Linear Algebraic Groups   Jan. 20, 2026

$k = \bar{k}$ alg. closed field.

Def A LAG is a group G that is is also an affine variety, such that multiplication $\mu : G \times G \to G$ and inverse elt. fcn. $i : G \to G$ are morphisms of varieties.

Challenge: $\frac{1}{2}$ of class do research on alg. geo.

$\frac{1}{2}$ of class does not know what affine variety is!

Example: $G = GL_n$

$G = SL_n = \{ g \in GL_n \mid \det(g) = 1 \}$

$G = O(n) = \{ g \in GL_n \mid g^T g = 1 \}$

Fact: Any subgroup $G \subseteq GL_n$ defined by poly. equs. is a LAG. Every LAG is $\cong$ such a subgroup.

Students w/o alg. geo.:

Ok to ignore discussions about AG. aspects.

LAG = subgp of GL_n def. by poly equs.

Accept: AG $\Rightarrow$ "this map is surjective"

AG $\Rightarrow$ "this vector space has $\dim < \infty$".

Alg. Geo. I, Fall 2026.

Example

$A \in GL_n$ any element.

$k = \bar{k} \Rightarrow A = QJQ^{-1}$, J Jordan normal form.

$J = D + N$. $D \in GL_n$ diag. $N \in \mathrm{Mat}(n \times n)$ nilpotent.

$DN = ND$.

$J = J_s J_u$:

$J_s = D$ semisimple part.

$J_u = D^{-1} J$ unipotent part.

Note: $J_u - I = D^{-1}J - I = D^{-1}(J - D) = D^{-1}N$

$J_u - I$ nilpotent $\Leftrightarrow$ J_u unipotent.

$A = A_s A_u$, $A_s = Q J_s Q^{-1}$ ss part.

$A_u = Q J_u Q^{-1}$ unipot. part.

Fact: A_s, A_u are unique.

If $G \subseteq GL_n$ LAG, then $A_s, A_u \in G$.

LAG 2 2026-01-22

Algebraic variety: *separated* SWF with finite open covering by affine varieties.

Irreducible variety X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed $\Rightarrow X = X_1$ or $X = X_2$.

$X = \operatorname{Spec}(R)$ affine: X irred $\Leftrightarrow$ R domain.

Connected variety X:

$X = X_1 \cup X_2$, $X_i \subseteq X$ closed, $X_1 \cap X_2 = \varnothing \Rightarrow X = X_1$ or $X = X_2$.

Product of varieties $X \times Y$:

product in category of alg. varieties.

$$\begin{tikzcd}
 & & X \\
Z \arrow[rru, bend left, "\phi_X"] \arrow[rrd, bend right, "\phi_Y"'] \arrow[r, dashed, "\exists!", "\phi"'] & X \times Y \arrow[ru, "\pi_X"] \arrow[rd, "\pi_Y"] & \\
 & & Y
\end{tikzcd}$$

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ as *sets*!

Example: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ (not product topology!)

$X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$ affine

$\Rightarrow X \times Y = \operatorname{Spec}(R \otimes_k S)$.

$f \otimes g \in R \otimes S$: $(f \otimes g)(x,y) = f(x)g(y)$.

Def SWF X is separated

$\Updownarrow$

$\Delta_X = \{(x,x) \mid x \in X\} \subseteq X \times X$ closed.

Projective space

$$\begin{aligned} \mathbb{P}^n &= \{\text{lines through } 0 \text{ in } \mathbb{A}^{n+1}\} \\ &= \{[x_0 : x_1 : \dots : x_n] \mid (x_0, \dots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}\} \end{aligned}$$

Proj. coord. ring: $k[x_0, \dots, x_n]$.

$f_1, \dots, f_m \in k[x_0, \dots, x_n]$ homogeneous polys:

$Z(f_1, \dots, f_m) \subseteq \mathbb{P}^n$ closed.

$D_+(x_i) = \{[x_0 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n]\} \cong \mathbb{A}^n$.

$\mathbb{P}^n = D_+(x_0) \cup \dots \cup D_+(x_n)$ alg. var.

Dimension:

X variety.

$$\dim(X) = \max\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists\, X_0 \subsetneq X_1 \subsetneq \dots \subsetneq X_d \subseteq X \\ \text{s.t. } X_i \text{ closed \& irreducible} \end{array} \right\}$$

$X = \operatorname{Spec}(R)$ irred. affine variety

$$\Rightarrow \dim(X) = \operatorname{tr.deg.}_k(K(R)).$$

Examples:

- $\dim(\mathbb{A}^n) = \operatorname{tr.deg.}_k k(x_1, \dots, x_n) = n$.
- $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Thm $\phi : X \longrightarrow Y$ morphism of varieties.
Then $\phi(X)$ contains a dense open subset of $\overline{\phi(X)}$.

Fact: $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$.

$$\begin{aligned} \{\text{morphisms } X \longrightarrow Y\} &\longleftrightarrow \{k\text{-alg. hom. } S \longrightarrow R\} \\ \phi &\longmapsto \phi^* \end{aligned}$$

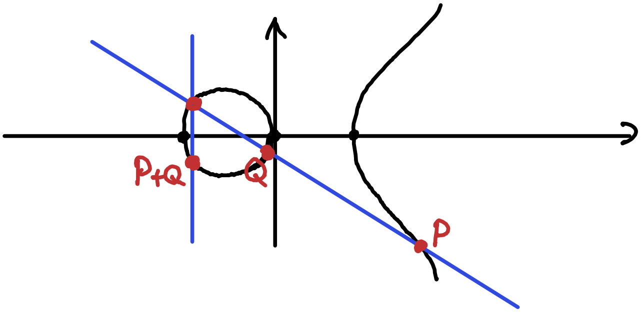
Algebraic Groups

Alg. group: alg. variety G that is also a group:
$\mu: G \times G \longrightarrow G$ and $i: G \to G$ morphisms.

Consequence: Each elt. $x \in G$ defines two automorphisms:
$G \xrightarrow{\cong} G$, $y \mapsto x \cdot y$, $y \mapsto y \cdot x$

Example: Elliptic curve $E = Z(zy^2 - x^3 + xz^2) \subseteq \mathbb{P}^2$

Draw in $\mathbb{A}^2 = \{z=1\}$: $y^2 = x^3 - x$



homomorphism of alg. groups $\phi: G \longrightarrow G'$:
morphism + group hom.

products: $G \times G'$

closed subgroup: $H \subseteq G$

LAG: affine algebraic group.

Hopf algebra

G LAG. $\mu : G \times G \longrightarrow G.$ $i : G \to G.$

$A = k[G] = A(G) = \mathcal{O}_G(G).$

$k[G \times G] = A \otimes_k A.$

Comult: $\Delta = \mu^* : A \longrightarrow A \otimes A, \quad f \mapsto f\mu$

$\Delta f(x,y) = f\mu(x,y) = f(x \cdot y)$

Antipode: $\iota = i^* : A \longrightarrow A, \quad f \mapsto fi$

$\iota f(x) = fi(x) = f(x^{-1})$

Example

$M_n = Mat(n \times n, k).$

$k[M_n] = k[T_{ij}, \ 1 \le i,j \le n].$

$G = GL_n = \{ x \in M_n \mid det(x) \neq 0 \}$

$A = k[G] = k[M_n]_{det} = k[T_{ij}, det^{-1}].$

$\mu : G \times G \longrightarrow G$ mult. $\quad \Delta : A \longrightarrow A \otimes A$

$$\begin{array}{c} (x \cdot y)_{ij} = \sum_k x_{ik} y_{kj}. \\ \shortparallel \\ T_{ij}(\mu(x,y)) = \Delta T_{ij}(x,y) \end{array} \qquad \Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$$

$$\iota : A \longrightarrow A, \quad \iota(T_{ij}) = f_{ij} : \quad \begin{array}{c} (x^{-1})_{ij} = f_{ij}(x). \\ \shortparallel \\ T_{ij}(i(x)) = \iota T_{ij}(x) \end{array}$$

Mult: $m : A\otimes A \longrightarrow A$, $f\otimes g \mapsto fg$.

$m = \delta^*$, $\delta : G \longrightarrow G\times G$, $x \mapsto (x,x)$.

$$\delta^*(f\otimes g)(x) = (f\otimes g)(\delta(x)) = f(x)g(x) = (fg)(x) = m(f\otimes g)(x).$$

Id. elt: $e \in G$. $e : A \longrightarrow k$, $f \mapsto f(e)$.

$\varepsilon : A \xrightarrow{e} k \xrightarrow{\subseteq} A$.

$\varepsilon f(x) = f(e)$.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "id\otimes\Delta"] \\
A\otimes A \arrow[r, "\Delta\otimes id"'] & A\otimes A\otimes A
\end{tikzcd}$$

$\Updownarrow$

$x\cdot(y\cdot z) = (x\cdot y)\cdot z$

$$\begin{tikzcd}
A\otimes A \arrow[r, "\iota\otimes id"] & A\otimes A \arrow[d, "m"] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & A \\
A\otimes A \arrow[r, "id\otimes\iota"'] & A\otimes A \arrow[u, "m"']
\end{tikzcd}$$

$\Updownarrow$

$x^{-1}\cdot x = e = x\cdot x^{-1}$.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & A\otimes A \arrow[d, "id\otimes e"] \\
A\otimes A \arrow[r, "e\otimes id"'] & A
\end{tikzcd}$$

$\Updownarrow$

$e\cdot x = x = x\cdot e$

Basic results

G alg. group.

Lemma: $\exists !$ irred. comp. $G^\circ \subseteq G$ with $e \in G^\circ$.

G° is closed normal subgroup of finite index.

Pf $X, Y \subseteq G$ irred. comps, $e \in X$, $e \in Y$.

$X \times Y$ irred. $\Rightarrow$ $XY = \mu(X \times Y)$ irred.

$\Rightarrow$ $\overline{XY}$ irred.

$X, Y \subseteq \overline{XY}$ $\Rightarrow$ $X = Y = \overline{XY}$.

X closed under mult.

$i^{-1}(X)$ irred comp., $e \in i^{-1}(X)$ $\Rightarrow$ $i^{-1}(X) = X$.

$\therefore$ $G^\circ = X$ closed subgroup.

$[G : G^\circ] = \#$ irred. comps $< \infty$.

□

Cor G irred $\Leftrightarrow$ G connected.

Pf G connected but not irred.

$\Rightarrow \exists x \in G$, x in two irred. comps

$\Rightarrow$ e in two comps. $\quad x : G \xrightarrow{\cong} G$, $y \mapsto xy$

□

Cor $H \subseteq G$ closed subgp. with $[G:H] < \infty$

$\Rightarrow$ $G^\circ \subseteq H$.

Pf $H^\circ \subseteq G^\circ$ closed with $[G^\circ : H^\circ] < \infty$.

$G^\circ = x_1 H^\circ \cup x_2 H^\circ \cup \dots \cup x_\ell H^\circ$. G° irred $\Rightarrow$ $G^\circ = H^\circ$.

□

Lemma $U, V \subseteq G$ dense open subsets $\Rightarrow UV = G$

Proof Let $x \in G$.

$U, \; xV^{-1} \subseteq G$ dense open.

$\Rightarrow U \cap xV^{-1} \neq \emptyset$. $\quad xv^{-1} \in U$ for some $v \in V$.

□

Lemma $H \leq G$ any subgroup.

