset k^n \mid \Sigma \text{ is a vector subspace with } \dim(\Sigma) = m\}\,.$$

Our first goal is to show that $\mathrm{Gr}(m,n)$ has a structure of algebraic variety.

1.2. **Space with functions.** Let $\mathrm{FR}(n,m) = \{A \in \mathrm{Mat}(n \times m) \mid \mathrm{rank}(A) = m\}$ be the set of all $n \times m$ matrices of full rank, and let $\pi : \mathrm{FR}(n,m) \to \mathrm{Gr}(m,n)$ be the map defined by $\pi(A) = \mathrm{Span}(A)$, the column span of A. We define a topology on $\mathrm{Gr}(m,n)$ be declaring the a subset $U \subset \mathrm{Gr}(m,n)$ is open if and only if $\pi^{-1}(U)$ is open in $\mathrm{FR}(n,m)$, and further declare that a function $f : U \to k$ is regular if and only if $f \circ \pi$ is a regular function on $\pi^{-1}(U)$. This gives $\mathrm{Gr}(m,n)$ the structure of a space with functions.

Exercise 1.1. (1) The map $\pi : \mathrm{FR}(n,m) \to \mathrm{Gr}(m,n)$ is a morphism of spaces with functions. (2) Let X be a space with functions and $\phi : \mathrm{Gr}(m,n) \to X$ a map. Then ϕ is a morphism if and only if $\phi \circ \pi : \mathrm{FR}(n,m) \to X$ is a morphism.

Note 1.2. The group $\mathrm{GL}(m) = \mathrm{GL}(m;k)$ acts on the $\mathrm{FR}(n,m)$ from the right, and $\mathrm{Gr}(m,n)$ can be identified with the set of orbits $\mathrm{FR}(n,m)/\,\mathrm{GL}(m)$.

Note 1.3. The group $\mathrm{GL}(n)$ acts transitively on $\mathrm{FR}(n,m)$ and on $\mathrm{Gr}(m,n)$ from the left. Let $g \in \mathrm{GL}(n)$, $A \in \mathrm{FR}(n,m)$, and $\Sigma \in \mathrm{Gr}(m,n)$. Then $g.A = gA$ is the usual matrix product and $g.\Sigma = \{g.v \mid v \in \Sigma\}$. The morphism $\pi : \mathrm{FR}(n,m) \to \mathrm{Gr}(m,n)$ is $\mathrm{GL}(n)$-equivariant.

1.3. **Local coordinate charts.** Define a *Schubert symbol* for $\mathrm{Gr}(m,n)$ to be any subset I of cardinality m in the integer interval $[1,n]$. For any Schubert symbol I we set $\Sigma_I = \mathrm{Span}\{e_i \mid i \in I\}$.

Exercise 1.4. Let $T \subset \mathrm{GL}(n)$ be the 'maximal torus' of invertible diagonal matrices. The T-fixed points in $\mathrm{Gr}(m,n)$ are exactly the points of the form Σ_I where I is a Schubert symbol.

Given $A \in \mathrm{FR}(n,m)$ and a Schubert symbol I, let $A_I \in \mathrm{Mat}(m \times m)$ denote the submatrix of the rows in A determined by I. We set $x_I(A) = \det(A_I)$. The function x_I is called a *Plucker coordinate*. Notice that the non-vanishing set $D(x_I) \subset \mathrm{FR}(n,m)$ is an open affine subvariety.

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Set $\Sigma_{I^c} = \operatorname{Span}(e_i \mid i \notin I\}$. This is a point of the dual Grassmannian $\operatorname{Gr}(n-m,n)$. Consider the set $U_I = \{\Sigma \in \operatorname{Gr}(m,n) \mid \Sigma \cap \Sigma_{I^c} = 0\}$. Since $\pi^{-1}(U_I) = D(x_I) \subset \operatorname{FR}(n,m)$, it follows that $U_I \subset \operatorname{Gr}(m,n)$ is open.

Lemma 1.5. *The subspace U_I is isomorphic to the affine space $\mathbb{A}^{m(n-m)}$.*

Proof. We may assume that $I = \{1,2,\dots,m\}$. Since $\pi : \operatorname{FR}(n,m) \to \operatorname{Gr}(m,n)$ is a morphism of spaces with functions, it follows that the map $\operatorname{Mat}(n-m\times m) \to U_I$ given by

$$B \quad \mapsto \quad \pi \begin{bmatrix} 1_m \\ B \end{bmatrix}$$

is also a morphism. Here 1_m denotes the identity matrix of size m. On the other hand, we may define a morphism $D(x_I) \to \operatorname{Mat}(n-m\times m)$ by

$$\begin{bmatrix} A \\ B \end{bmatrix} \quad \mapsto \quad BA^{-1}\,.$$

This map factors through U_I, and it follows from Exercise 1.1 that the resulting map $U_I \to \operatorname{Mat}(n-m\times m)$ is a morphism. □

Exercise 1.6. Let $\Sigma_1, \Sigma_2 \in \operatorname{Gr}(m,n)$ be two points, and set $I = \{1,2,\dots,m\}$. There exists $g \in \operatorname{GL}(n)$ such that both $g.\Sigma_1$ and $g.\Sigma_2$ belong to U_I.

Notice that for any element $g \in \operatorname{GL}(n)$, the map $\operatorname{Gr}(m,n) \to \operatorname{Gr}(m,n)$ defined by $\Sigma \mapsto g.\Sigma$ is an automorphism in the category of spaces with functions. This follows from Exercise 1.1.

Theorem 1.7. *The Grassmannian $\operatorname{Gr}(m,n)$ is a non-singular rational variety of dimension $m(n-m)$.*

Proof. It follows from Lemma 1.5 that $\operatorname{Gr}(m,n)$ is a prevariety. Exercise 1.6 implies that any two points of $\operatorname{Gr}(m,n)$ are contained in a common open affine subvariety. It follows that $\operatorname{Gr}(m,n)$ is separated. □

Note 1.8. The action map $\operatorname{GL}(n) \times \operatorname{Gr}(m,n) \to \operatorname{Gr}(m,n)$ is a morphism of algebraic varieties. To see this, it is enough to show that each restriction $\operatorname{GL}(n)\times U_I \to \operatorname{Gr}(m,n)$ is a morphism. We will assume that $I = \{1,2,\dots,m\}$. By identifying U_I with $\operatorname{Mat}(n-m,m)$, the restricted map is given by

$$(g,B) \quad \mapsto \quad \pi\left(g \begin{bmatrix} 1_m \\ B \end{bmatrix}\right)\,.$$

1.4. **The Plucker embedding.** Recall that we set $x_I(A) = \det(A_I)$ whenever $A \in \operatorname{FR}(n,m)$ and I is a Schubert symbol for $\operatorname{Gr}(m,n)$. Set $N = \binom{n}{m} - 1$. Consider the map $\operatorname{FR}(n,m) \to \mathbb{P}^N$ defined by mapping each matrix A to the point of $\mathbb{P}^N$ given by its Plucker coordinates in some fixed order. This map factors through $\operatorname{Gr}(m,n)$, and the resulting map $x : \operatorname{Gr}(m,n) \to \mathbb{P}^N$ is a morphism of varieties by Exercise 1.1.

Exercise 1.9. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of varieties such that $g \circ f$ is an isomorphism of X with Z. Show that $f(X)$ is a closed subset of Y and that $f : X \to f(X)$ is an isomorphism.

Theorem 1.10. *The map $x : \operatorname{Gr}(m,n) \to \mathbb{P}^N$ is an isomorphism of $\operatorname{Gr}(m,n)$ with a closed subvariety of $\mathbb{P}^N$. In particular, $\operatorname{Gr}(m,n)$ is a projective variety.*

Proof. Let z_I be the homogeneous coordinate on $\mathbb{P}^N$ corresponding to the Plucker coordinate x_I. Then we have $x^{-1}(D_+(z_I)) = U_I$. It is enough to show that $x : U_I \to D_+(z_I)$ is a closed embedding for each Schubert symbol I. There are exactly $m(n-m)$ Schubert symbols J for which $I \cap J$ has cardinality $m-1$. Let $g : D_+(z_I) \to \mathbb{A}^{m(n-m)}$ be the map defined by the coordinate functions z_J/z_I for these symbols J. By Exercise 1.9 it suffices to show that the composed map $U_I \xrightarrow{x} D_+(z_I) \xrightarrow{g} \mathbb{A}^{m(n-m)}$ is an isomorphism. Assume that $I = \{1, 2, \dots, m\}$ and identify U_I with $\mathrm{Mat}(n-m \times m)$. For $B \in \mathrm{Mat}(n-m \times m)$ and $J = I - \{j\} \cup \{i\}$, the constant

$$x_J\left(\begin{bmatrix} 1_m \\ B \end{bmatrix}\right)$$

is equal to plus or minus the $(i-m, j)$-entry in B. This shows that $g \circ x$ is an isomorphism, as required. $\square$

Exercise 1.11. Set $V = k^n$. The exterior power $\bigwedge^m V$ has dimension $\binom{n}{m}$. Show that the the Plucker embedding is the map $\mathrm{Gr}(m, V) \to \mathrm{Gr}(1, \bigwedge^m V) = \mathbb{P}^N$ defined by $\Sigma \mapsto \bigwedge^m \Sigma$.

