Knutson-Vakil puzzles compute equivariant $K$-theory of Grassmannians

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Consider the Grassmannian $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$. Their $B$-orbit closures are the Schubert varieties, and the Schubert classes $\sigma_\lambda$ give a $\mathbb{Z}$-basis of $H^\ast(X)$. The product $\sigma_\lambda \cdot \sigma_\mu = \sum \sigma_\nu c^{\nu}_{\lambda,\mu}$.
Consider the Grassmannian $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$.

$$B = \begin{bmatrix}
\star \\
\star & \star \\
\star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star & \star
\end{bmatrix} \simeq X$$
Setup

- Consider the **Grassmannian** $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$

$$B = \begin{bmatrix} \star \\ \star & \star \\ \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix} \curvearrowright X \quad \text{and} \quad T = \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix} \curvearrowright X$$
Consider the Grassmannian $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$.

$B = \begin{bmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{bmatrix} \sim X \quad T = \begin{bmatrix} \ast & \ast \\ \ast & \ast \end{bmatrix} \sim X$

$T$-fixed points $\leftrightarrow$ partitions in $k \times (n - k)$
Consider the Grassmannian $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$.

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- $T$-fixed points $\iff$ partitions in $k \times (n - k)$

- Their $B$-orbit closures are the Schubert varieties $X_\lambda$
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 \star & \star & & \\
 \star & \star & \star & \\
 \star & \star & \star & \star \end{bmatrix} \cdot X$
- $T = \begin{bmatrix} \star & & \\
 & \star & \star \\
 & & \star \\
 & & & \star \end{bmatrix} \cdot X$

$T$-fixed points $\leftrightarrow$ partitions in $k \times (n-k)$

Their $B$-orbit closures are the Schubert varieties $X_\lambda$

Schubert classes $\sigma_\lambda$ give a $\mathbb{Z}$-basis of $H^*(X)$

$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} \sigma_\nu$
Consider the **Grassmannian** $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$.

- **B** = \[
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix}
\] $\rtimes X$ \hspace{1cm} **T** = \[
\begin{bmatrix}
\ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\] $\rtimes X$

- **T**-fixed points $\iff$ partitions in $k \times (n - k)$

Their **B**-orbit closures are the Schubert varieties $X_\lambda$.

**Schubert classes** $\sigma_\lambda$ give a $\mathbb{Z}$-basis of $H^*(X)$.

$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda, \mu} \sigma_\nu$ \hspace{1cm} (**Littlewood-Richardson coefficients**)
Setup

- Consider the **Grassmannian** $X = \text{Gr}_k(\mathbb{C}^n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$.

$$B = \begin{bmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{bmatrix} \bowtie X \quad T = \begin{bmatrix} \ast & \ast \end{bmatrix} \bowtie X$$

- $T$-fixed points $\iff$ partitions in $k \times (n - k)$

- Their $B$-orbit closures are the Schubert varieties $X_\lambda$

- Schubert classes $\sigma_\lambda$ give a $\mathbb{Z}$-basis of $H^*(X)$

$$c_{\lambda, \mu}^\nu \in \mathbb{Z}_{\geq 0}$$
Partitions inside $k \times (n - k) \leftrightarrow$ binary strings of length $n$ with $k$ 1’s
Cohomological puzzles

- Partitions inside $k \times (n - k) \leftrightarrow$ binary strings of length $n$ with $k$ 1’s

Let $\Delta_{\lambda,\mu,\nu}$ be an equilateral triangle of side length $n$ with the boundary labeled by

- $\lambda$ as read $\uparrow$ along the left side;
- $\mu$ as read $\downarrow$ along the right side; and
- $\nu$ as read $\rightarrow$ along the bottom side.
Cohomological puzzles

- Partitions inside $k \times (n - k) \leftrightarrow$ binary strings of length $n$ with $k$ 1's

- Let $\Delta_{\lambda,\mu,\nu}$ be an equilateral triangle of side length $n$ with the boundary labeled by
  - $\lambda$ as read $\uparrow$ along the left side;
  - $\mu$ as read $\downarrow$ along the right side; and
  - $\nu$ as read $\rightarrow$ along the bottom side.