(1) $\overline{H} \leq G$ is a closed subgroup. (EXER)

(2) If H contains nonempty open subset of $\overline{H}$, then $H = \overline{H}$. $(H \cdot H = \overline{H}.)$

Prop $\phi : G \longrightarrow G'$ hom. of alg. groups.

(1) $\operatorname{Ker}(\phi) \leq G$ closed normal subgp.

(2) $\phi(G) \leq G'$ closed subgp

(3) $\phi(G^\circ) = \phi(G)^\circ$

Pf: (2): $\phi(G)$ contains dense open subset of $\overline{\phi(G)}$.

(3): $[\phi(G) : \phi(G^\circ)] < \infty \Rightarrow \phi(G)^\circ \subseteq \phi(G^\circ)$.

□

Prop $\{\phi_i : X_i \longrightarrow G\}_{i\in I}$ family of morphisms.

Assume X_i irred. and $e \in Y_i = \phi_i(X_i)$ $\forall i \in I$.

$H \leq G$ smallest closed subgroup with $Y_i \subseteq H$ $\forall i$

(1) H is connected.

(2) $H = Y_{a(1)}^{\varepsilon(1)} Y_{a(2)}^{\varepsilon(2)} \cdots Y_{a(n)}^{\varepsilon(n)}$ for some $a(1), \dots, a(n) \in I$, $\varepsilon(1), \dots, \varepsilon(n) \in \{\pm 1\}$.

Eg. $H = Y_1 Y_2^{-1} Y_3 Y_1 Y_2$

Proof WLOG: $\forall i \in I \ \exists j \in I : Y_i^{-1} = Y_j$.

Given $a = (a(1), \dots, a(n)) \in I^n$,

set $Y_a = Y_{a(1)} Y_{a(2)} \cdots Y_{a(n)}$.

Then $\overline{Y_a} \subseteq G$ irred. closed subset.

$Y_b \cdot Y_c = Y_{(b,c)}$.

EXER: $\overline{Y_b} \cdot \overline{Y_c} \subseteq \overline{Y_{(b,c)}}$.

Choose a such that $\dim \overline{Y_a}$ is maximal.

$\forall b$: $\overline{Y_a} \subseteq \overline{Y_a} \cdot \overline{Y_b} \subseteq \overline{Y_{(a,b)}}$.

$\dim \overline{Y_a} = \dim \overline{Y_{(a,b)}}$, both closed irred

$\Rightarrow \overline{Y_a} = \overline{Y_{(a,b)}}$.

$\therefore$ $H = \overline{Y_a} \leq G$ connected closed subgroup.

□

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G alg. group.

G-variety: Variety X with action (morphism) $a : G \times X \to X$.

$a(g,x) = g.x$.

Orbit of x: $G.x$

X **homogeneous** $\Leftrightarrow$ $X = G.x_0$.

G-equivariant morphism: $\phi : X \longrightarrow Y$, $\phi(g.x) = g.\phi(x)$.

Rational representation:

Alg. group hom. $r : G \longrightarrow GL(V)$, $\dim(V) < \infty$.

$G \circlearrowright V$: $g.v = r(g)(v)$.

Lemma X G-variety.

(1) $G.x \subseteq \overline{G.x}$ is open $\forall x \in X$.

(2) $\exists$ closed orbits in X.

Pf (1) $G \longrightarrow X$, $g \mapsto g.x$.

Image $G.x \supseteq$ dense open $U \subseteq \overline{G.x}$.

$G.x = \bigcup_{g \in G} g.U \subseteq \overline{G.x}$ open.

(2) $\overline{G.x} - G.x$ = union of orbits.

Choose $x \in X$ with $\dim G.x$ minimal.

□

__Now:__ G LAG, X affine G-variety.

$a : G \times X \longrightarrow X$

$a^* : k[X] \longrightarrow k[G] \otimes k[X]$. $\quad a^*(f)(g, x) = f(g.x)$.

__Def:__ $s : G \longrightarrow GL(k[X])$ $\qquad$ __rep. of abstract gps.__

$$(s(g)f)(x) = f(g^{-1}.x)$$

__Relation:__ let $f \in k[X]$.

$$a^*(f) = \sum_{i=1}^{n} u_i \otimes f_i \quad \in \quad k[G] \otimes k[X].$$

$$(s(g)f)x = f(g^{-1}.x) = a^*(f)(g^{-1}, x) = \sum u_i(g^{-1}) f_i(x)$$

$$\Rightarrow \quad s(g)f = \sum u_i(g^{-1}) f_i$$

Assume $V \subseteq k[X]$, $\dim(V) < \infty$.

__Lemma__ $\exists\, V \subseteq W \subseteq k[X] : \dim(W) < \infty,\ s(g)(W) \subseteq W \quad \forall g \in G$.

__Pf__ WLOG $V = \operatorname{Span}\{f\}$.

Relation $\Rightarrow$ $s(g)f \in \operatorname{Span}\{f_1, \dots, f_n\}$ $\forall\, g \in G$.

$\Rightarrow \dim\left(W = \operatorname{Span}\{ s(g)f \mid g \in G\}\right) < \infty$.

□

__Lemma__ $V \subseteq k[X]$ $s(G)$-stable $\Leftrightarrow$ $a^*(V) \subseteq k[G] \otimes V$.

$\Rightarrow$ V rational rep. of G, $s : G \times V \longrightarrow V$.

- similar.

Action by translation

$G \circlearrowright G$, $\quad g.x = gx$ (left), $\quad g.x = xg^{-1}$ (right)

$\lambda : G \longrightarrow GL(k[G])$, $\quad (\lambda(g)f)(x) = f(g^{-1}x)$

$\rho : G \longrightarrow GL(k[G])$, $\quad (\rho(g)f)(x) = f(xg)$.

Note: λ and ρ are faithful (= injective):

$\lambda(g)$ determines $G \longrightarrow G$, $x \mapsto g^{-1}x$

determines $g \in G$.

Thm $G \cong$ closed subgroup $\subseteq GL_n$.

Pf Choose $f_1, \dots, f_n \in k[G]$:

- $k[G] = k[f_1, \dots, f_n]$
- $V = \operatorname{span}\{f_1, \dots, f_n\}$ is $\rho(G)$-stable
- $\{f_1, \dots, f_n\}$ lin. indep.

$\exists\ m_{ij} \in k[G]$, $1 \le i,j \le n$ such that

$$\rho(g) f_j = \sum_{i=1}^{n} m_{ij}(g) f_i$$

Check: $a : G \times G \longrightarrow G$, $a(g,x) = xg$.

$a^*(f_j) = \sum_i m_{ij} \otimes f_i$ for some $m_{ij} \in k[G]$.

$(\rho(g) f_j)(x) = f_j(xg) = a^*(f_j)(g,x) = \sum_i m_{ij}(g) f_i(x)$.

$\phi : G \longrightarrow GL_n$, $\phi(g) = (m_{ij}(g))_{i,j}$ alg. gp. hom.

$k[GL_n] = k[T_{ij}, \det^{-1}]$.

$$\phi^* : k[T_{ij}, \det^{-1}] \longrightarrow k[G]$$

$$T_{ij} \longmapsto m_{ij}$$

$$\det^{-1} \longmapsto \det(m_{ij})^{-1}.$$

Surjective:

$$f_j(g) = f_j(eg) = (\rho(g) f_j)(e) = \sum_i m_{ij}(g) f_i(e)$$

$$f_j = \sum_i f_i(e)\, m_{ij} \in \mathrm{Im}(\phi^*).$$

$\therefore$ $k[G] = k[GL_n]/I$, $I = \ker(\phi^*)$

$\Updownarrow$ $G \cong Z(I) \subseteq GL_n$.

□

Jordan decomposition

V vector space, $\dim(V) < \infty$.

$a \in \mathrm{End}_k(V)$.

a **semi-simple**: $\exists$ basis of eigenvectors.

a **nilpotent**: $a^n = 0$ for some $n \geq 0$.

a **unipotent**: $a - 1$ nilpotent.

Note: $\mathrm{char}(k) = p > 0$: a unipotent $\Leftrightarrow a^{p^s} = 1$, $s \in \mathbb{N}$.

$M_n = \mathrm{Mat}(n \times n) = \mathrm{End}(k^n)$.

Lemma $S \subseteq M_n$ set of pairwise commuting matrices.

(1) $\exists\, x \in GL_n$: xSx^{-1} upper Δ.

(2) All elts. of S semi-simple $\Rightarrow xSx^{-1}$ diagonal.

Proof Simultaneous Jordan decomp (almost)! $\square$

Lemma

(1) $a, b \in \mathrm{End}(V)$, $ab = ba$.
a, b both ss/nilpot/unipot $\Rightarrow$ so is ab.

(2) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot/unipot
$\Rightarrow$ so is $a \oplus b \in \mathrm{End}(V \oplus W)$ and $a \otimes b \in \mathrm{End}(V \otimes W)$.

(3) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot
$\Rightarrow$ so is $a \otimes 1 + 1 \otimes b \in \mathrm{End}(V \otimes W)$.

Note: $a \otimes 1 + 1 \otimes b - 2(1 \otimes 1)$ is nilpotent!

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Prop $a \in \mathrm{End}(V)$, $\dim(V) < \infty$.

(1) $\exists!\ a_s, a_n \in \mathrm{End}(V)$: a_s ss, a_n nilpot, $a_s a_n = a_n a_s$, $a = a_s + a_n$.

(2) $a_s, a_n \in k[a]$ (polys. in a).

(3) $W \subseteq V$, $a(W) \subseteq W \Rightarrow a_s(W) \subseteq W$ and $a_n(W) \subseteq W$.

$(a|_W)_s = a_s|_W$ and $(a|_W)_n = a_n|_W$.

$\bar{a} : V/W \longrightarrow V/W$. $(\bar{a})_s = \overline{a_s}$ and $(\bar{a})_n = \overline{a_n}$.

(4) $\phi : V \longrightarrow W$ linear, $b \in \mathrm{End}(W)$.

$\phi a = b \phi \Rightarrow \phi a_s = b_s \phi$ and $\phi a_n = b_n \phi$.

Proof of (4):

$$\begin{tikzcd} \overset{a}{V} \arrow[r] & \overset{a \oplus b}{V \oplus W} \arrow[r] & \overset{b}{W} \end{tikzcd}$$

$v \mapsto (v, \phi(v))$

Cor $a \in GL(V)$, $\dim(V) < \infty$.

$\exists!\ a_s, a_u \in GL(V)$: a_s ss, a_u unipotent.

$a = a_s a_u = a_u a_s$

Proof $a = a_s + a_n = a_s(1 + a_s^{-1} a_n)$.

Locally finite

V any vector space, $a \in \mathrm{End}(V)$.

a is locally finite: $\forall v \in V \ \exists v \in W \subseteq V : a(W) \subseteq W, \dim(W) < \infty$.

Assume a locally finite.

a semi-simple: $a|_W$ simisimple $\forall W \subseteq V$, $a(W) \subseteq W$, $\dim < \infty$

a locally nilpotent: $a|_W$ nilpot. —"—

a locally unipotent: $a|_W$ unipot. —"—

Example $a = J_1(0) \oplus J_2(0) \oplus J_3(0) \oplus \cdots$ locally nilpot., not nilpot.