1.5. **Tautological vector bundles.** Set $V = k^n$ and let $\mathcal{V} = \mathrm{Gr}(m, n) \times V$ denote the trivial vector bundle over $\mathrm{Gr}(m, n)$ of rank n. The tautological subbundle $\mathcal{S}$ on $\mathrm{Gr}(m, n)$ is defined by

$$\mathcal{S} = \{(\Sigma, v) \in \mathcal{V} \mid v \in \Sigma\}.$$

Exercise 1.12. The set $\mathcal{S}$ is a subbundle of $\mathcal{V}$ of rank m.

The tautological quotient bundle on $\mathrm{Gr}(m, n)$ is defined by $\mathcal{Q} = \mathcal{V}/\mathcal{S}$. We obtain the tautological exact sequence

$$0 \to \mathcal{S} \to \mathcal{V} \to \mathcal{Q} \to 0.$$

Exercise 1.13. (1) With the identification $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$, the sheaf of sections of $\mathcal{S}$ is $\mathcal{O}(-1)$. (2) With the identification $\mathrm{Gr}(n-1, n) = \mathbb{P}^{n-1}$, the sheaf of sections of $\mathcal{Q}$ is $\mathcal{O}(1)$.

Exercise 1.14. Let $x : \mathrm{Gr}(m, V) \to \mathrm{Gr}(1, \bigwedge^m V)$ be the Plucker embedding from Exercise 1.11. Show that $x^*\mathcal{O}(-1)$ is the sheaf of sections of $\bigwedge^m \mathcal{S}$ and $x^*\mathcal{O}(1)$ is the sheaf of sections of $\bigwedge^{n-m} \mathcal{Q}$.

Exercise 1.15. Let Y be a variety and let $\mathcal{A} \subset Y \times V$ be a subbundle of rank m in the trivial vector bundle $Y \times V$. For $y \in Y$ we let $\mathcal{A}(y) \subset V$ denote the fiber of $\mathcal{A}$ over y. Show that the map $f : Y \to \mathrm{Gr}(m, n)$ defined by $f(y) = \mathcal{A}(y)$ is a morphism of varieties and that $f^*\mathcal{S} = \mathcal{A}$. This gives a one-to-one correspondence between morphisms $Y \to \mathrm{Gr}(m, n)$ and subbundles of rank m in $Y \times V$.

2. Schubert varieties

2.1. **Subgroups.** Let $X = \mathrm{Gr}(m, n)$ denote a fixed Grassmannian and set $G = \mathrm{GL}(n)$. We also let $T \subset G$ be the *maximal torus* of diagonal matrices and $B \subset G$ the *Borel subgroup* of upper triangular matrices. In this section we will study the orbits of the action of B on X. These orbits are called the *Schubert cells* of X, and their closures are the *Schubert varieties*.

2.2. **Schubert cells.** The standard flag of $V = k^n$ is defined by $F_\bullet = (F_1 \subset F_2 \subset \cdots \subset F_n)$, where $F_i = \operatorname{Span}\{e_1, e_2, \dots, e_i\}$. This is the unique flag of vector subspaces in V that is preserved by the action of B. Given a point $\Sigma \in X$, define an associated Schubert symbol by $I(\Sigma) = \{i \in [1,n] \mid \Sigma \cap F_i \supset \Sigma \cap F_{i-1}\}$. Given any Schubert symbol J for X we set $\mathring{X}_J = \{\Sigma \in X \mid I(\Sigma) = J\}$. The Grassmannian X is the disjoint union of these subsets. Notice that for $J = \{n-m+1, n-m+2, \dots, n\}$ we have $\mathring{X}_J = U_J$. Set $|J| = (\sum_{i \in J} i) - \binom{m+1}{2}$, and recall the notation $\Sigma_J = \operatorname{Span}\{e_i \mid i \in J\}$. The following lemma shows that X contains finitely many B-orbits, one for each T-fixed point, and each orbit is isomorphic to an affine space.

Lemma 2.1. *For each Schubert symbol J for X we have $B.\Sigma_J = \mathring{X}_J \cong \mathbb{A}^{|J|}$.*

Proof. Since B stabilizes the flag $F_\bullet$, it follows that $I(b.\Sigma) = I(\Sigma)$ for all $\Sigma \in X$ and $b \in B$. The inclusion $B.\Sigma_J \subset \mathring{X}_J$ follows from this and the observation that $I(\Sigma_J) = J$. On the other hand, given any point $\Sigma \in \mathring{X}_J$, we may choose a vector $v_j \in (\Sigma \cap F_j) \smallsetminus (\Sigma \cap F_{j-1})$ for each $j \in J$. Now write $v_j = \sum_{i=1}^{j} b_{i,j} e_i$. After rescaling we may assume that $b_{j,j} = 1$, and by subtracting multiples of the vectors v_i for $i < j$, we may also assume that $b_{i,j} = 0$ for $i \in J$. For $i > j$ we set $b_{i,j} = 0$, and for $j \notin J$ we set $b_{i,j} = \delta_{i,j}$. The entries $b_{i,j}$ define a matrix $b = (b_{i,j}) \in B$ such that $b.\Sigma_J = \Sigma$. Since we can freely choose the entries $b_{i,j}$ for which $i < j$, $j \in J$, and $i \notin J$, we obtain the isomorphism $\mathring{X}_J \cong \mathbb{A}^{|J|}$. $\square$

Example 2.2. Let $X = \operatorname{Gr}(3,7)$ and $J = \{2,5,7\}$. The identity $b.\Sigma_J = \Sigma$ in the above proof corresponds to the following identity of matrices:

$$\begin{bmatrix} 1 & b_{1,2} & 0 & 0 & b_{1,5} & 0 & b_{1,7} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & b_{3,5} & 0 & b_{3,7} \\ 0 & 0 & 0 & 1 & b_{4,5} & 0 & b_{4,7} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & b_{6,7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{1,2} & b_{1,5} & b_{1,7} \\ 1 & 0 & 0 \\ 0 & b_{3,5} & b_{3,7} \\ 0 & b_{4,5} & b_{4,7} \\ 0 & 1 & 0 \\ 0 & 0 & b_{6,7} \\ 0 & 0 & 1 \end{bmatrix}$$

2.3. **Schubert varieties.** Given Schubert symbols $I = \{i_1 < i_2 < \cdots < i_m\}$ and $J = \{j_1 < j_2 < \cdots < j_m\}$ for X, we will write $I \leq J$ if and only if $i_r \leq j_r$ for $1 \leq r \leq m$. This defines the *Bruhat order* on the set of all Schubert symbols for X. Write $I < J$ if $I \leq J$ and $I \neq J$. Define the *Schubert variety*

$$\begin{aligned} X_J &= \{\Sigma \in X \mid \dim(\Sigma \cap F_{j_r}) \geq r \text{ for } 1 \leq r \leq m\} \\ &= \{\Sigma \in X \mid \dim(\Sigma \cap F_p) \geq \#(J \cap [1,p]) \text{ for } 1 \leq p \leq n\}\,. \end{aligned} \tag{1}$$

Given $\Sigma \in X$, set $I = I(\Sigma)$. Then we have $\dim(\Sigma \cap F_p) = \#(I \cap [1,p])$, so it follows that $\Sigma \in X_J$ if and only if $\#(I \cap [1,p]) \geq \#(J \cap [1,p])$ for each p, which is equivalent to $I \leq J$. We obtain the alternative definition

$$X_J \;=\; \{\Sigma \in X \mid I(\Sigma) \leq J\} \;=\; \bigcup_{I \leq J} \mathring{X}_I\,. \tag{2}$$