Theorem (A. Knutson–T. Tao 1999)

$c_{\lambda,\mu,\nu}$ counts tilings of $\Delta_{\lambda,\mu,\nu}$ by the following puzzle pieces:
Example puzzle calculation

\[ c_{\square, \square} = 2 \] is calculated by the tilings:
Puzzles in richer cohomology theories

- In $K$-theory, structure coefficients are computed by puzzles with an extra (non-rotatable) piece due to A. Buch:

```
0 1
1 0
0 1
```

It has weight $-1$.

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Puzzles and equivariant $K$-theory
In $K$-theory, structure coefficients are computed by puzzles with an extra (non-rotatable) piece due to A. Buch:

```
0 1
1 0
0 1
```

It has weight $-1$.

In $T$-equivariant cohomology, structure coefficients are computed by puzzles with an extra (non-rotatable) piece due to A. Knutson–T. Tao:

```
0 1
1 0
1 0
```

It has weight $t_i - t_j$, where $i, j$ depend on the location.
The Knutson-Vakil conjecture

- The equivariant green rhombus now has weight $1 - \frac{t_i}{t_j}$.
- The purple and yellow gashed triangles have weight $-1$.
The Knutson-Vakil conjecture

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- The purple and yellow gashed triangles have weight $-1$.
- The yellow gashed triangle may only appear with attached to its left, as
The Knutson-Vakil conjecture

- The equivariant green rhombus now has weight $1 - \frac{t_i}{t_j}$
- The purple and yellow gashed triangles have weight $-1$
- The yellow gashed triangle may only appear with $\blacklozenge$ attached to its left, as $\blacklozenge$
- There is a ‘non-local’ requirement for using $\blacktriangledown$:

  “It may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0’s followed by a 1”
The Knutson-Vakil conjecture

- The equivariant green rhombus now has weight $1 - \frac{t_i}{t_j}$.
- The purple and yellow gashed triangles have weight $-1$.
- The yellow gashed triangle may only appear with $\updownarrow$ attached to its left, as $\updownarrow$.
- There is a ‘non-local’ requirement for using $\updownarrow$: 

  “It may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0’s followed by a 1”


The $T$-equivariant $K$-theory coefficient $c_{\lambda,\mu}^{\nu}$ is the weighted count of all such puzzle fillings of $\Delta_{\lambda,\mu,\nu}$.
Counterexample

For $c_{\mathbb{C}^5}$ for $\text{Gr}_2(\mathbb{C}^5)$, there are six KV-puzzles $P_1, P_2, \ldots, P_6$.

$\text{wt}(P_1) = -1$
$\text{wt}(P_2) = -1$
$\text{wt}(P_3) = (-1)^2(1 - \frac{t_3}{t_4})$

$\text{wt}(P_4) = (-1)^2(1 - \frac{t_2}{t_3})$
$\text{wt}(P_5) = (-1)^2(1 - \frac{t_2}{t_3})$
$\text{wt}(P_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$
Counterexample

For $c_{2,0}$ for $\text{Gr}_2(\mathbb{C}^5)$, there are six KV-puzzles $P_1, P_2, \ldots, P_6$.

However

$c_{2,0} = -(1 - \frac{t_2}{t_4}) = \text{wt}(P_2) + \text{wt}(P_3) + \text{wt}(P_5) + \text{wt}(P_6)$

Knutson-Vakil conjecture is false
Modified KV-puzzles compute $c_{\lambda,\mu}^\nu$

- But the Knutson-Vakil conjecture is almost correct.
- Replace the complicated ‘non-local’ condition on \( \triangledown \) with the condition that \( \triangledown \) only appears in the combination pieces.

\[
\text{Theorem (P.-Yong 2015)}
\]

The $T$-equivariant $K$-theory coefficient $c_{\lambda,\mu}^\nu$ is the weighted count of all modified KV-puzzles with boundary $\Delta_{\lambda,\mu,\nu}$. 

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Puzzles and equivariant $K$-theory
Modified KV-puzzles compute $c_{\lambda,\mu}^\nu$

- But the Knutson-Vakil conjecture is almost correct
- Replace the complicated ‘non-local’ condition on $\nabla$ with the condition that $\nabla$ only appears in the combination pieces

and

Theorem (P.–Yong 2015)

The $\mathbb{T}$-equivariant $K$-theory coefficient $c_{\lambda,\mu}^\nu$ is the weighted count of all modified KV-puzzles with boundary $\Delta_{\lambda,\mu,\nu}$. 
Bijection to genomic tableaux

THANK YOU!

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Bijection to genomic tableaux

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Puzzles and equivariant $K$-theory