Cor $a \in \mathrm{End}(V)$ loc. finite $\Rightarrow$

$\exists!\ a_s, a_n \in \mathrm{End}(V)$ loc. finite: a_s ss, a_u loc. nilpot.

$$a = a_s + a_n, \quad a_s a_n = a_n a_s.$$

$$a_s|_W = (a|_W)_s, \quad a_u|_W = (a|_W)_u.$$

Cor $a \in GL(V)$ loc. finite

$\exists!\ a_s, a_u \in GL(V)$ loc. finite: a_s ss, a_u loc. unipot.

$$a = a_s a_u = a_u a_s.$$

G LAG.

Recall: $\rho(g): k[G] \longrightarrow k[G]$ locally finite $\forall g \in G$.

Lemma $G = GL(V)$, $\dim(V) < \infty$.

$g \in G$ is ss/unipot $\Leftrightarrow$ $\rho(g)$ ss/loc. unipot.

Proof

For $f \in V^*$, def. $\tilde{f}: V \longrightarrow k[G]$, $\tilde{f}(v)(g) = f(gv)$

$\tilde{f}(gv) = \rho(g)\tilde{f}(v)$:

$$\tilde{f}(gv)(h) = f(hgv) = \tilde{f}(v)(hg) = \big(\rho(g)\tilde{f}(v)\big)(h).$$

$$\rho(g)_s \tilde{f}(v) \underset{\substack{\uparrow \\ \text{Prop (4)}}}{=} \tilde{f}(g_s v) = \rho(g_s)\tilde{f}(v).$$

$k[G]$ gen. by $\{\tilde{f}(v) \mid f \in V^*, v \in V\}$

$\square \quad \Rightarrow \rho(g)_s = \rho(g_s)$.

Def $g \in G$ is semi-simple if $\rho(g)$ semi-simple.

$g \in G$ is unipotent if $\rho(g)$ loc. unipot.

Thm G LAG, $g \in G$.

(1) $\exists!\, g_s, g_u \in G$: g_s ss, g_u unipot., $g = g_s g_u = g_u g_s$.

(2) $\phi: G \longrightarrow G'$ alg. group hom.

$\Rightarrow \phi(g)_s = \phi(g_s)$ and $\phi(g)_u = \phi(g_u)$.

(3) $G = GL_n \Rightarrow g_s, g_u$ are as above.

Recall: G simple: G solvable $\Leftrightarrow$ nilpotent $\Leftrightarrow$ abelian.

G nilpotent $\Leftrightarrow$ $G/Z(G)$ nilpotent.

$H \triangleleft G$ normal: G solvable $\Leftrightarrow$ G/H, H solvable.

G abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable.

Exer

$$U_n = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \subseteq B_n = \begin{bmatrix} * & & * \\ & * & \\ 0 & & \ddots \; * \end{bmatrix} \subseteq GL_n$$

U_n nilpotent

B_n solvable, not nilpotent.

Burnside's Thm

V vector space over $k = \bar{k}$, $\dim(V) < \infty$.

$R \subseteq \mathrm{End}_k(V)$ subalgebra.

V simple R-module $\Rightarrow R = \mathrm{End}_k(V)$.

Cor $G \subseteq GL(V)$ closed subgroup, $\dim(V) < \infty$.

If V irred. G-rep. then $\mathrm{End}_k(V) = \mathrm{span}(G)$.

Cor $G \subseteq GL_n$ consists of unipotent elts.

$\Rightarrow xGx^{-1} \subseteq U_n$ for some $x \in GL_n$.

Proof Enough: $xGx^{-1} \subseteq B_n$.

$V = k^n$ not irred: $\exists\, 0 \neq W \subsetneq V$, W G-stable.

Images of G in $GL(W)$, $GL(V/W)$ consist of unipot. elts.

Use induction on n.

WLOG V irreducible.

For $g, h \in G$: $\mathrm{Tr}((1-g)h) = \mathrm{Tr}(h) - \mathrm{Tr}(gh) = n - n = 0$.

Burnside $\Rightarrow$ $\mathrm{Tr}((1-g)h) = 0 \;\; \forall\, h \in \mathrm{End}(V) \Rightarrow g = e$.

□

Def G is <u>unipotent</u> if all elts of G are unipot.

Cor unipotent $\Rightarrow$ nilpotent.

Prop G unipotent, X affine G-variety.
$\Rightarrow$ $G.x \subseteq X$ is closed $\forall x \in X$.

Proof

$O \subseteq X$ orbit.

WLOG: $X = \overline{O}$. $\Rightarrow O \subseteq X$ is open.

$Y = X \setminus O \subseteq X$ closed, G-stable.

$s : G \longrightarrow GL(k[X])$, $\quad (s(g)f)(x) = f(g^{-1}.x)$

G acts locally finitely on $k[X]$.

$G.I(Y) \subseteq I(Y)$.

$\exists\, 0 \neq f \in I(Y) : \; s(g)f = f \;\; \forall g \in G :$

$\exists\, 0 \neq W \subseteq I(Y)$, $\dim(W) < \infty$, $G.W \subseteq W$.

G unipotent $\Rightarrow$ $s(G) \subseteq GL(W)$ unipotent

WLOG: $s(G) \subseteq U_m \subseteq GL_m = GL(W)$.

$\Rightarrow$ f constant on O

$\Rightarrow$ f constant on X

$I(Y) \supseteq \langle f \rangle = k[X] \Rightarrow Y = \emptyset$.

$\square$

Def G LAG.

$$\left.\begin{aligned} G_s &= \{ g \in G \mid g \text{ is semi-simple} \} \\ G_u &= \{ g \in G \mid g \text{ is unipotent} \} \end{aligned}\right\} \text{ subsets, usually not subgroups.}$$

Note: $G_u \subseteq G$ is closed.

$$(GL_n)_u = \{ g \in GL_n \mid \chi_g(t) = (t-1)^n \}$$

Thm G commutative LAG.

(1) G_s and G_u are closed subgroups.

(2) $\mu : G_s \times G_u \xrightarrow{\cong} G$ (product map).

Proof

$(gh)_s = g_s h_s = gh \Rightarrow G_s$ subgroup.

WLOG: $G \subseteq B_n \subseteq GL_n$.

$G_s = G \cap D_n$, $G_u = G \cap U_n$ closed. $D_n = \begin{bmatrix} * & & 0 \\ & * & \\ 0 & & * \end{bmatrix}$

$\mu : G_s \times G_u \longrightarrow G$, $(g, h) \longmapsto gh$ is bijective

(unique Jordan decomp.)

$\mu^{-1}(g) = (g_s, g_s^{-1} g)$ is morphism of varieties.

□

Cor G commutative & connected $\Rightarrow$ so are G_s, G_u.

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Prop G connected LAG, $\dim(G) = 1$.

(1) G is commutative.

(2) $G = G_s$ or $G = G_u$.

(3) $G = G_u$ and $\mathrm{char}(k) = p > 0 \Rightarrow g^p = e \quad \forall g \in G$.

Pf

(1) Assume G not commutative.

$(G,G) \neq e$ connected closed subgroup $\Rightarrow (G,G) = G$.

Let $g \in G \setminus Z(G)$.

$$G = \overline{\{xgx^{-1} \mid x \in G\}}$$

$G \subseteq GL_n$: $\chi_g(t)$ constant for $g \in G$.

$\Rightarrow \quad \chi_g(t) = (t-1)^n \quad \forall g \in G$

$\Rightarrow$ G is unipotent $\Rightarrow$ solvable $\Rightarrow (G,G) \neq G$ ↯

(2) clear.

(3) $G^{(p^h)} = \langle g^{p^h} \mid g \in G \rangle \subseteq G$ connected closed subgp.

$G \subseteq U_n$: $g^{p^h} = e$ for $h \geq n$.

Must have $G^{(p)} \neq G \Rightarrow G^{(p)} = e$.

□

Characters & cocharacters

$\mathbb{G}_m = k^\times$ mult. group.

Character: $\chi : G \longrightarrow \mathbb{G}_m$ alg. gp. hom.

$X^*(G) = \{\chi : G \longrightarrow \mathbb{G}_m\} \subseteq k[G]^\times$ (abelian) subgroup.

Dedekind: $X^*(G) \subseteq k[G]$ lin. independent.

Pf

Equation with n minimal: $\sum_{i=1}^{n} \lambda_i \chi_i(g) = 0.$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i \chi_n(h) \chi_i(g) = 0$$

$$\sum_{i=1}^{n} \lambda_i \chi_i(h) \chi_i(g) = 0$$

$$\Rightarrow \sum_{i=1}^{n-1} \lambda_i (\chi_n(h) - \chi_i(h)) \chi_i(g) = 0$$

Choose $h \in G$ with $\chi_n(h) \neq \chi_1(h)$. ↯

□

Cocharacter: $\lambda : \mathbb{G}_m \longrightarrow G$ alg. group hom.

$X_*(G) = \{\lambda : \mathbb{G}_m \longrightarrow G\}$ set of cocharacters.

G commutative $\Rightarrow X_*(G)$ (abelian) group.

For $n \in \mathbb{Z}$: $(n.\lambda)(a) = \lambda(a)^n$. Prop. (1) $\Rightarrow n.\lambda \in X_*(G)$

$-\lambda = (-1).\lambda$.

Example: $X^*(\mathbb{G}_m) = X_*(\mathbb{G}_m) = \mathbb{Z}$.

$\chi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ alg. group hom.

$\chi(a) = a^n$ for some $n \in \mathbb{Z}$.

Note: $k[\mathbb{G}_m] = k[t, t^{-1}]$ has basis $\{t^n : n \in \mathbb{Z}\}$.

$D_n = (\mathbb{G}_m)^n$. $X^*(D_n) \cong \mathbb{Z}^n \cong X_*(D_n)$.

$k[D_n] = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has basis $X^*(D_n)$.

<u>Def</u> G is <u>diagonalizable</u> $\Leftrightarrow$ $G \subseteq D_n$ closed.

G is a <u>torus</u> $\Leftrightarrow$ $G \cong D_n$

<u>Thm</u> G LAG. TFAE:

(1) G is diagonalizable.

(2) $X^*(G)$ is a basis of $k[G]$.

(3) $G \circlearrowright V$ rational rep. $\Rightarrow$ V direct sum of 1-dim. reps.

<u>Proof</u>

(1) $\Rightarrow$ (2): $G \subseteq D_n$. $k[D_n] \twoheadrightarrow k[G]$.

$k[G]$ spanned by image of $X^*(D_n)$.

$\therefore$ $k[G] = \mathrm{Span}(X^*(G))$.

(2) $\Rightarrow$ (3): $\phi : G \longrightarrow GL(V)$ rat. rep.

$\exists !\, A_\chi \in \mathrm{End}(V)$ for $\chi \in X^*(G)$:

$$\phi(g) = \sum_\chi \chi(g) A_\chi$$

$$\Big(\phi : G \longrightarrow GL(V) \subseteq \mathrm{End}(V) = M_n$$
$$\phi(g) = (\phi_{ij}(g)) \in M_n,\ \phi_{ij} \in k[G] = \mathrm{Span}\, X^*(G). \Big)$$

<u>Note</u>: $A_\chi \neq 0$ for finitely many χ.

$$1_V = \phi(e) = \sum_{\chi} A_\chi$$

$$g, h \in G: \sum_{\chi} \chi(g)\chi(h) A_\chi = \phi(gh) = \phi(g)\phi(h)$$

$$= \sum_{\chi,\psi} \chi(g)\psi(h) A_\chi A_\psi$$

$X^*(G \times G)$ linearly indep. $\Rightarrow A_\chi A_\psi = \delta_{\chi,\psi} A_\chi$

$\therefore V = \bigoplus_{\chi} A_\chi(V)$.