Lemma 2.3. *Let $I \leq J$ be Schubert symbols for X. Then $\Sigma_I \in \overline{B.\Sigma_J}$.*

Proof. The lemma is clear if $I = J$. Otherwise we can choose a Schubert symbol K such that $I \leq K < J$ and $|K| = |J| - 1$. By induction on $|J|$ we may assume that $\Sigma_I \in \overline{B.\Sigma_K}$. It is enough to show that $\Sigma_K \in \overline{B.\Sigma_J}$, since this implies that $B.\Sigma_K \subset$

$\overline{B.\Sigma_J}$, and hence $\overline{B.\Sigma_K} \subset \overline{B.\Sigma_J}$. The choice of K implies that $\#(K \cap J) = m-1$ and $K = (J \smallsetminus \{j\}) \cup \{j-1\}$ for some $j \in J$. Define a subvariety $Y \subset X$ by

$$Y = \{\Sigma_{K \cap J} \oplus L \mid L \subset \operatorname{Span}\{e_{j-1}, e_j\} \text{ and } \dim(L) = 1\}.$$

The lemma follows because $Y \cong \mathbb{P}^1$, $\Sigma_K \in Y$, and $Y \cap B.\Sigma_J = Y \smallsetminus \{\Sigma_K\}$. □

Theorem 2.4. *Let J be any Schubert symbol for X. Then we have $X_J = \overline{B.\Sigma_J}$, hence X_J is a rational projective variety of dimension $|J|$.*

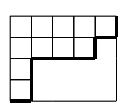
Proof. It follows from (1) that X_J is a B-stable closed subvariety of X and from (2) that $B.\Sigma_J \subset X_J$. This implies that $\overline{B.\Sigma_J} \subset X_J$. Let $\Sigma \in X_J$ be any point and set $I = I(\Sigma)$. Then $I \leq J$, so it follows from Lemma 2.3 that $\Sigma \in \overline{B.\Sigma_I} \subset \overline{B.\Sigma_J}$. □

Corollary 2.5. *We have $X_I \subset X_J$ if and only if $I \leq J$.*

Proof. This follows from Theorem 2.4 and equation (2). □

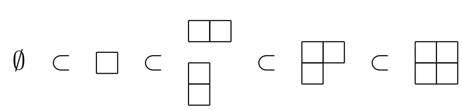
2.4. **Young diagrams.** Each Schubert symbols for $X = \operatorname{Gr}(m,n)$ corresponds to a Young diagram of boxes contained in the rectangle with m rows and $n-m$ columns. Given a Schubert symbol I, consider the path from the lower-left corner to the upper-right corner consisting of n steps, where the i-th step is vertical if $i \in I$ and horizontal if $i \notin I$. The Young diagram corresponding to I is the part of the rectangle that is north-west of this path. Notice that $|I|$ is the number of boxes in the Young diagram for I, and we have $I \leq J$ if and only if the Young diagram of I is a subset of the Young diagram of J. If λ is the Young diagram corresponding to J, then we will denote the Schubert variety X_J also by X_λ.

Example 2.6. The Schubert symbol $I = \{2,3,7,9\}$ for $\operatorname{Gr}(4,9)$ corresponds to the following Young diagram:



Exercise 2.7. Let λ_1 and λ_2 be two Young diagrams for X. Then $X_{\lambda_1} \cap X_{\lambda_2} = X_{\lambda_1 \cap \lambda_2}$.

Example 2.8. The Bruhat order of Schubert varieties in $\operatorname{Gr}(2,4)$ is determined by the inclusions of Young diagrams:



Example 2.9. The Grassmannian $X = \operatorname{Gr}(2,4)$ can be identified with the set of lines in $\mathbb{P}^3$. The $F_\bullet = (F_1 \subset F_2 \subset F_3 \subset V)$ be the standard flag in $V = k^4$. Then F_1 corresponds to a point in $\mathbb{P}^3$, F_2 is a line in $\mathbb{P}^3$, and F_3 is a plane in $\mathbb{P}^3$. The

Schubert varieties in X defined by

$$\begin{aligned}
X_{\emptyset} &= X_{\{1,2\}} = \{F_2\} \\
X_{\square} &= X_{\{1,3\}} = \{\Sigma \in X \mid F_1 \subset \Sigma \subset F_3\} \\
X_{\square\square} &= X_{\{1,4\}} = \{\Sigma \in X \mid F_1 \subset \Sigma\} \\
X_{\substack{\square\\\square}} &= X_{\{2,3\}} = \{\Sigma \in X \mid \Sigma \subset F_3\} \\
X_{\substack{\square\square\\\square}} &= X_{\{2,4\}} = \{\Sigma \in X \mid \Sigma \cap F_2 \neq 0\} \\
X_{\substack{\square\square\\\square\square}} &= X_{\{3,4\}} = X
\end{aligned}$$

Example 2.10. Given 4 fixed generic lines in a 3-dimensional space, how many lines meet all of them? The lines in $\mathbb{P}^3$ that meet the fixed line F_2 are exactly the points in the Schubert variety $X_{\{2,4\}}$ of $X = \mathrm{Gr}(2,4)$. Choose $g_1, g_2, g_3, g_4 \in \mathrm{GL}(4)$ such that the fixed lines are $g_1.F_2$, $g_2.F_2$, $g_3.F_2$, $g_4.F_2$. Then a line meets all four fixed lines if and only if it belongs to the intersection

$$g_1.X_{\{2,4\}} \cap g_2.X_{\{2,4\}} \cap g_3.X_{\{2,4\}} \cap g_4.X_{\{2,4\}}\,. \tag{3}$$

The Plucker embedding $X \subset \mathbb{P}^5$ identifies X with the closed subset $X = V_+(z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23})$. Notice also that $X_{\{2,4\}} = X \cap V_+(z_{34})$ is the intersection of X with a hyperplane in $\mathbb{P}^5$. It follows that the intersection (3) is the intersection of 4 hyperplanes and one quadratic hypersurface in $\mathbb{P}^5$. We deduce that exactly 2 lines meet the 4 fixed lines.

Exercise 2.11. Consider the Schubert variety $X_{\{2,4\}}$ in $X = \mathrm{Gr}(2,4)$.
(1) The variety $X_{\{2,4\}}$ is normal.
(2) $\mathrm{C}\ell(X_{\{2,4\}})$ is the free abelian group generated by $[X_{\{2,3\}}]$ and $[X_{\{1,4\}}]$.
(3) The divisor $[X_{\{2,3\}}] + [X_{\{1,4\}}]$ is a Cartier divisor, but $[X_{\{2,3\}}]$ is not.

Exercise 2.12. Let λ be a Young diagram for $X = \mathrm{Gr}(m,n)$. The Schubert variety X_λ is non-singular if and only if λ is a rectangle, in which case X_λ is isomorphic to a smaller Grassmannian.

2.5. **Opposite Schubert varieties.** Let $B^- \subset G = \mathrm{GL}(n)$ denote the opposite Borel subgroup of lower triangular matrices. Given a Schubert symbol I for $X = \mathrm{Gr}(m,n)$, define the opposite Schubert cell $\mathring{X}^I = B^-.\Sigma_I$ and the opposite Schubert variety $X^I = \overline{B^-.\Sigma_I}$. Let $w_0 \in G$ be the element defined by $w_0(e_i) = e_{n+1-i}$. We then have $B^- = w_0 B w_0 \subset G$. Furthermore we have $w_0(\Sigma_I) = \Sigma_{I^\vee}$ where $I^\vee = \{n+1-i \mid i \in I\}$ denotes the dual Schubert symbol of I. We obtain $X^I = \overline{w_0 B w_0.\Sigma_I} = w_0.X_{I^\vee}$. In particular, X^I is a closed irreducible subvariety of X. Since $|I| + |I^\vee| = \dim(X)$, we obtain $\mathrm{codim}(X^I, X) = |I|$. Notice also that the Young diagram of $I^\vee$ is the complement of the Young diagram of I in the $m \times (n-m)$ rectangle, rotated by 180 degrees. Define the opposite flag $F^\bullet = (F^n \subset F^{n-1} \subset \cdots \subset F^1)$ by $F^p = \mathrm{Span}\{e_p, e_{p+1}, \ldots, e_n\}$. Equations (1) and (2) imply that

$$X^J = \bigcup_{I \geq J} \mathring{X}^I = \{\Sigma \in X \mid \dim(\Sigma \cap F^p) \geq \#(J \cap [p,n]) \text{ for } 1 \leq p \leq n\}\,. \tag{4}$$

