Example of the Simplex Algorithm

Consider the linear problem:

Maximize $z = 6x_1 + 5x_2$ subject to

$$\begin{aligned} 5x_1 + 2x_2 + x_3 \qquad\qquad &= 20 \\ -2x_1 + x_2 \qquad + x_4 \qquad &= 1 \\ x_1 + x_2 \qquad\qquad + x_5 &= 5 \end{aligned}$$

Rewrite this problem in tableau form. We can easily do this because there is an obvious BFS.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_3$ | 5 | 2 | 1 | 0 | 0 | 0 | 20 |
| $x_4$ | -2 | 1 | 0 | 1 | 0 | 0 | 1 |
| $x_5$ | 1 | 1 | 0 | 0 | 1 | 0 | 5 |
| | -6 | -5 | 0 | 0 | 0 | 1 | 0 |

The objective row has a negative number (-6) in the column of $x_1$. So we can choose $x_1$ as our entering variables. To find the departing variable, calculate the $\theta$-ratios:

$\theta_1 = \frac{20}{5} = 4$

$\theta_2 = \frac{1}{-2} = -\frac{1}{2}$

$\theta_3 = \frac{5}{1} = 5.$

The smallest positive $\theta$-ratio is $\theta_1 = 4$. So the departing variable is the basic variable of row 1, namely $x_3$. We create a pseudo-pivot in row 1, column 1:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 2/5 | 1/5 | 0 | 0 | 0 | 4 |
| $x_4$ | 0 | 9/5 | 2/5 | 1 | 0 | 0 | 9 |
| $x_5$ | 0 | 3/5 | -1/5 | 0 | 1 | 0 | 1 |
| | 0 | -13/5 | 6/5 | 0 | 0 | 1 | 24 |

This time the objective row has a negative entry in the column of $x_2$, so we use $x_2$ as entering variable. To find the departing variable, compute the $\theta$-ratios:

$\theta_1 = \frac{4}{2/5} = 10$

$\theta_2 = \frac{9}{9/5} = 5$

$\theta_3 = \frac{1}{3/5} = 5/3$

The smallest positive $\theta$-ratio is $\theta_3 = 5/3$, so the departing variable is $x_5$, the basic variable of row 3. We create a pseudo-pivot in row 3, column 2:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 1/3 | 0 | -2/3 | 0 | 10/3 |
| $x_4$ | 0 | 0 | 1 | 1 | -3 | 0 | 6 |
| $x_2$ | 0 | 1 | -1/3 | 0 | 5/3 | 0 | 5/3 |
| | 0 | 0 | 1/3 | 0 | 13/3 | 1 | 85/3 |

This time all entries of the objective row are non-negative, so we have found an optimal solution: $x = (10/3, 5/3, 0, 6, 0)^T$, $z(x) = 85/3$.

**Notice:** Since the '$z$'-column never changes, we can safely drop it. This is done in most examples and exercises in the book.