Note: $\phi(g).v = \chi(g)v$ for $v \in A_\chi(V)$.

(3) $\Rightarrow$ (1): $G \subseteq GL(V) = GL_n$ closed. Clear.

□

Cor Assume G is diagonalizable.

(1) $X^*(G)$ f.g. abelian group.

(2) $k[G] = k[X^*(G)]$ group algebra.

(3) $\mathrm{char}(k) = p > 0 \Rightarrow X^*(G)$ has no p-torsion.

Pf

(1) $G \subseteq D_n$ closed $\Rightarrow \mathbb{Z}^n = X^*(D_n) \twoheadrightarrow X^*(G)$.

(3) $\chi^p = 1 \Rightarrow \chi(g)^p = 1 \in k \ \forall g \Rightarrow \chi = 1$.

□

Diagonalizable LAG $\longleftrightarrow$ f.g. abelian group

M f.g. abelian group.

$k[M]$ = k-vector space with basis $\{e(m) : m \in M\}$.

$$e(m)\,e(n) = e(m+n).$$

Assume M has no p-torsion.

$\Leftrightarrow$ $k[M]$ reduced f.g. k-alg.

$\mathcal{G}(M) = \operatorname{Spec}(k[M])$ affine variety.

$$\Delta : k[M] \longrightarrow k[M] \otimes k[M], \quad \Delta(e(m)) = e(m) \otimes e(m).$$

$$\iota : k[M] \longrightarrow k[M] \qquad \iota(e(m)) = e(-m).$$

$$\varepsilon : k[M] \longrightarrow k \qquad \varepsilon(e(m)) = 1.$$

Prop

(1) $\mathcal{G}(M)$ is diagonalizable LAG.

(2) $X^*(\mathcal{G}(M)) = M$

(3) G diagonalizable LAG $\Rightarrow$ $\mathcal{G}(X^*(G)) = G$.

Note: M_1, M_2 f.g. abelian groups.

$$k[M_1 \oplus M_2] = k[M_1] \otimes_k k[M_2]$$

$$\mathcal{G}(M_1 \oplus M_2) = \mathcal{G}(M_1) \times \mathcal{G}(M_2).$$

Exer: M finite $\Rightarrow$ $\mathcal{G}(M) \cong M$.

Cor G diagonalizable LAG.

(1) $G \cong D_n \times F$, F finite abelian w/o p-torsion.

(2) G torus $\Leftrightarrow$ G connected $\Leftrightarrow$ $X^*(G)$ free abelian.

Prop (Rigidity)

G, H diagonalizable LAGs. V connected affine var.

$\phi : V \times G \longrightarrow H$ morphism.

Assume $g \longmapsto \phi(v,g)$ is alg. gp. hom. $\forall v \in V$.

Then $\phi(v,g)$ is independent of v.

Proof

Let $\psi \in X^*(H) \subseteq k[H]$.

$$\phi^*(\psi) = \sum_{\chi \in X^*(G)} f_{\chi,\psi} \otimes \chi \in k[V] \otimes k[G].$$

$$\psi(\phi(v,g)) = \sum_{\chi} f_{\chi,\psi}(v)\, \chi(g)$$

$v \in V$ fixed: LHS $\in X^*(G)$.

$$\Rightarrow \quad f_{\chi,\psi}(v) = \begin{cases} 1 & \text{if } \chi = \text{LHS} \\ 0 & \text{else.} \end{cases}$$

V connected $\Rightarrow$ $f_{\chi,\psi}$ constant.

□

G alg. group, $H \subseteq G$ closed subgroup.

$Z_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}$

$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$

Exer $G = GL_n$

$Z_G(D_n) = D_n$

$N_G(D_n) = S_n D_n$, $S_n \subseteq G$ perm. matrices.

$N_G(D_n)/Z_G(D_n) = S_n$ Weyl group of GL_n.

Cor G LAG, $H \subseteq G$ diagonalizable closed subgroup.
Then $N_G(H)^\circ = Z_G(H)^\circ$ and $N_G(H)/Z_G(H)$ is finite.

Proof

The morphism

$$N_G(H)^\circ \times H \longrightarrow H, \quad (g, h) \longmapsto ghg^{-1}$$

is independent of g.

$\Rightarrow ghg^{-1} = h \quad \forall \ g \in N_G(H), \ h \in H$

$\Rightarrow N_G(H)^\circ \subseteq Z_G(H).$

□

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T torus.

$X^*(T) = \{\chi : T \to \mathbb{G}_m\}$ group of characters.

$X_*(T) = \{\lambda : \mathbb{G}_m \to T\}$ group of cocharacters.

Pairing: $X^*(T) \times X_*(T) \longrightarrow X^*(\mathbb{G}_m) = \mathbb{Z}$

$$(\chi, \lambda) \longmapsto \chi\lambda \longleftrightarrow \langle \chi, \lambda \rangle$$

$\chi\lambda(a) = a^{\langle \chi, \lambda \rangle}$ for $a \in \mathbb{G}_m$.

Exer: Perfect pairing.

$\mathbb{G}_m = k^\times = \mathbb{A}^1 \smallsetminus \{0\} = \mathbb{P}^1 \smallsetminus \{0, \infty\}$.

Def $\phi : \mathbb{G}_m \longrightarrow Z$ morphism, $z \in Z$ point.

$\lim\limits_{a \to 0} \phi(a) = z \iff \exists$ morphism $\tilde{\phi} : \mathbb{A}^1 \longrightarrow Z$:
$\tilde{\phi}(a) = \phi(a)$ for $a \in \mathbb{G}_m$, $\tilde{\phi}(0) = z$.

$$\lim_{a \to \infty} \phi(a) = z \iff \lim_{a \to 0} \phi(a^{-1}) = z.$$

Always exist if Z is projective (or complete).

Assume Z affine.

$\phi^* : k[Z] \longrightarrow k[t, t^{-1}]$.

$\lim\limits_{a \to 0} \phi(a)$ exists $\iff \phi^*(k[Z]) \subseteq k[t]$

$\iff \forall f \in k[Z] : f\phi \in k(\mathbb{A}^1)$ is defined at 0.

Def T torus, V T-variety, $\lambda \in X_*(T)$.

$$V(\lambda) = \{ v \in V \mid \lim_{a \to 0} \lambda(a).v \text{ exists} \}$$

Note: $V(-\lambda) = \{ v \in V \mid \lim_{a \to \infty} \lambda(a).v \text{ exists} \}$

Lemma T torus, V affine T-variety, $\lambda \in X_*(T)$.

(1) $V(\lambda) \subseteq V$ is closed.

(2) $V(\lambda) \cap V(-\lambda) = V^{\lambda(\mathbb{G}_m)} = \{ v \in V \mid \lambda(a).v = v \ \forall a \in \mathbb{G}_m \}$.

Proof

$T \circlearrowright k[V]$ locally finite.

$$(s(t).f)(v) = f(t^{-1}.v).$$

$$k[V] = \bigoplus_{\chi} k[V]_\chi$$

$$f = \sum_\chi f_\chi \quad \Rightarrow \quad s(t).f = \sum_\chi \chi(t) f_\chi .$$

Let $v \in V$.

$$\phi : \mathbb{G}_m \longrightarrow V, \quad \phi(a) = \lambda(a).v$$

$$\phi^*(f)(a) = f(\lambda(a).v) = (s(\lambda(a)^{-1}).f)(v)$$
$$= \sum_\chi a^{-\langle \chi, \lambda \rangle} f_\chi(v).$$

Defined at $a = 0 \Leftrightarrow f_\chi(v) = 0$ when $\langle \chi, \lambda \rangle > 0$.

$$V(\lambda) = Z\Big(\bigoplus_{\langle \chi, \lambda \rangle > 0} k[V]_\chi \Big)$$

(not ideal!)

$$f(\lambda(a).v) = \sum_{\chi} a^{-\langle \chi,\lambda\rangle} f_\chi(v)$$

$$\begin{aligned} v \in V(\lambda) \cap V(-\lambda) &\Leftrightarrow \forall f \in k[V] : f_\chi(v) = 0 \text{ for } \langle \chi,\lambda\rangle \neq 0 \\ &\Leftrightarrow \forall f \in k[V] : f(\lambda(a).v) = f(v) \\ &\Leftrightarrow v \in V^{\lambda(\mathbb{G}_m)} \end{aligned}$$

$\square$

Example

$\mathbb{G}_m \circlearrowright \mathbb{A}^2, \quad a.(x,y) = (ax, a^{-1}y).$

$\mathbb{G}_m \circlearrowright k[\mathbb{A}^2] = k[X,Y].$

$(a.f)(x,y) = f(a^{-1}.(x,y)) = f(a^{-1}x, ay).$

$a.X = a^{-1}X, \quad a.Y = aY$

$\lambda = id : \mathbb{G}_m \longrightarrow \mathbb{G}_m.$

$(x,y) \in \mathbb{A}^2(\lambda) \Leftrightarrow \lim\limits_{a\to 0} a.(x,y) \text{ exists} \Leftrightarrow y = 0.$

$\mathbb{A}^2(\lambda) = Z(Y), \quad \mathbb{A}^2(-\lambda) = Z(X).$

$\mathbb{A}^2(\lambda) \cap \mathbb{A}^2(-\lambda) = \{(0,0)\} = (\mathbb{A}^2)^{\mathbb{G}_m}$

$k[\mathbb{A}^2]_\chi = \mathrm{Span}\{X^iY^j \mid j-i = \chi\}$

$\bigoplus\limits_{\langle \chi,\lambda\rangle > 0} k[\mathbb{A}^2]_\chi = \mathrm{Span}\{X^iY^j \mid j-i > 0\}$ (not ideal!)

Generates $I(\mathbb{A}^2(\lambda)) = \langle Y \rangle \subseteq k[\mathbb{A}^2].$

Quiz

G LAG, $g \in G$ torsion elt. $g^m = e$.

$g = g_s g_u = ?$

$p = \mathrm{char}(k) = 0$: $g = g_s$.

Assume $p > 0$:

$p \nmid m \Rightarrow g = g_s$.

$m = p^j \Rightarrow g = g_u$.

$m = n p^j$, $p \nmid n$.

g^n is unipotent.

g^{p^j} is semi-simple.

$an + bp^j = 1$, $a, b \in \mathbb{Z}$.

$g = (g^{bp^j})(g^{an}) = g_s g_u$.

Additive functions

G LAG. $p = \mathrm{char}(k)$.

$\mathbb{G}_a = k$ (additive group)

Def Additive functions on G:

$$\mathcal{A}(G) = \{ f \in k[G] \mid f : G \longrightarrow \mathbb{G}_a \text{ group hom.} \}$$

Example $G = \mathbb{G}_a^n$ vector group.

$k[G] = k[T_1, \dots, T_n]$

$f \in k[G]$ additive $\Leftrightarrow$ $f(xy) = f(x) + f(y)$ $\forall\, x, y \in G$

$\Leftrightarrow$ $f(T_1 + U_1, \dots, T_n + U_n) = f(T_1, \dots, T_n) + f(U_1, \dots, U_n)$.