Proposition 2.13. *We have $X^I \cap X_J \neq \emptyset$ if and only if $I \leq J$.*

Proof. If $I \leq J$, then $\Sigma_I \in X^I \cap X_J$. Assume that $X^I \cap X_J \neq \emptyset$. We must show that $\#(J \cap [1,p]) \leq \#(I \cap [1,p])$ for $1 \leq p \leq n$. Equivalently, we must show that

$\#(J \cap [1,p]) + \#(I \cap [p+1,n]) \leq m$ for each $p \in [1,n]$. Set $U_1 = \Sigma \cap F_p$ and $U_2 = \Sigma \cap F^{p+1}$. Since both U_1 and U_2 are subspaces of the m-dimensional vector space Σ, and we have $U_1 \cap U_2 = 0$, it follows that $\dim(U_1) + \dim(U_2) \leq m$. The proposition now follows because $\dim(U_1) \geq \#(J \cap [1,p])$ and $\dim(U_2) \geq \#(I \cap [p+1,n])$. □

Remark 2.14. The second part of the proof of Proposition 2.13 can also be derived from Borel's fixed point theorem: if H is any irreducible solvable algebraic group that acts on a non-empty complete variety Y, then Y contains an H-fixed point. If $X^I \cap X_J \neq \emptyset$, then this fact implies that $X^I \cap X_J$ contains a T-fixed point Σ_K, in which case $I \leq K \leq J$.

Proposition 2.15. *We have $X^I \cap X_I = \{\Sigma_I\}$ for each Schubert symbol I.*

Proof. Let $\Sigma \in X^I \cap X_I$ be any point. Fix $p \in I$ and set $U_1 = \Sigma \cap F_p$ and $U_2 = \Sigma \cap F^p$. Then $\dim(U_1) + \dim(U_2) \geq \#(I \cap [1,p]) + \#(I \cap [p,n]) = m+1$. Since U_1 and U_2 are subspaces of the same m-dimensional vector space Σ, this implies that $\dim(U_1 \cap U_2) \geq 1$. Since we also have $U_1 \cap U_2 \subset F_p \cap F^p = \mathrm{Span}\{e_p\}$, we deduce that $e_p \in \Sigma$. This shows that $\Sigma = \Sigma_I$. □

Any non-empty intersection $X^I \cap X_J$ is called a *Richardson variety.* Notice that Schubert varieties are special cases of Richardson varieties.

Fact 2.16. *Given Schubert symbols $I \leq J$, the Richardson variety $X^I \cap X_J$ is a rational projective variety of dimension $|J| - |I|$. In addition, $X^I \cap X_J$ is Cohen-Macaulay and has rational singularities. In particular, Richardson varieties are normal.*

Given a Young diagram λ for $\mathrm{Gr}(m,n)$, let λ^t be the transposed Young diagram for the dual Grassmannian $\mathrm{Gr}(n-m,n)$ obtained by interchanging rows and columns in λ. If λ is the Young diagram of the Schubert symbol I, then λ^t is the diagram of $[1,n] \smallsetminus I^\vee$.

Exercise 2.17. Construct an isomorphism $\iota : X = \mathrm{Gr}(m,n) \xrightarrow{\sim} Y = \mathrm{Gr}(n-m,n)$ such that $\iota(X_\lambda) = Y_{\lambda^t}$ and $\iota(X^\lambda) = Y^{\lambda^t}$ for each Young diagram λ.

3. Schubert calculus

3.1. **Cohomology.** In the rest of these notes we will work with varieties over the field $\mathbb{C}$ of complex numbers. However, any statements about closed or open sets will refer to the Zariski topology. In this section we state some basic facts about the cohomology ring of a projective non-singular complex variety. More details and proofs can be found in e.g. [2, App. B]. All (co)homology groups will be taken with integer coefficients.

Let X be a projective non-singular variety of (complex) dimension n. If $V \subset X$ is any irreducible closed subvariety of codimension c, then one can show that $H^{2c}(X, X \smallsetminus V)$ is a free abelian group of rank one generated by a fundamental class $[V]$. The group $H^{2c}(X, X \smallsetminus V)$ is the top Borel-Moore homology group of V. The image of $[V]$ under the map $H^{2c}(X, X \smallsetminus V) \to H^{2c}(X)$ will also be denoted by $[V]$, and we will call it simply the *cohomology class* of V.

Since X is oriented, we have the Poincare duality map $H^k(X) \xrightarrow{\sim} H_{2n-k}(X)$ defined by $\alpha \mapsto \alpha \cap [X]$. Let $\int_X : H^*(X) \to \mathbb{Z}$ be the map defined by $\int_X \alpha =$

$f_*(\alpha \cap [X]) \in H_*(\text{point}) = \mathbb{Z}$, where $f : X \to \{\text{point}\}$ is the structure morphism. For any irreducible closed subvariety $V \subset X$ we then have

$$\int_X [V] = \begin{cases} 1 & \text{if } \dim(V) = 0; \\ 0 & \text{if } \dim(V) > 0. \end{cases} \tag{5}$$

Let $V, W \subset X$ be closed irreducible subvarieties, and write the intersection $V \cap W$ as the union of its irreducible components:

$$V \cap W = Z_1 \cup Z_2 \cup \cdots \cup Z_r \,.$$

The intersection of V and W is called *proper* if $\operatorname{codim}(Z_i, X) = \operatorname{codim}(V, X) + \operatorname{codim}(W, X)$ for each i. We will say that V and W *meet transversely* if their intersection is proper and each component Z_i contains a dense open subset of points z for which $T_z Z_i = T_z V \cap T_z W \subset T_z X$.

Fact 3.1. *If V and W meet transversely, then $[V] \cdot [W] = [Z_1] + [Z_2] + \cdots + [Z_r]$ holds in the cohomology ring $H^*(X)$.*

Using appropriate definitions, this statement can be reformulated as $[V \cap W] = [V] \cdot [W]$. In other words, we can study the intersection $V \cap W$ by studying the product of the cohomology classes of V and W. We also need the following.

Fact 3.2. *Assume that X has a filtration $\emptyset = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_s = X$ by closed subsets such that each difference $U_i = X_i \smallsetminus X_{i-1}$ is isomorphic to an affine space $\mathbb{C}^{n_i}$. Then the cohomology classes $[\overline{U_i}]$ form an additive basis of $H^*(X)$ over $\mathbb{Z}$.*

Notice that if X satisfies the condition of Fact 3.2, then $H^p(X) = 0$ for every odd integer p. In particular, $H^*(X)$ is a commutative ring.

3.2. **Kleiman's transversality theorem.** We need the following result due to Kleiman. The version we state is only true in characteristic zero. Several more general statements can be found in [3].

Fact 3.3. *Let G be a connected complex algebraic group that acts transitively on the variety X, and let V and W be irreducible subvarieties of X. Then G contains a dense open subset U such that V and $g.W$ meet transversely for all $g \in U$.*

Notice also that if G is a connected group acting on X and $g \in G$, then the map $g : X \to X$ defined by $g(x) = g.x$ is homotopic to the identity map on X. This implies that for any closed irreducible subvariety $V \subset X$ we have $[g.V] = g_*[V] = [V] \in H^*(X)$. In other words, the cohomology class of V is invariant under translation.

3.3. **Cohomology of Grassmannians.** Set $V = \mathbb{C}^n$ and let $X = \operatorname{Gr}(m, n)$ be the Grassmannian of m-dimensional vector subspaces of V. We let G, B, B^-, and T be as in Section 2. The cohomology classes $[X_I]$ and $[X^I]$ of the Schubert varieties in X are called *Schubert classes*. We have $[X^I] \in H^{2|I|}(X)$ and $[X_I] = [w_0.X^{I^\vee}] = [X^{I^\vee}]$.

Proposition 3.4. *The Schubert classes $[X^I]$ form an additive basis for $H^*(X)$.*

Proof. Let $I_1, I_2, \ldots, I_s$ be any ordering of the Schubert symbols for X such that $|I_r| \geq |I_{r+1}|$ for each r, and define $Y_p = \bigcup_{r=1}^p X^{I_r}$ for $p \in [0, s]$. Then Y_p is a closed subset of X and $Y_p \smallsetminus Y_{p-1} = \mathring{X}^{I_p} \cong \mathbb{C}^{|I_p^\vee|}$ for each r. The proposition now follows from Fact 3.2. $\square$

For applications to enumerative geometry we are particularly interested in the multiplicative structure of the cohomology ring $H^*(X)$. This structure is encoded in the *Schubert structure constants* $C^K_{I,J}$ of X, which are defined by the identity

$$[X^I] \cdot [X^J] \;=\; \sum_K C^K_{I,J}\,[X^K]\,. \tag{6}$$

The constants $C^K_{I,J}$ are also called the *Littlewood-Richardson coefficients*. Since $H^*(X)$ is a graded ring it follows that $C^K_{I,J}$ is non-zero only if $|K| = |I| + |J|$. If λ, μ, and ν are the Young diagrams of I, J, and K, then we also write $C^\nu_{\lambda,\mu} = C^K_{I,J}$.