Claim:

$$\mathcal{A}(G) = \begin{cases} \mathrm{Span}_k \{T_1, \dots, T_n\} & \text{if } p = 0 \\ \mathrm{Span}_k \{ T_i^{p^j} \mid 1 \le i \le n,\ j \ge 0 \} & \text{if } p > 0 \end{cases}$$

Proof

$$\frac{\partial f}{\partial T_i}(T_1 + U_1, \dots, T_n + U_n) = \frac{\partial f}{\partial T_i}(T_1, \dots, T_n)$$

$$\Rightarrow \frac{\partial f}{\partial T_i}(U_1, \dots, U_n) = c_i \in k \text{ constant.}$$

$g = f - \sum_{i=1}^{n} c_i T_i$. $\qquad \frac{\partial g}{\partial T_i} = 0 \ \forall i$

$p = 0$: $g = 0$

$p > 0$: $g = h(T_1^p, \dots, T_n^p)$, $\quad h \in k[G]$

$g \in \mathcal{A}(G) \Rightarrow h \in \mathcal{A}(G)$.

Induction on $\deg(f)$.

□

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Def G LAG.

$G \cong \mathbb{G}_a^n$: G is a *vector group*

$G \subseteq \mathbb{G}_a^n$ closed: G is an *elementary unipotent group.*

Theorem G LAG. TFAE:

(1) G is elementary unipotent.

(2) G is unipotent, abelian, and $pG = 0$.

(3) $G = \mathbb{G}_a^n \times F$, F finite elementary unipotent.

(4) $k[G]$ is generated by $\mathcal{A}(G)$ as k-algebra.

Note $p = 0 \Rightarrow F = 0$

$p > 0 \Rightarrow F = (\mathbb{Z}/p\mathbb{Z})^m$

Cor G connected LAG, $\dim(G) = 1$

$\Rightarrow G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.

Module structure on $\mathcal{A}(G)$

$\mathcal{A}(G) \subseteq k[G]$ vector subspace.

$p = 0$: $R = k$, $\mathcal{A}(G)$ is an R-module.

Assume $p > 0$:

$\mathcal{A}(\mathbb{G}_a) = \mathrm{Span}_k\{T^{p^j} \mid j \geq 0\}$, $\dim \mathcal{A}(\mathbb{G}_a) = \infty$.

$f \in \mathcal{A}(G) \Rightarrow f^p \in \mathcal{A}(G)$.

$R = k[T]$ as <u>additive group</u>

$$(aT^i)\cdot(bT^j) := a b^{p^i} T^{i+j}.$$

$$Tb = b^p T.$$

Properties:

(1) R associative, non-commutative.

(2) R is "Euclidean": division algorithm works.

(3) All left/right ideals are principal.

(4) Any f.g. left R-module is direct sum of cyclic modules.

R-module structure on $\mathcal{A}(G)$:

$a.f = af$ for $a \in k$, $f \in \mathcal{A}(G)$.

$T.f = f^p$

$(aT^i) = a f^{p^i}$

Exer: $\mathcal{A}(\mathbb{G}_a^n) =$ free left R-module, basis $\{T_1, \ldots, T_n\}$.

Thm G elementary unipotent.

(1) $\mathcal{A}(G)$ f.g. left R-module.

(2) G connected $\Leftrightarrow$ $\mathcal{A}(G)$ free left R-module.

Derivations

R com. ring. A com. R-algebra. M A-module.

R-derivation $D : A \longrightarrow M$:

(1) R-linear.

(2) $D(ab) = a.D(b) + b.D(a)$, $a, b \in A$.

Note: If D satisfies (2), then (1) $\Leftrightarrow$ $D(R) = 0$.

$\mathrm{Der}_R(A, M) = \{ D : A \to M \text{ } R\text{-derivation} \}$

A-module: $(b.D + D')(a) = b.D(a) + D'(a)$.

Example $A = k[x_1, \ldots, x_n]$, $D : A \longrightarrow M$ any k-derivation.

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i).$$

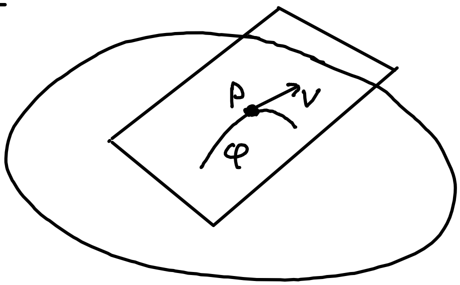
$\mathrm{Der}_k(A, M) \cong M^{\oplus n}$ as A-module.

$\phi : A \longrightarrow B$ R-algebra hom, N B-module.

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_R(B, N) \xrightarrow{\phi_\circ} \mathrm{Der}_R(A, N).$$

$$D \longmapsto D \circ \phi$$

Tangent and cotangent vectors



X manifold, $p \in X$.

Tangent vector $v \in T_pX$:

Equiv. class of param. curves
$\varphi : \mathbb{R} \longrightarrow X$ with $\varphi(0) = p$.

Given C^∞ $f : X \longrightarrow \mathbb{R}$:

$$D_v(f) = \frac{d}{dt} f(\varphi(t))\Big|_{t=0}.$$

$C^\infty(X)$-module: $\mathbb{R}(p) = \mathbb{R}$, $f \cdot a = f(p)a$.

$D_v \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), \mathbb{R}(p)) =: T_pX$.

$df_p \in (T_pX)^*$ **cotangent vector**: $df_p(v) = D_v(f)$.

Local ring of variety

X irred. variety, $p \in X$.

$\mathcal{F} = \{(U, f) \mid p \in U \subseteq X \text{ open}, \ f : U \to k \text{ regular}\}$

Equiv. rel: $(U, f) \sim (U', f') \Leftrightarrow f|_{U \cap U'} = f'|_{U \cap U'}$

Local ring at p: $\mathcal{O}_{X,p} = \mathcal{F}/\sim = \{f \in k(X) \mid f \text{ def. at } p\}$

$\mathfrak{m}_x = \{f \in \mathcal{O}_{X,p} \mid f(p) = 0\} \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \cong k$ is $\mathcal{O}_{X,p}$-module: $f \cdot a = f(p)a$

Example: X affine, $p \in X$.

$I(p) \subseteq k[X]$ max. ideal.

$$\mathcal{O}_{X,p} = k[X]_{I(p)} = (k[X] \setminus I(p))^{-1} k[X].$$

this def. is valid when X not irred.

Zariski tangent space

$T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$ tangent space.

$T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$. cotangent space.

Note: $D \in T_pX$, $f \in \mathfrak{m}_p^2 \Rightarrow D(f) = 0$.

$g, h \in \mathfrak{m}_p \Rightarrow D(gh) = g(p)D(h) + h(p)D(g) = 0$.

Note:

$$\begin{array}{ccc} \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) & \xrightarrow{\cong} & (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \\ D & \longmapsto & [f + \mathfrak{m}_p^2 \mapsto D(f)] \\ [f \mapsto \bar{D}(f - f(p) + \mathfrak{m}_p^2)] & \longleftarrow\!\shortmid & \bar{D} \end{array}$$

$\therefore$ **Perfect pairing**

$$\begin{array}{c} T_p^*X \times T_pX \longrightarrow k \\ (f + \mathfrak{m}_p^2, D) \longmapsto D(f) \end{array}$$

Def $p \in X$ is a non-sing. point if $\dim_k(T_pX) = \dim(X)$.

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X irred. variety, $p \in X$.

$\mathcal{O}_{X,p} = \{ f \in k(X) \mid f \text{ def. at } p \}$

$\mathfrak{m}_p \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$T_p^*X = \mathfrak{m}_p / \mathfrak{m}_p^2$. $\quad T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$

<u>Exer</u>: $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2)$ = min. # generators of ideal $\mathfrak{m}_p$.

<u>Principal Ideal Theorem</u>:

min. # gens of $\mathfrak{m}_p \geq \dim \mathcal{O}_{X,p}$ (Krull dim.)

<u>Def</u> $\mathcal{O}_{X,p}$ is a <u>regular local ring</u> if $\mathfrak{m}_p$ is gen. by $\dim(\mathcal{O}_{X,p})$ elts.

X irred. $\Rightarrow$ $\dim(X) = \dim(\mathcal{O}_{X,p})$.

$\therefore \dim_k(T_p^*X) \geq \dim(X)$.

<u>Def</u> $p \in X$ non-sing point $\Leftrightarrow$ $\mathcal{O}_{X,p}$ regular local

$\Leftrightarrow \dim_k(T_p^*X) = \dim(X)$.

<u>Theorem</u> $X_{\text{sing}} \subsetneq X$ proper closed subset.

<u>Exer</u> X affine, $p \in X$. $k[X] \longrightarrow \mathcal{O}_{X,p}$ k-alg. hom.

$$\mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) \xrightarrow{\cong} \mathrm{Der}_k(k[X], k(p))$$

$$I(p)/I(p)^2 \xrightarrow{\cong} \mathfrak{m}_p/\mathfrak{m}_p^2$$

$$f/g(p) + I(p)^2 \longmapsfrom f/g + \mathfrak{m}_p^2$$

$f, g \in k[X]$,
$f(p) = 0, \ g(p) \neq 0$.

Differentiation:

$\phi : X \longrightarrow Y$ morphism. $\phi^* : \mathcal{O}_{Y,\phi(p)} \longrightarrow \mathcal{O}_{X,p}$

$$d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y.$$
$$D \longmapsto D\phi^*$$

$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$: $d(\psi\phi)_p = d\psi_{\phi(p)} \circ d\phi_p$

Differentials

R com. ring. A com. R-algebra.

$\exists$ <u>universal R-derivation</u> $d_A : A \longrightarrow \Omega_{A/R}$:

For any R-derivation $D : A \longrightarrow M$

$\exists!$ A-linear map $\tilde{D} : \Omega_{A/R} \longrightarrow M$ s.t. $D = \tilde{D} \circ d_A$.

$$\begin{tikzcd}
A \arrow[rr, "D"] \arrow[dr, "d_A"'] & & M \\
& \Omega_{A/R} \arrow[ur, dashed, "\exists!"', "\tilde{D}"] &
\end{tikzcd}$$

Construction:

$$\Omega_{A/R} = \left(\text{free } A\text{-module gen. by } \{d_A(b) : b \in A\}\right)$$
$$\Big/ \left\langle \begin{array}{l|l} d_A(a+b) = d_A(a) + d_A(b) & a, b \in A \\ d_A(ab) = a\, d_A(b) + b\, d_A(a) & r \in R \\ d_A(r) = 0 & \end{array} \right\rangle$$

X affine variety, $p \in X$.

<u>Notation</u>: M $k[X]$-module.

$$M(p) = M/I(p)M = M \otimes_{k[X]} k(p).$$

$$M \longrightarrow M(p), \quad m \mapsto m(p) = m + I(p)M.$$

<u>Exer</u> $\mathrm{Hom}_{k[X]}(M, k(p)) = \mathrm{Hom}_k(M(p), k)$

$$M \longrightarrow M(p) \longrightarrow k(p).$$

<u>Def</u>: $\Omega_X = \Omega_{k[X]/k} = \{\text{covector fields on } X\}$

$d = d_X : k[X] \longrightarrow \Omega_X$ universal k-derivation.

$$T_pX = \mathrm{Der}_k(k[X], k(p)) = \mathrm{Hom}_{k[X]}(\Omega_X, k(p))$$
$$= \mathrm{Hom}_k(\Omega_X(p), k) = \Omega_X(p)^*$$

$$\therefore \; \Omega_X(p) = T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$$
$$df(p) \longleftrightarrow f - f(p) + \mathfrak{m}_p^2.$$

<u>Exer</u> $k[\mathbb{A}^n] = k[T_1, \dots, T_n]$.

$\Omega_{\mathbb{A}^n} = \mathrm{Span}_{k[\mathbb{A}^n]}\{dT_1, \dots, dT_n\}$ (free!)

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i$$

$$T_p^*\mathbb{A}^n = \mathrm{Span}_k\{dT_1, \dots, dT_n\}$$

$$df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(p)\, dT_i$$

Exer $X \subseteq \mathbb{A}^n$ closed. $I(X) = \langle f_1, .., f_m \rangle \subseteq k[\mathbb{A}^n]$.

$t_i = \overline{T_i} \in k[X] = k[\mathbb{A}^n]/I(X)$.

$$\Omega_X = \operatorname{Span}_{k[X]}\{dt_1, .., dt_n\} / \langle \overline{df_1}, .., \overline{df_m} \rangle$$

$$= \left(\Omega_{\mathbb{A}^n}/\langle df_1, .., df_m \rangle\right) \otimes_{k[\mathbb{A}^n]} k[X].$$

Notation: $\overline{df} = \sum_{i=1}^{n} \overline{\frac{\partial f}{\partial T_i}}\, dt_i \in \operatorname{Span}_{k[X]}\{dt_1, .., dt_n\}$

$T_p^* X = T_p^* \mathbb{A}^n / \langle df_1(p), .., df_m(p) \rangle$

$T_p X = \langle df_1(p), .., df_m(p) \rangle^{\perp} \subseteq T_p \mathbb{A}^n$

Jacobi matrix: $J = \left(\frac{\partial f_i}{\partial T_j}\right) \in \operatorname{Mat}(n \times m, k[X])$

$$k[X]^{\oplus m} \xrightarrow{J} k[X]^{\oplus n} \longrightarrow \Omega_X \longrightarrow 0$$

$$k^{\oplus m} \xrightarrow{J(p)} k^{\oplus n} \longrightarrow T_p^* X \longrightarrow 0$$

$\therefore$ $\operatorname{rank} J(p) \leq n - \dim(X)$

Equality $\Leftrightarrow$ $p \in X$ nonsing. point.

Cor $X_{\text{sing}} = \{ \operatorname{rank}(J) < \operatorname{codim}(X, \mathbb{A}^n) \} \subseteq X$ closed.

Vector fields: $\operatorname{Der}_k(k[X], k[X]) = \operatorname{Hom}_{k[X]}(\Omega_X, k[X])$

X non-singular $\Rightarrow$ Ω_X locally free $k[X]$-module

$\Rightarrow \operatorname{Hom}_{k[X]}(\Omega_X, k[X])(p) = \operatorname{Hom}_k(\Omega_X(p), k) = T_p X.$

Separable field extensions

E/F field extension. $(F \subseteq E)$ $p = \mathrm{char}(F)$.

Def: E/F is separably algebraic if

$\forall a \in E\ \exists f \in F[T] : f(a) = 0$ and f has no multible roots.

Note: $b \in E$ is a multiple root $\Leftrightarrow$ $f(b) = f'(b) = 0$.

$p = 0 \Rightarrow E/F$ always separable.

WLOG: $f \in F[T]$ irred.

$f(b) = f'(b) = 0 \Rightarrow f'(T) = 0 \in F[T]$.

$p = 0 \Rightarrow f'(T) \neq 0$.

Def: Transcendence basis of E/F:

$B \subseteq E$ such that B is alg. indep. $/F$

and $E/F(B)$ is algebraic.

$\mathrm{tr.deg}_F(E) = \# B$

Def E/F is separably generated if $\exists$ tr. basis B

such that $E/F(B)$ is separably algebraic.

Def F is perfect if $p = 0$ or $\forall r \in F\ \exists s \in F : s^p = r$.

alg. closed $\Rightarrow$ perfect.

Theorem F perfect $\Rightarrow$ E/F separably generated.

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Rational functions

X irred. variety.

$k(X) = \{(U,f) \mid \emptyset \neq U \subseteq X,\ f: U \to k \text{ regular}\}/\sim$

$= \{f : X \dashrightarrow k\}$ field of rat. fcns. on X.

X affine $\Rightarrow$ $k(X) = K(k[X])$ field of fractions.

$\phi : X \longrightarrow Y$ morphism of irred. varieties.

Def ϕ is dominant if $\overline{\phi(X)} = Y$.

Assume $\phi : X \longrightarrow Y$ dominant.

$\phi^* : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ is injective.

$\phi^* : k(Y) \longrightarrow k(X)$, $\phi^* f = f\phi : X \to Y \dashrightarrow k$

Def: ϕ is separable if $k(X)/k(Y)$ is separably generated.

Thm $\phi : X \longrightarrow Y$ morphism of irred. varieties.

(1) Assume $p \in X$ is non-sing, $\phi(p) \in Y$ is non-sing., and $d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y$ is surjective.
Then ϕ is dominant and separable.

(2) Assume ϕ is dominant and separable.
Then assumption of (1) holds for all points p in dense open $\subseteq X$.

Let G be a connected alg. group.

<u>Cor</u> Any homogeneous G-variety X is irred. and non-singular.

<u>Cor</u> $\phi : X \longrightarrow Y$ equivariant morphism of homogeneous G-varieties. TFAE:

(1) ϕ is separable.

(2) $d\phi_p : T_pX \longrightarrow T_{\phi(p)}Y$ is surjective for some $p \in X$.

(3) $d\phi_p$ is surjective for all $p \in X$.

<u>Cor</u> $\phi : G \longrightarrow G'$ surjective homomorphism of alg. groups. ϕ separable $\Leftrightarrow$ $d\phi_e$ surjective.

Tangent spaces

X affine variety, $p \in X$.

$k(p) = k[X]/I(p)$.

$d_X : k[X] \longrightarrow \Omega_X = \Omega_{k[X]/k}$.

$$\Omega_X \longrightarrow \Omega_X(p) = T_p^*X = I(p)/I(p)^2$$

$$d_Xf \longmapsto d_Xf(p) \longleftrightarrow f - f(p) + I(p)^2$$

$T_pX = \mathrm{Der}_k(k[X], k(p))$

Perfect pairing: $T_p^*X \times T_pX \longrightarrow k$

$$(f + I(p)^2, D) \longmapsto D(f)$$

Differentiation

$\phi : X \longrightarrow Y$ morphism of affine varieties, $p \in X$.

$$\begin{tikzcd}
k[Y] \arrow[r, "\phi^*"] \arrow[d, "d_Y"'] & k[X] \arrow[d, "d_X"] \\
\Omega_Y \arrow[r, "\phi^*"] \arrow[d] & \Omega_X \arrow[d] \\
T^*_{\phi(p)}Y \arrow[r, "\phi^*"] & T^*_p X
\end{tikzcd}$$

$$\phi^*(d_Y(f)) = d_X(\phi^* f)$$

$$\phi^*(f + I(\phi(p))^2) = \phi^*(f) + I(p)^2.$$

$$\phi^*(d_Y f(\phi(p))) = d_X(\phi^* f)(p)$$

$$\begin{aligned} d\phi_p : T_pX &\longrightarrow T_{\phi(p)}Y \\ D &\longmapsto D\phi^* \end{aligned}$$

$D \in T_pX$, $u \in T^*_{\phi(p)}Y \Rightarrow (u, d\phi_p D) = (\phi^* u, D)$

Products

Let $(p,q) \in X \times Y$

$D \in T_{(p,q)}(X \times Y) = \mathrm{Der}_k(k[X] \otimes k[Y], k(p,q))$

$D(f \otimes g) = g(q) D(f \otimes 1) + f(p) D(1 \otimes g)$.

$j_q : X \longrightarrow X \times Y$, $j_p : Y \longrightarrow X \times Y$

$T_{(p,q)}(X \times Y) = T_pX \oplus T_qY = dj_q(T_pX) \oplus dj_p(T_qY)$

Lemma G alg. group. $X, Y \in T_eG$.

$\mu : G \times G \longrightarrow G$ mult. $i : G \longrightarrow G$ inverse.

$$d\mu_{(e,e)} : T_eG \oplus T_eG \longrightarrow T_eG, \quad (X,Y) \mapsto X+Y$$

$$di_e : T_eG \longrightarrow T_eG, \quad X \mapsto -X.$$

Proof

$$\begin{tikzcd} G \arrow[r, "j_1"] & G \times G \arrow[r, "\mu"] & G \\ x \arrow[r, mapsto] & (x,e) \arrow[r, mapsto] & x \end{tikzcd}$$

$$d\mu(X,0) = d\mu(dj_1(X)) = d(\mu j_1)(X) = X.$$

$$\begin{tikzcd} G \arrow[r, "\phi"] & G \times G \arrow[r, "\mu"] & G \\ x \arrow[r, mapsto] & (x, x^{-1}) \arrow[r, mapsto] & e \end{tikzcd}$$

$$\begin{tikzcd} T_eG \arrow[r, "d\phi"] & T_eG \oplus T_eG \arrow[r, "d\mu"] & T_eG \\ X \arrow[r, mapsto] & (X, di_e(X)) \arrow[r, mapsto] & X + di_e(X) = 0. \end{tikzcd}$$

$\square$

Adjoint Representation

G LAG.

$$\hat{\lambda}, \hat{\rho} : G \longrightarrow \mathrm{Aut}_{var}(G)$$

$$\hat{\lambda}(x)(y) = xy, \quad \hat{\rho}(x)(y) = yx^{-1}.$$

$$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[G])$$

$$\lambda(x) = \hat{\lambda}(x^{-1})^*, \quad \rho(x) = \hat{\rho}(x^{-1})^*$$

$$(\lambda(x) f)(y) = f(x^{-1}y), \quad (\rho(x) f)(y) = f(yx).$$

<u>Note</u>: $\lambda(x)\rho(y) = \rho(y)\lambda(x) \quad \forall\, x, y \in G.$

$$\lambda(x) = \hat{\lambda}(x^{-1})^* : T^*_{x^{-1}y}G \longrightarrow T^*_y G$$

$$\rho(x) = \hat{\rho}(x^{-1})^* : T^*_{yx} G \longrightarrow T^*_y G$$

$$\lambda(x).df(x^{-1}y) = d(\lambda(x).f)(y).$$

$$\rho(x).df(yx) = d(\rho(x).f)(y).$$

$$\mathrm{Int} : G \longrightarrow \mathrm{Aut}(G)$$

$$\mathrm{Int}(x) = \hat{\lambda}(x)\hat{\rho}(x). \quad \mathrm{Int}(x)(y) = xyx^{-1}.$$

$$\mathrm{Int}(x)^* = \lambda(x^{-1})\rho(x^{-1}) : k[G] \longrightarrow k[G]$$

$$(\mathrm{Int}(x)^*.f)(y) = f(xyx^{-1}).$$

$$\mathrm{Ad} : G \longrightarrow GL(T_e G)$$

$$\mathrm{Ad}(x) = d\,\mathrm{Int}(x)_e$$

$$\mathrm{Ad}(x).X = X\,\mathrm{Int}(x)^* = X\,\lambda(x^{-1})\rho(x^{-1}).$$

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Dual adjoint representation

$Ad^*: G \longrightarrow Aut_{k\text{-alg}}(k[G])$

$Ad^*(x) = Int(x^{-1})^* = \lambda(x)\rho(x) : k[G] \longrightarrow k[G].$

$Ad^*: G \longrightarrow GL(T_e^*G)$

$Ad^*(x) = Int(x^{-1})^* = \lambda(x)\rho(x) : T_e^*G \longrightarrow T_e^*G$

$Ad^*(x).df(e) = \lambda(x).d(\rho(x).f)(x^{-1})$

$= d(Ad^*(x).f)(e).$

For $u \in T_e^*G$, $X \in T_eG$: $(Ad^*(x).u, X) = (u, Ad(x^{-1}).X)$

because $(Int(x^{-1})^*u, X) = (u, dInt(x^{-1})_e . X)$

Rationality

$\mu^2 : G \times G \times G \longrightarrow G$ mult.