Exercise 3.5. The set $B^-B = \{b'b \mid b' \in B^- \text{ and } b \in B\}$ is a dense open subset of G. Hint: We have $g \in B^-B$ if and only if $g.F_\bullet$ and $F^\bullet$ are opposite flags.

Lemma 3.6. *Any two opposite Schubert varieties X^I and X_J meet transversely.*

Proof. By Fact 3.3 there is a dense open subset $U \subset G$ such that X^I and $g.X_J$ meet transversely for all $g \in U$, and Exercise 3.5 shows that $U \cap B^-B \neq \emptyset$. Choose $b' \in B^-$ and $b \in B$ such that $b'b \in U$. Then X^I and $b'b.X_J = b'.X_J$ meet transversely. Since the automorphism of X defined by $\Sigma \mapsto b'.\Sigma$ maps X^I to itself and maps X_J to $b'.X_J$, it follows that X^I and X_J also meet transversely. □

Lemma 3.6 implies that all components of the Richardson variety $X^I \cap X_J$ have dimension $|J| - |I|$. This statement is weaker than Fact 2.16 but sufficient for the proof of the following proposition.

Proposition 3.7. *For Schubert symbols I and J we have $\int_X [X^I] \cdot [X_J] = \delta_{I,J}$.*

Proof. Since X^I and X_J meet transversely, we may use Fact 3.1 to compute the product $[X^I] \cdot [X_J]$. If $I \not\leq J$ then $X^I \cap X_J = \emptyset$ and $[X^I] \cdot [X_J] = 0$. Otherwise $[X^I] \cdot [X_J] = [X^I \cap X_J]$ is the class of a Richardson variety, and it follows from Proposition 2.15 and equation (5) that $\int_X [X^I \cap X_J]$ is equal to 1 if $I = J$ and equal to zero otherwise. □

Corollary 3.8. *We have $C^K_{I,J} = \int_X [X^I] \cdot [X^J] \cdot [X_K]$.*

Since we have $C^K_{I,J} = \int_X [X^I] \cdot [X^J \cap X_K] = \int_X [X^J] \cdot [X^I \cap X_K]$, the constant $C^K_{I,J}$ is non-zero only if $I \leq K$ and $J \leq K$.

3.4. **The Pieri formula.** For $0 \leq p \leq n-m$ we define the special Schubert variety $X^p = X^{I(p)}$ where $I(p) = \{1, 2, \dots, m-1, m+p\}$. Equivalently we have

$$X^p = \{\Sigma \in X \mid \Sigma \cap F^{m+p} \neq 0\}\,.$$

The corresponding class $[X^p] \in H^{2p}(X)$ is called a special Schubert class. It is equal to the p-th Chern class $c_p(\mathcal{Q})$ of the tautological quotient bundle $\mathcal{Q}$. Notice that the Young diagram of $I(p)$ consists of p boxes in a single row.

Theorem 3.9. *Let $I = \{i_1 < i_2 < \cdots < i_m\}$ be a Schubert symbol for X and let $p \in [0, n-m]$. Then we have*

$$[X^p] \cdot [X^I] = \sum_J [X^J]$$

where the sum is over all Schubert symbols $J = \{j_1 < j_2 < \cdots < j_m\}$ for which $|J| = |I| + p$ and $i_r \leq j_r < i_{r+1}$ for $1 \leq r \leq m$. Here we set $i_{m+1} = n+1$.

Proof. Let J be any Schubert symbol such that $J \geq I$ and $|J| = |I| + p$. We have $C^J_{I(p),I} = \int_X [X^p] \cdot [X^I] \cdot [X_J] = \int_X [X^p] \cdot [X^I \cap X_J]$. For $1 \leq r \leq m$ we set $E_r = F^{i_r} \cap F_{j_r} = \operatorname{Span}\{e_{i_r}, \dots, e_{j_r}\}$, and we let $E = E_1 + \cdots + E_m$ be the span of these vector spaces. Notice that $E = \bigcap_{r=0}^m (F_{j_r} + F^{i_{r+1}})$, where we set $j_0 = 0$. In fact, both vector spaces are spanned by the same set of basis vectors e_i of V. We claim that for any point $\Sigma \in X^I \cap X_J$ we have $\Sigma \subset E$. It is enough to show that $\Sigma \subset F_{j_r} + F^{i_{r+1}}$ for each r. This is clear if $j_r \geq i_{r+1}$. Otherwise the Schubert conditions $\dim(\Sigma \cap F_{j_r}) \geq r$ and $\dim(\Sigma \cap F^{i_{r+1}}) \geq m - r$ imply that $\Sigma = (\Sigma \cap F_{j_r}) \oplus (\Sigma \cap F^{r_{r+1}}) \subset F_{j_r} + F^{i_{r+1}}$, as claimed.

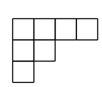
Choose a generic element $g \in G$. If $C^J_{I(p),J} \neq 0$, then Fact 3.1 implies that $g.X^p \cap X^I \cap X_J \neq \emptyset$, so we may choose a point $\Sigma \in g.X^p \cap X^I \cap X_J$. Since $\Sigma \cap g.F^{m+p} \neq 0$ and $\Sigma \subset E$, it follows that $E \cap g.F^{m+p} \neq 0$. By the choice of g this implies that $\dim(E) \geq m + p$. But we also have $\dim(E) \leq \sum_{r=1}^m \dim(E_r) = |J| - |I| + m = m + p$. We deduce that E is the direct sum of the vector spaces E_r, or equivalently $j_r < i_{r+1}$ for each r.

Finally assume that J is an arbitrary Schubert symbol such that $J \geq I$, $|J| = |I| + p$, and $j_r < i_{r+1}$ for each r. It remains to show that $g.X^p \cap X^I \cap X_J$ contains exactly one point. Our assumption implies that $E = E_1 \oplus \cdots \oplus E_m$ is a direct sum of dimension $m + p$, hence $E \cap g.F^{m+p}$ has dimension 1. Let $v \in E \cap g.F^{m+p}$ be a generator, and write $v = v_1 + \cdots + v_m$ with $v_r \in E_r$. Since g is generic we have $v_r \neq 0$ for each r, so $\operatorname{Span}\{v_1, \dots, v_m\}$ is a point of $g.X^p \cap X^I \cap X_J$. Let $\Sigma \in g.X^p \cap X^I \cap X_J$ be any point. Since $\dim(\Sigma \cap F_{j_r}) \geq r$ and $\dim(\Sigma \cap F^{i_r}) \geq m + 1 - r$, we deduce that $\Sigma \cap E_r \neq 0$, so $\Sigma = (\Sigma \cap E_1) \oplus \cdots \oplus (\Sigma \cap E_m)$. Since $v \in \Sigma$, this implies that $v_r \in \Sigma$ for each r, hence $\Sigma = \operatorname{Span}\{v_1, \dots, v_m\}$. This completes the proof. $\square$

Given Young diagrams $\lambda \subset \nu$, let ν/λ denote the *skew diagram* of boxes in ν that are not in λ. A skew diagram is called a *horizontal strip* if each column contains at most one box. The Pieri formula can be reformulated as follows.

Exercise 3.10. Let λ be a Young diagram for X and $p \in [0, n-m]$. Then we have $[X^p] \cdot [X^\lambda] = \sum_\nu [X^\nu]$ where the sum is over all Young diagrams ν for which ν/λ is a horizontal strip of p boxes.

Due to the simple statement of the Pieri formula in Exercise 3.10, we will use Young diagrams to index Schubert classes for combinatorial purposes. We will identify a Young diagram λ with the corresponding partition $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$, where λ_i is the number of boxes in the i-th row of λ. For example we have

$(4, 2, 1) =$  .