$(\mu^2)^*f = \sum_i f_i \otimes g_i \otimes h_i$: $f(xyz) = \sum_i f_i(x) g_i(y) h_i(z)$

$(Ad^*(x).f)(y) = f(Int(x^{-1})(y)) = f(x^{-1}yx)$

$Ad^*(x).f = \sum_i f_i(x^{-1}) h_i(x) g_i$

$Ad^*(x).df(e) = \sum_i f_i(x^{-1}) h_i(x) dg_i(e).$

$\therefore$ $Ad^*: G \longrightarrow GL(T_e^*)$, $Ad: G \longrightarrow GL(T_eG)$ are rational representations of G.

Lie algebra

G LAG.

$\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G]) = \{\text{tangent vector fields on } G\}$

$D, D' \in \mathcal{D}_G \Rightarrow [D, D'] = DD' - D'D \in \mathcal{D}_G$

$\lambda, \rho : G \longrightarrow \mathrm{Aut}_{k[G]}(\mathcal{D}_G)$

$$\left.\begin{aligned} \lambda(x).D &= \lambda(x)\, D\, \lambda(x^{-1}) \\ \rho(x).D &= \rho(x)\, D\, \rho(x^{-1}) \end{aligned}\right\} \text{ translation of vector fields.}$$

$L(G) = \{ D \in \mathcal{D}_G \mid \lambda(x).D = D \ \forall x \in G \} \subseteq \mathcal{D}_G$ Lie subalg.

Note: $\rho(x).L(G) = L(G)$.

Def $X \in T_eG$, $f \in k[G]$, $y \in G$: $(\overline{X}f)(y) = X(\lambda(y^{-1}).f)$

Lemma $\overline{X} \in L(G)$

Proof

$\overline{X}f \in k[X]$: $\mu^*(f) = \sum g_i \otimes h_i$: $f(xy) = \sum g_i(x) h_i(y)$.

$\lambda(x^{-1}).f = \sum g_i(x) h_i \in k[G]$

$X(\lambda(x^{-1}).f) = \sum g_i(x) X(h_i)$ reg. fcn. of $x \in G$.

$\overline{X} \in \mathcal{D}_G$: $\overline{X}(fg) = f\cdot(\overline{X}g) + g\cdot(\overline{X}f)$

$\overline{X} \in L(G)$: $(\lambda(x)\,\overline{X}\,\lambda(x^{-1}).f)(y) = (\overline{X}\,\lambda(x^{-1}).f)(x^{-1}y)$

$= X(\lambda(y^{-1}x)\,\lambda(x^{-1}).f) = X(\lambda(y^{-1}).f) = (\overline{X}f)(y)$

□

Def $\alpha : D_G \longrightarrow T_eG$, $(\alpha D).f = (Df)(e)$

Prop $\alpha : L(G) \xrightarrow{\cong} T_eG$ iso. of vector spaces with inverse $X \mapsto \overline{X}$.

Proof

$\alpha(\overline{X}) = X$: $\alpha(\overline{X}).f = (\overline{X}f)(e) = X(\lambda(e^{-1}).f) = X(f)$.

$D \in L(G) \Rightarrow \overline{\alpha D} = D$:

$$(\overline{\alpha D}.f)(x) = (\alpha D)(\lambda(x^{-1}).f) = D(\lambda(x^{-1}).f)(e)$$

$$= (\lambda(x^{-1})D.f)(e) = Df(x).$$

□

Lemma $\alpha \circ \rho(y) \circ \alpha^{-1} = Ad(y) : T_eG \longrightarrow T_eG$

$$\begin{tikzcd} L(G) \arrow[r, "\alpha"] \arrow[loop below, "\rho(y)"] & T_eG \arrow[loop below, "Ad(y)"] \end{tikzcd}$$

Proof

$$(\alpha \circ \rho(y) \circ \alpha^{-1})(X)(f) = ((\rho(y).\overline{X}).f)(e)$$

$$= (\rho(y)\overline{X}\rho(y^{-1}).f)(e) = (\overline{X}\rho(y^{-1}).f)(y)$$

$$= X(\lambda(y^{-1})\rho(y^{-1}).f) = (Ad(y).X)(f).$$

□

Lie algebra of subgroup

G LAG, $H \subseteq G$ closed subgroup.

$k[H] = k[G]/I(H)$

$T_eH = \{X \in T_eG \mid X(I(H)) = 0\} \subseteq T_eG$

Def: $\mathcal{D}_{G,H} = \{D \in \mathcal{D}_G \mid D(I(H)) \subseteq I(H)\} \subseteq \mathcal{D}_G$ Lie subalg.

Lie algebra hom: $\phi : \mathcal{D}_{G,H} \longrightarrow \mathcal{D}_H$:

$D \in \mathcal{D}_{G,H}$: $D : k[G] \longrightarrow k[G]$ k-derivation,

$\phi D : k[H] \longrightarrow k[H]$, $(\phi D)(\bar{f}) = \overline{Df}$.

Lemma $\phi : \mathcal{D}_{G,H} \cap L(G) \xrightarrow{\cong} L(H)$ iso. of Lie algebras.

Proof

$$\begin{tikzcd}
\mathcal{D}_{G,H} \arrow[r, "\alpha_G"] \arrow[d, "\phi"'] & T_eG \\
\mathcal{D}_H \arrow[r, "\alpha_H"'] & T_eH \arrow[u, "\cup|"']
\end{tikzcd}$$

Note: $x \in H \Rightarrow$
$\lambda(x)(I(H)) \subseteq I(H)$.

$\phi(\mathcal{D}_{G,H} \cap L(G)) \subseteq L(H)$:

$\lambda(x)\, D = D\, \lambda(x)$: $k[G] \longrightarrow k[G]$ $\forall x \in G$

$\Rightarrow \lambda(x)\, \phi(D) = \phi(D)\, \lambda(x)$: $k[G]/I(H) \longrightarrow k[G]/I(H)$ $\forall x \in H$

$$\begin{tikzcd}
\mathcal{D}_{G,H} \cap L(G) \arrow[r, "\alpha_G", "\subseteq"'] \arrow[d, "\phi"', "\cap|"] & T_eG \\
L(H) \arrow[r, "\cong", "\alpha_H"'] & T_eH \arrow[u, "\cup|"']
\end{tikzcd}$$

Show: $X \in T_eH \Rightarrow \bar{X} \in \mathcal{D}_{G,H}$

$X \in T_eH$, $f \in I(H)$, $y \in H$:

$(\bar{X}f)(y) = X(\lambda(y^{-1}).f) = 0$ since $\lambda(y^{-1}).f \in I(H)$.

$\therefore$ $\bar{X}(I(H)) \subseteq I(H)$

□

Lie algebra homomorphism

$\phi : G \longrightarrow H$ homomorphism of LAGs.

$d\phi = d\phi_e : L(G) \longrightarrow L(H)$, $d\phi(\bar{X}) = \overline{d\phi_e(X)}$

Lemma

$D \in L(G) \Rightarrow D \circ \phi^* = \phi^* \circ d\phi(D) : k[H] \longrightarrow k[G]$

$$\begin{tikzcd}
k[H] \arrow[r, "\phi^*"] \arrow[d, "d\phi(D)"'] & k[G] \arrow[d, "D"] \\
k[H] \arrow[r, "\phi^*"] & k[G]
\end{tikzcd}$$

Proof

$X \in T_eG$, $f \in k[H]$, $y \in G$.

$$(\bar{X} \circ \phi^*(f))(y) = X(\lambda(y^{-1}).\phi^*(f)) = X(\phi^*(\lambda(\phi(y)^{-1}).f))$$

$$= d\phi(X)(\lambda(\phi(y)^{-1}).f) = (\overline{d\phi(X)}.f)(\phi(y)) = (\phi^* \circ \overline{d\phi(X)}(f))(y)$$

□

Prop $d\phi : L(G) \longrightarrow L(H)$ is a Lie alg. hom.

Proof

$$\begin{tikzcd}
G \arrow[r, "\phi"] & \phi(G) \arrow[r, "\subseteq"] & H \\
L(G) \arrow[r, "d\phi"] & L(\phi(G)) \arrow[r, "\subseteq", "\text{Lie subalg.}"'] & L(H)
\end{tikzcd}$$

WLOG: ϕ surjective $\Rightarrow \phi^* : k[H] \longrightarrow k[G]$ injective.

$$\phi^* \circ d\phi([D, D']) = [D, D'] \circ \phi^* = (DD' - D'D) \circ \phi^*$$

$$= \phi^* \circ (d\phi(D) d\phi(D') - d\phi(D') d\phi(D)) = \phi^* \circ [d\phi(D), d\phi(D')]$$

$$\Rightarrow d\phi([D, D']) = [d\phi(D), d\phi(D')]$$

□

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Lie algebra of $GL(V)$

E vector space $/k$, $\dim_k(E) < \infty$.

As variety: $k[E] = \operatorname{Sym}^\bullet(E^*)$

$E^* \subseteq k[E]$ linear fcns.

$p \in E$: $T_pE = \operatorname{Der}_k(k[E], k(p)) = \operatorname{Hom}_k(E^*, k) = E$

$X \in T_pE$, $f \in E^* \subseteq k[E]$: $X(f) = (f, X)$

Assume $E = \operatorname{End}_k(V)$, $\dim(V) < \infty$.

Perfect pairing: $E \times E \longrightarrow k$, $(A, B) \longmapsto tr(AB)$

For $A \in E$, def. $f_A \in E^*$ by $(f_A, B) = tr(AB)$.

$GL(V) \subseteq End(V)$ LAG.

$\mathfrak{gl}(V) = End(V)$ Lie algebra: $[X, Y] = XY - YX$.

$T_e GL(V) = T_e E = \mathfrak{gl}(V)$

Prop $\mathfrak{gl}(V) \xrightarrow{\cong} L(GL(V))$, $X \longmapsto \bar{X}$ iso. of Lie algebras.

Proof

$X, A \in E$, $B, C \in GL(V)$.

$\lambda(B^{-1}).f_A = f_{AB}$: $(\lambda(B^{-1}).f_A)(C) = f_A(BC) = tr(ABC) = f_{AB}(C)$

$\bar{X} f_A = f_{XA}$: $\bar{X} f_A(B) = X(\lambda(B^{-1}).f_A) = X(f_{AB}) = tr(XAB) = f_{XA}(B)$

$[\bar{X}, \bar{Y}].f_A = (\bar{X}\bar{Y} - \bar{Y}\bar{X}).f_A = f_{XYA} - f_{YXA} = \overline{[X, Y]}.f_A$.