Example 3.11. Let $X = \operatorname{Gr}(2, 4)$. Given 4 generic elements $g_1, g_2, g_3, g_4 \in \operatorname{GL}(4)$ we have

$$\left[g_1.X^{(1)} \cap g_2.X^{(1)} \cap g_3.X^{(1)} \cap g_4.X^{(1)}\right] = \left[X^{(1)}\right]^4 \in H^*(X)\,.$$

Using the Pieri formula we obtain $[X^{(1)}]^2 = [X^{(2)}] + [X^{(1,1)}]$, $[X^{(1)}]^3 = 2\,[X^{(2,1)}]$, and $[X^{(1)}]^4 = 2\,[X^{(2,2)}]$. Since $[X^{(2,2)}]$ is the class of a single point in X, this gives a cohomological solution to Example 2.10.

Exercise 3.12. Show that the degree of $X = \operatorname{Gr}(m, n)$ in the Plucker embedding $X \subset \mathbb{P}^N$, $N = \binom{n}{m} - 1$, is equal to the number of standard tableaux on the rectangle with m rows and $n-m$ columns, i.e. fillings of the boxes in this rectangle with distinct integers from 1 to $m(n-m)$, such that every row is increasing form left to right and every column is increasing from top to bottom. For example, $\mathrm{Gr}(2,5)$ has degree 5 in its Plucker embedding due to the following list of standard tableaux:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

For $0 \leq q \leq m$ we let $(1^q) = (1,1,\dots,1)$ denote the Young diagram consisting of a single column with q boxes. The corresponding Schubert class $[X^{(1^q)}]$ is equal to the Chern class $c_q(\mathcal{S}^\vee)$ of the dualized tautological subbundle. These Schubert classes are also called special.

Exercise 3.13. For $k \geq 1$ we have $\sum_{p+q=k} (-1)^p\,[X^p]\cdot[X^{(1^q)}] = 0$ in $H^*(X)$.

Exercise 3.14. Formulate a Pieri formula for multiplying with the classes $[X^{(1^q)}]$.

Exercise 3.15 (Presentation)**.** The cohomology ring $H^*(X)$ is generated as a ring by the special Schubert classes $[X^p]$ for $p \in [1, n-m]$. More precisely, consider the polynomial ring $\mathbb{Z}[\sigma_1,\dots,\sigma_{n-m}]$ in $n-m$ independent variables. For $m < p \leq n$ we let $\Delta_p = \det(\sigma_{1+j-i})_{p\times p}$ denote the determinant of the $p\times p$ matrix whose (i,j)-th entry is σ_{1+j-i}. Here we set $\sigma_0 = 1$, and we set $\sigma_p = 0$ for $p < 0$ or $p > n-m$. For example we have

$$\Delta_3 \;=\; \det\begin{bmatrix}\sigma_1 & \sigma_2 & \sigma_3\\ 1 & \sigma_1 & \sigma_2\\ 0 & 1 & \sigma_1\end{bmatrix} \;=\; \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3\,.$$

Then we have an isomorphism $\mathbb{Z}[\sigma_1,\dots,\sigma_{n-m}]/(\Delta_{m+1},\dots,\Delta_n) \xrightarrow{\sim} H^*(X)$ given by $\sigma_p \mapsto [X^p]$.

Exercise 3.16 (Jacobi-Trudi formula)**.** With the notation of Exercise 3.15, show that the class $[X^\lambda]$ is the image of the determinant $\det(\sigma_{\lambda_i+j-i})_{m\times m}$. Hint: Expand the determinant along the m-th column.

3.5. **The Littlewood-Richardson rule.** Let ν/λ be a skew Young diagram. A *semistandard Young tableau* of shape ν/λ is a labeling T of the boxes in ν/λ with positive integers such that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. The *content* of T is the sequence $(a_1, a_2, \dots, a_\ell)$ for which a_x is the number of boxes in T that contain the integer x. The *row word* of T is the list of integers obtained by reading the rows of T from left to right, starting with the bottom row and moving up. We will say that T is a *Littlewood-Richardson tableau* (LR tableau) if every occurrence of an integer $x \geq 2$ in the row word is followed by more occurrences of $x-1$ than of x. Notice that the content of any LR tableau is a partition corresponding to a Young diagram

Example 3.17. Both of the following tableaux are semistandard Young tableaux with content $(4,2,1)$. The first tableau is an LR tableau, but the second is not because the 3 in the row word $(1,2,2,1,3,1,1)$ is not followed by any 2s.

$$\begin{array}{ccccc} & & & \boxed{1} & \boxed{1}\\ & & \boxed{1} & \boxed{2} & \\ \boxed{1} & \boxed{2} & \boxed{3} & & \end{array}\qquad\qquad\begin{array}{ccccc} & & & \boxed{1} & \boxed{1}\\ & & \boxed{1} & \boxed{3} & \\ \boxed{1} & \boxed{2} & \boxed{2} & & \end{array}$$

Theorem 3.18. *Let λ, μ, and ν be Young diagrams. Then $C^{\nu}_{\lambda,\mu}$ is the number of LR tableaux of shape ν/λ and content μ.*

Example 3.19. We have $C^{(3,2,1)}_{(2,1),(2,1)} = 2$ due to the following list of LR tableaux:

$$\begin{array}{ccc} & & 1 \\ & 1 & \\ 2 & & \end{array} \qquad \begin{array}{ccc} & & 1 \\ & 2 & \\ 1 & & \end{array}$$

Theorem 3.18 was first stated in a paper of Littlewood and Richardson from 1934 [4], but rigorous proofs did not appear until the 1970's. We will prove Theorem 3.18 based on the following fact.

Exercise 3.20. Let R be an associative ring with unit 1, let $S \subset R$ be a subset that generates R as a $\mathbb{Z}$-algebra, and let M be a left R-module. Let $\phi : R \times M \to M$ be any $\mathbb{Z}$-bilinear map satisfying that, for all $r \in R$, $s \in S$, and $m \in M$ we have $\phi(1, m) = m$ and $\phi(sr, m) = s\phi(r, m)$. Then $\phi(r, m) = rm$ for all $(r, m) \in R \times M$.

We will apply this exercise to the action on $H^*(X)$ on itself. Let $S \subset H^*(X)$ be the set of special Schubert classes $[X^p]$ for $1 \leq p \leq n - m$, and define a bilinear map $\phi : H^*(X) \times H^*(X) \to H^*(X)$ by $\phi([X^\lambda], [X^\mu]) = \sum_\nu \widehat{C}^{\nu}_{\lambda,\mu}[X^\nu]$, where $\widehat{C}^{\nu}_{\lambda,\mu}$ denotes the number of LR tableaux of shape ν/λ and content μ. We must show that the map ϕ satisfies the conditions of Exercise 3.20.

Exercise 3.21. We have $\phi(1, [X^\mu]) = [X^\mu]$ for each Young diagram μ.

3.6. **Jeu de taquin slides.** Let T be a semistandard tableau of shape ν/λ and let b be an outer corner of λ, i.e. a box of λ such λ contains no box directly below b or directly to the right of b. We can apply a *jeu de taquin slide* to T starting from b as follows. Move the smallest of the integers in T directly below and directly to the right of b into the box b. If T contains only one integer below or to the right of b, then move that integer, and if the same integer appears below and to the right of b, then move the integer below b into b. This creates a new empty box in T. Now move the smaller of the integers directly below and to the right of the empty box into the empty box in the same way. Continue until the empty box is an outer box b' of ν. The result of the jeu de taquin slide is a new semistandard tableau T' of some shape ν'/λ', where λ' is obtained by removing b from λ, and ν' is obtained by removing b' from ν.

Example 3.22. The following is an example of a jeu de taquin slide.

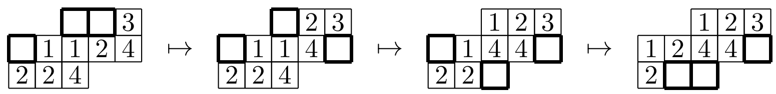
$$\begin{array}{ccccc} & \boxed{} & 1 & 2 & 4 \\ & 1 & 3 & 5 & \\ 2 & 4 & 4 & & \\ 3 & & & & \end{array} \mapsto \begin{array}{ccccc} & 1 & 1 & 2 & 4 \\ & \boxed{} & 3 & 5 & \\ 2 & 4 & 4 & & \\ 3 & & & & \end{array} \mapsto \begin{array}{ccccc} & 1 & 1 & 2 & 4 \\ & 3 & \boxed{} & 5 & \\ 2 & 4 & 4 & & \\ 3 & & & & \end{array} \mapsto \begin{array}{ccccc} & 1 & 1 & 2 & 4 \\ & 3 & 4 & 5 & \\ 2 & 4 & \boxed{} & & \\ 3 & & & & \end{array}$$

Notice that the jeu de taquin slide can be reversed. Given a semistandard tableau T' of shape ν'/λ' and a box b' outside ν' such that $\nu' \cup \{b'\}$ is a Young diagram, one can move the largest integer directly above or to the left of b' into b' and continue until the empty box emerges at the north-west side of the tableau. This is called a *reverse jeu de taquin slide.*

Exercise 3.23. Let T' be the tableau obtained by performing a jeu de taquin slide on T. Then T' is an LR tableau if and only if T is an LR tableau.

Let again T be a semistandard Young tableau of shape ν/λ, and let λ' be a Young diagram contained in λ such that λ/λ' is a horizontal strip. Let T' be the tableau obtained by performing a sequence of jeu de taquin slides to T, starting form the boxes of λ/λ' in right-to-left order. Let ν'/λ' be the shape of T'.

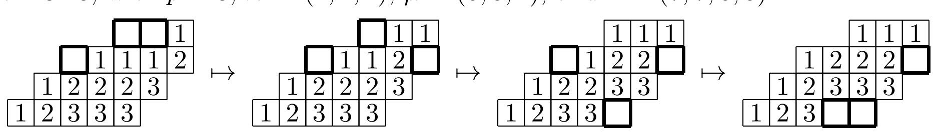
Example 3.24. The following is a sequence of jeu de taquin slides starting from the boxes in a horizontal strip in right-to-left order.



Exercise 3.25. The skew diagram ν/ν' is a horizontal strip. Furthermore, the tableau T is recovered by applying a sequence of reverse jeu de taquin slides to T', starting from the boxes of ν/ν' in left-to-right order.

Proof of Theorem 3.18. By Exercise 3.20 it suffices to prove that the map $\phi : H^*(X) \times H^*(X) \to H^*(X)$ satisfies $\phi(1, [X^\mu]) = [X^\mu]$ and $\phi([X^p] \cdot [X^{\lambda'}], [X^\mu]) = [X^p] \cdot \phi([X^{\lambda'}], [X^\mu])$ for all Young diagrams λ' and μ and integers $p \in [1, n-m]$. The first of these identities is Exercise 3.21. Fix a Young diagram ν. It follows from the Pieri rule (Theorem 3.9) and the definition of ϕ that the coefficient of $[X^\nu]$ in the expansion of $\phi([X^p] \cdot [X^{\lambda'}], [X^\mu])$ is equal to the number of LR tableaux T with content μ and shape ν/λ, where λ is some Young diagram such that λ/λ' is a horizontal strip of p boxes. Similarly, the coefficient of $[X^\nu]$ in the expansion of $[X^p] \cdot \phi([X^{\lambda'}], [X^\mu])$ is equal to the number of LR tableaux T' with content μ and shape ν'/λ' for some Young diagram ν' such that ν/ν' is a horizontal strip of p boxes. Since Exercise 3.23 and Exercise 3.25 give an explicit bijection between the set of tableaux T and the set of tableaux T', it follows that the coefficient of $[X^\nu]$ is the same in both expansions. $\square$

Example 3.26. Here is an example of the bijection used in the proof of Theorem 3.18, with $p = 3$, $\lambda' = (4, 2, 1)$, $\mu = (6, 5, 4)$, and $\nu = (7, 7, 6, 5)$.



4. Quantum cohomology

4.1. **Gromov-Witten invariants.** Define a *rational curve* in the Grassmannian $X = \mathrm{Gr}(m, n)$ to be the image of any morphism of varieties $\mathbb{P}^1 \to X$. The degree of a curve $C \subset X$ is defined by $\deg(C) = \int_X [C] \cdot [X^{(1)}]$. A point in X will be considered as a rational curve of degree zero.

Definition 4.1. Given Young diagrams λ, μ, ν, and a degree d for which $|\lambda| + |\mu| = |\nu| + nd$, define the (3 point, genus 0) *Gromov-Witten invariant* $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d$ to be the number of rational curves $C \subset X$ of degree d that meet the Schubert varieties X^λ, $g.X^\mu$, and X_ν. Here $g \in G = \mathrm{GL}(n)$ is a fixed generic element.

This Gromov-Witten invariant can also be defined using the Kontsevich moduli space $\overline{\mathcal{M}}_{0,3}(X, d)$ of stable maps to X. We have

$$\begin{aligned} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d &:= \int_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*[X^\lambda] \cdot \mathrm{ev}_2^*[X^\mu] \cdot \mathrm{ev}_3^*[X_\nu] \\ &= \#\big(\mathrm{ev}_1^{-1}(X^\lambda) \cap \mathrm{ev}_2^{-1}(g.X^\mu) \cap \mathrm{ev}_3^{-1}(X_\nu)\big)\,. \end{aligned}$$

The first identity defines the Gromov-Witten invariant without depending on $g \in G$, and the second equality follows from intersection theory applied to $\overline{\mathcal{M}}_{0,3}(X,d)$ together with a more general version of Kleiman's transversality theorem.

The requirement $|\lambda|+|\mu| = |\nu|+nd$ reflects that $\dim \overline{\mathcal{M}}_{0,3}(X,d) = \dim(X)+nd$, and hence $\dim(\mathrm{ev}_1^{-1}(X^\lambda) \cap \mathrm{ev}_2^{-1}(g.X^\mu) \cap \mathrm{ev}_3^{-1}(X_\nu)) = |\nu| + nd - |\lambda| - |\mu|$. For $|\lambda| + |\mu| \neq |\nu| + nd$ we set $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d = 0$.

Notice that the Gromov-Witten invariants of degree zero are the Schubert structure constants of X: $C^\nu_{\lambda,\mu} = \#(X^\lambda \cap g.X^\mu \cap X_\nu) = \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_0$.

4.2. **Small quantum ring.** The (small) quantum cohomology ring of X is an algebra over the polynomial ring $\mathbb{Z}[q]$. The variable q is called the deformation parameter. As a $\mathbb{Z}[q]$-module, the quantum ring is defined by $\mathrm{QH}(X) = H^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The multiplicative structure is defined by

$$[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} \langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d \, q^d \, [X^\nu]$$

where the sum is over all Young diagrams ν and degrees $d \geq 0$.

Theorem 4.2 (Ruan–Tian, Kontsevich–Manin)**.** *The small quantum cohomology ring* $\mathrm{QH}(X)$ *is associative.*

Notice also that $\mathrm{QH}(X)$ is a graded ring if we set $\deg(q) = 2n$. In addition $\mathrm{QH}(X)$ is a deformation of the ordinary cohomology ring $H^*(X)$, which is recovered by setting $q = 0$.

4.3. **Kernel and Span.** Given a curve $C \subset X$, define $\mathrm{Ker}(C)$ to be the intersection of the m-dimensional subspaces of $V = \mathbb{C}^n$ given by its points, and let $\mathrm{Span}(C)$ be the linear span of these subspaces.

$$\mathrm{Ker}(C) = \bigcap_{\Sigma \in C} \Sigma \quad \text{and} \quad \mathrm{Span}(C) = \sum_{\Sigma \in C} \Sigma$$

Exercise 4.3. Every vector bundle on $\mathbb{P}^1$ of finite rank is a direct sum of line bundles. Hint: Any vector bundle is trivial when restricted to $D_+(x_0)$ and $D_+(x_1)$. Let $A \in \mathrm{GL}(k[t,t^{-1}])$ be the change of basis matrix on the overlap. Show that there exist $P \in \mathrm{GL}(k[t])$ and $Q \in \mathrm{GL}(k[t^{-1}])$ for which PAQ is a diagonal matrix.

Exercise 4.4. Let $C \subset X = \mathrm{Gr}(m,n)$ be a rational curve of degree d. Then $\dim \mathrm{Ker}(C) \geq m - d$ and $\dim \mathrm{Span}(C) \leq m + d$. Hint: Use Exercise 1.15 and Exercise 4.3.

Given a Young diagram λ for $X = \mathrm{Gr}(m,n)$ we have

$$X^\lambda = \{\Sigma \in X \mid \dim(\Sigma \cap F^{m+1-i+\lambda_i}) \geq i \ \forall 1 \leq i \leq m\} .$$

Let $\overline{\lambda}$ denote the result of removing the first d columns of λ, i.e. we have $\overline{\lambda}_i = \max(\lambda_i - d, 0)$ for each i. We consider $\overline{\lambda}$ as a Young diagram for the Grassmannian $Z = \mathrm{Gr}(m+d,n)$. Set $k = n - m$.

Lemma 4.5. *Let $C \subset X$ be a rational curve of degree $d \leq k$, and let $S \in Z$ be such that* $\mathrm{Span}(C) \subset S$. *If C meets X^λ, then $S \in Z^{\overline{\lambda}}$.*

Proof. Let $\Sigma \in C \cap X^\lambda$. Since $\Sigma \subset S$, the Schubert conditions on Σ imply that $\dim(S \cap F^{m+1-i+\lambda_i}) \geq i$ for $i \in [1,m]$. It follows that $S \in Z^{\overline{\lambda}}$. □

4.4. **Quantum Pieri formula.** Given a Young diagram λ, let $\widehat{\lambda}$ denote the result of removing the top row and the leftmost column of λ. In other words, we have $\lambda_i = \max(\lambda_{i+1} - 1, 0)$ for each i. The structure of the quantum cohomology ring $\mathrm{QH}(X)$ is determined by the following result.

Theorem 4.6 (Bertram)**.** *Let λ be a Young diagram for $X = \mathrm{Gr}(m, n)$ and let $p \in [1, n-m]$. Then*

$$[X^p] \star [X^\lambda] \;=\; \sum [X^\mu] + q \sum [X^\nu]$$

where the first sum is over all Young diagrams μ for which μ/λ is a horizontal strip of p boxes, and the second sum is over all Young diagrams ν containing $\widehat{\lambda}$ such that $\nu/\widehat{\lambda}$ is a horizontal strip, $\nu_1 < \lambda_1$, and $|\nu| = |\lambda| + p - n$.

Notice that a Young diagram μ occurs in the first sum if and only if $|\mu| = |\lambda| + p$ and $k \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_m \geq \lambda_m$. A Young diagram ν occurs in the second sum if and only if $|\nu| = |\lambda| + p - n$ and $\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \nu_2 \geq \cdots \geq \lambda_m - 1 \geq \nu_m \geq 0$.

Proof. The first sum is dictated by the classical Pieri formula (Theorem 3.9). This classical case is equivalent to the following statement:

If α and β are Young diagrams such that $|\alpha| + |\beta| + p = mk$, then

$$\int_X [X^p] \cdot [X^\alpha] \cdot [X^\beta] \;=\; \begin{cases} 1 & \text{if } \alpha_i + \beta_j \geq k \text{ for } i + j = m \\ & \text{and } \alpha_i + \beta_j \leq k \text{ for } i + j = m+1; \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose $|\alpha| + |\beta| + p = mk + nd$ for some $d \geq 1$ and let C be a rational curve of degree d in X which meets each of the varieties $g.X^p$, X^α, and $w_0.X^\beta$, where $g \in \mathrm{GL}(n)$ is a generic element. Notice that $dn \leq mk + k$, so we must have $k \leq m$. Let $S \in Z = \mathrm{Gr}(m + d, n)$ be a be such that $\mathrm{Span}(C) \subset S$. Lemma 4.5 then shows that $S \in g.Z^{\overline{p}} \cap Z^{\overline{\alpha}} \cap w_0.Z^{\overline{\beta}}$, which implies $\overline{p} + |\overline{\alpha}| + |\overline{\beta}| \leq \dim(Z) = (m+d)(k-d)$. Since we also have

$$\overline{p} + |\overline{\alpha}| + |\overline{\beta}| \geq p - d + |\alpha| - md + |\beta| - md = (m + d)(k - d) + d^2 - d\,,$$

we deduce that $d = 1$, and both α and β have m rows. The quantum Pieri formula is therefore equivalent to the following statement:

If $|\alpha| + |\beta| + p = mk + n$, then

$$\langle [X^p], [X^\alpha], [X^\beta] \rangle_1 \;=\; \begin{cases} 1 & \text{if } \alpha_i + \beta_j \geq k + 1 \text{ for } i + j = m + 1 \\ & \text{and } \alpha_i + \beta_j \leq k+1 \text{ for } i + j = m+2; \\ 0 & \text{otherwise.} \end{cases}$$

In other words the quantum Pieri formula is equivalent to the identity

$$\langle [X^p], [X^\alpha], [X^\beta] \rangle_1 = \int_Z [Z^{p-1}] \cdot [Z^{\overline{\alpha}}] \cdot [Z^{\overline{\beta}}]\,,$$

where $Z = \mathrm{Gr}(m + 1, n)$.

If $\int_Z [Z^{p-1}] \cdot [Z^{\overline{\alpha}}] \cdot [Z^{\overline{\beta}}] = 0$, then the point $S \in Z$ does not exist, hence C does not exist, so $\langle [X^p], [X^\alpha], [X^\beta] \rangle_1 = 0$. Assume that $\int_Z [Z^{p-1}] \cdot [Z^{\overline{\alpha}}] \cdot [Z^{\overline{\beta}}] = 1$. Then there exists a unique point $S \in g.Z^{p-1} \cap Z^{\overline{\alpha}} \cap w_0.Z^{\overline{\beta}}$. Since g is generic, it follows that S belongs to the intersection of Schubert cells $g.\mathring{Z}^{p-1} \cap \mathring{Z}^{\overline{\alpha}} \cap w_0.\mathring{Z}^{\overline{\beta}}$. This implies that both $\Sigma_1 = S \cap F^{1+\alpha_m}$ and $\Sigma_2 = S \cap w_0.F^{1+\beta_m}$ have dimension

m. Notice that $\Sigma_1 \in X^\alpha$ and $\Sigma_2 \in w_0.X^\beta$. Since $X^\alpha \cap w_0.X^\beta = \emptyset$, it follows that $\Sigma_1 \neq \Sigma_2$. Therefore $K = \Sigma_1 \cap \Sigma_2$ has dimension $m-1$. Finally, the unique rational curve of degree 1 in X that meets $g.X^p$, X^α, and $w_0.X^\beta$ is given by $C = \{\Sigma \in X \mid K \subset \Sigma \subset S\}$. $\square$

Exercise 4.7 (Witten)**.** With the notation of Exercise 3.15 we have an isomorphism $\mathbb{Z}[\sigma_1, \ldots, \sigma_{n-m}, q]/(\Delta_{m+1}, \ldots, \Delta_{n-1}, \Delta_n - (-1)^m q) \xrightarrow{\sim} \mathrm{QH}(X)$ given by $\sigma_p \mapsto [X^p]$. The determinants Δ_p must be expanded using the ring structure of $\mathrm{QH}(X)$.

Exercise 4.8 (Bertram)**.** The class $[X^\lambda] \in \mathrm{QH}(X)$ is the image of the determinant $\det(\sigma_{\lambda_i+j-i})_{m\times m}$.

4.5. **Two-step flag varieties.** Fix a degree d and define the two-step flag variety

$$Y_d = \mathrm{Fl}(m-d, m+d; n) = \{(A,B) \mid A \subset B \subset V,\ \dim(A) = m-d,\ \dim(B) = m+d\}\,.$$

This is the variety of Kernel-Span pairs of general rational curves in X of degree d. Given a subvariety $\Omega \subset X$, define

$$\widetilde{\Omega} = \{(A,B) \in Y_d \mid \exists\, \Sigma \in \Omega : A \subset \Sigma \subset B\}\,.$$

This is the variety of Kernel-Span pairs of general rational curves in X that meet Ω. If Ω is a Schubert variety of X, then $\widetilde{\Omega}$ is a Schubert variety of Y. The following was proved in [1].

Theorem 4.9. *Assume that $|\lambda|+|\mu| = |\nu|+nd$. The map $C \mapsto (\mathrm{Ker}(C), \mathrm{Span}(C))$ gives an explicit bijection between the set of curves counted by the Gromov-Witten invariant $\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d$ and the points in the intersection $\widetilde{X^\lambda} \cap g.\widetilde{X^\mu} \cap \widetilde{X_\nu}$.*

Corollary 4.10. *The (3 point, genus 0) Gromov-Witten invariants of X are determined by*

$$\langle [X^\lambda], [X^\mu], [X_\nu] \rangle_d = \int_{Y_d} [\widetilde{X^\lambda}] \cdot [\widetilde{X^\mu}] \cdot [\widetilde{X_\nu}]\,.$$

References

[1] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915. MR 1992829

[2] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. MR 1464693 (99f:05119)

[3] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297. MR 0360616 (50 #13063)

[4] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Phil. Trans. R. Soc., A **233** (1934), 99–141.