□

Lie algebra of LAG

Note: $\phi : V \longrightarrow W$ k-linear map, V, W finite dim.

Then $d\phi_v = \phi$ for all $v \in V$.

$$\begin{tikzcd}
T_v V \arrow[r, phantom, "="] \arrow[d, "d\phi_v"] & \mathrm{Der}_k(k[V], k(v)) \arrow[r, phantom, "="] \arrow[d, "d\phi_v"] & \mathrm{Hom}_k(V^*, k) \arrow[r, phantom, "="] \arrow[d, "\phi^{**}"] & V \arrow[d, "\phi"] \\
T_{\phi(v)} W \arrow[r, phantom, "="] & \mathrm{Der}_k(k[W], k(\phi(v))) \arrow[r, phantom, "="] & \mathrm{Hom}_k(W^*, k) \arrow[r, phantom, "="] & W
\end{tikzcd}$$

G LAG, $\nu : G \longrightarrow GL(V)$ rat. rep.

$d\nu : L(G) \longrightarrow \mathfrak{gl}(V)$ Lie algebra homomorphism.

Lemma $\phi : \mathrm{End}(V) \longrightarrow k$ k-linear map, $X \in T_e G$.

Then $\phi(d\nu(X)) = X(\nu^*(\phi))$.

Proof

$\phi \circ \nu : G \longrightarrow \mathbb{A}^1$ morphism, $T_{\phi(e)} \mathbb{A}^1 = k$.

$\phi(d\nu(X)) = d\phi_e(d\nu(X)) = d(\phi \circ \nu)_e(X) = X(\phi \circ \nu)$.

□

Assume V has basis $\{v_1, \dots, v_n\}$.

$GL(V) = GL_n$, $\mathfrak{gl}(V) = \mathfrak{gl}_n = \mathrm{Mat}(n \times n, k)$.

$A \in \mathrm{End}(V)$: $A = (a_{ij})$, $A.v_j = \sum_i a_{ij} v_i$

$k[\mathrm{End}(V)] = k[T_{ij}]$: $T_{ij}(A) = a_{ij}$.

$\nu : G \longrightarrow GL_n$, $\nu(g) = (\nu_{ij}(g))$. $\nu_{ij} = \nu^*(T_{ij}) \in k[G]$.

$X \in T_e G$. $d\nu(X) = (b_{ij}) \in \mathfrak{gl}_n$.

$b_{ij} = T_{ij}(d\nu(X)) = X(\nu^*(T_{ij})) = X(\nu_{ij})$.

Prop G LAG, $V \subseteq k[G]$, $\dim(V) < \infty$, $\rho(x).V = V \ \forall\, x \in G$.

$\rho : G \longrightarrow GL(V)$ rat. rep., $d\rho : T_eG \longrightarrow \mathrm{End}(V)$.

Then $\overline{X}f = d\rho(X).f \ \ \forall\, X \in T_eG,\ f \in V$.

Proof

Fix $g \in G$, $f \in V$. Def. $\phi : \mathrm{End}(V) \longrightarrow k$, $\phi(Y) = (Y.f)(g)$.

$\lambda(g^{-1}).f = \rho^*\phi \in k[G]$:

$$(\lambda(g^{-1}).f)(x) = f(gx) = (\rho(x).f)(g) = \phi(\rho(x)).$$

$$(\overline{X}f)(g) = X(\lambda(g^{-1}).f) = X(\rho^*\phi) = \phi(d\rho(X)) = (d\rho(X).f)(g).$$

□

Cor $\overline{X} : k[G] \longrightarrow k[G]$ is locally finite $\forall\, X \in T_eG$.

Exer: G LAG. $\mathrm{Ad} : G \longrightarrow GL(T_eG)$, $d\mathrm{Ad} : T_eG \longrightarrow \mathrm{End}(T_eG)$.

$d\mathrm{Ad}(X)(Y) = [X, Y] \ \ \forall\, X, Y \in T_eG$.

Exer: $\nu : G \longrightarrow GL(V)$ rat. rep.

$\Lambda^n \nu : G \longrightarrow GL(\Lambda^n V)$, $d(\Lambda^n \nu) : T_eG \longrightarrow \mathrm{End}(\Lambda^n V)$.

$$d(\Lambda^n \nu)(X).(v_1 \wedge \dots \wedge v_n) = \sum_{i=1}^{n} v_1 \wedge \dots \wedge d\nu(X).v_i \wedge \dots \wedge v_n.$$

Jordan decomp in $L(G)$

G LAG, $X \in T_eG$.

$\overline{X} : k[G] \longrightarrow k[G]$ locally finite.

$\overline{X} = \overline{X}_s + \overline{X}_n$, $\overline{X}_s$ semi-simple, $\overline{X}_n$ nilpotent, $\overline{X}_s \overline{X}_n = \overline{X}_n \overline{X}_s$.

Thm (1) $\overline{X}_s, \overline{X}_n \in L(G)$ and $[\overline{X}_s, \overline{X}_n] = 0$.

(2) $\phi : G \longrightarrow G'$ hom. of LAGs $\Rightarrow$

$d\phi(X_s) = d\phi(X)_s$, $d\phi(X_n) = d\phi(X)_n$

(3) $G = GL_n \Rightarrow X = X_s + X_n$ is usual Jordan decomp. in M_n.

Fibers of morphisms

$\phi : X \longrightarrow Y$ dominant, X, Y irred. affine.

$\phi^* : k[Y] \subseteq k[X]$, $k(Y) \subseteq k(X)$.

$k[X] = k[Y][f_1, \dots, f_m] = k[Y][T_1, \dots, T_m]/I$

$X \cong Z(I) \subseteq Y \times \mathbb{A}^m$ closed subvariety.

$x \longmapsto (\phi(x), f_1(x), \dots, f_m(x))$

WLOG: $\{f_1, \dots, f_r\}$ transcendence basis of $k(X)/k(Y)$.

$r = \dim X - \dim Y$ *relative dimension*.

$k[Y] \subseteq k[Y][f_1, \dots, f_r] \subseteq k[X]$

$$\begin{tikzcd}
Y & Y \times \mathbb{A}^r \arrow[l] & X \arrow[l, "\text{gen. finite}"'] \\
\phi(x) & (\phi(x), f_1(x), \dots, f_r(x)) \arrow[l, mapsto] & x \arrow[l, mapsto]
\end{tikzcd}$$

Fact: $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq r$.

Def Assume $\phi : X \longrightarrow Y$ dominant.

ϕ is *generically finite*: $k(X)/k(Y)$ finite ext.

ϕ is *finite*: $k[X]$ f.g. $k[Y]$-module.

Finite $\Rightarrow$ generically finite.

Assume $\phi : X \longrightarrow Y$ gen. finite, $k[X] = k[Y][f]$.
$f \in k(X)$ algebraic over $k(Y)$.

$f^d + a_{d-1} f^{d-1} + \cdots + a_1 f + a_0 = 0, \quad a_i \in k(Y)$

$d = [k(X) : k(Y)]$

Choose $0 \neq h \in k[Y]$ s.t. $a_i \in k[Y]_h \ \forall i$.
$Y_h = \{ y \in Y \mid h(y) \neq 0 \} \subseteq Y$ open affine.
$k[Y_h] = k[Y]_h$

$k[X_h] \cong k[Y_h][T] / \langle T^d + \cdots + a_1 T + a_0 \rangle$
free $k[Y_h]$-module gen. by $\{1, T, \ldots, T^{d-1}\}$.

$$\begin{tikzcd} X \arrow[r, "\text{gen. finite}"] & Y \\ X_h \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\text{finite}"] & Y_h \arrow[u, phantom, "\cup\ \text{open}"] \end{tikzcd}$$

<u>Note:</u> $\phi : X_h \longrightarrow Y_h$ surjective with finite fibers:
$X_h \cong \{ (y,t) \in Y_h \times \mathbb{A}^1 \mid t^d + \cdots + a_1(y) t + a_0(y) = 0 \}$.
$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d + \cdots + a_1(y) t + a_0(y) = 0 \}$

Assume $f \in k(X)$ separable over $k(Y)$.
$T^d + \cdots + a_1 T + a_0$ has d distinct roots in $\overline{k(Y)}$.
$\Rightarrow \forall y \in$ dense open $\subseteq Y_h$:
$T^d + \cdots + a_1(y) T + a_0(y)$ has d distinct roots in k.

Assume $f \in k(X)$ purely inseparable over $k(Y)$:
$d = p^j, \quad f^d = a \in k(Y)$.
$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d = a(y) \} = \{ \sqrt[d]{a(y)} \}$.

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Thm

$\phi : X \longrightarrow Y$ dominant of irred. varieties.

$r = \dim X - \dim Y$.

$\exists$ dense open $U \subseteq X$ such that:

(1) $\phi \times 1_Z : U \times Z \longrightarrow Y \times Z$ is an open morphism $\forall Z$.

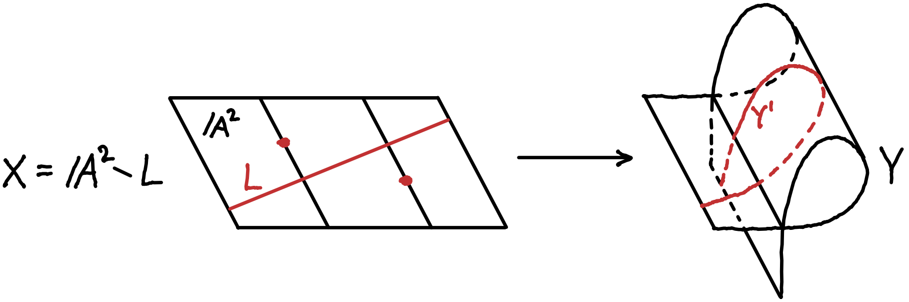
(2) $Y' \subseteq Y$ irred. closed, $X' \subseteq \phi^{-1}(Y')$ irred. comp.,
$X' \cap U \neq \emptyset \Rightarrow \dim(X') = \dim(Y') + r$.

(3) Assume $\dim X = \dim Y$.
$\forall y \in \phi(U) : \# \phi^{-1}(y) = [k(X)_s : k(Y)]$
$k(X)_s = \{ f \in k(X) \mid f \text{ separable } / k(Y) \}$.

Caution: $\phi : X \longrightarrow Y$ dominant, $r = \dim(X) - \dim(Y)$.

True: $y \in Y$ point, $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq r$.

False: $Y' \subseteq Y$ closed, irred, $\phi^{-1}(Y') \neq \emptyset \Rightarrow \dim \phi^{-1}(Y') \geq \dim Y' + r$



Integral extensions

A ring, B A-algebra.

$b \in B$ is <u>integral</u> over A if $\exists\ b^n + a_1 b^{n-1} + \dots + a_n = 0$, $a_i \in A$.

B <u>integral</u> over A $\Leftrightarrow$ All elts. integral over A.

B finite over A $\Leftrightarrow$ B f.g. as A-module.

<u>Exer</u>: B finite $/A$ $\Leftrightarrow$ B integral $/A$ & f.g. as A-algebra.

$\bar{A} = \{b \in B \mid b \text{ integral } /A\} \subseteq B$ subalgebra.

<u>Def</u> A domain A is <u>normal</u> if $A = \bar{A} \subseteq K(A)$.

$\phi : X \longrightarrow Y$ morphism, X, Y affine.

ϕ is <u>finite</u> $\Leftrightarrow$ $k[X]$ is finite over $k[Y]$.

<u>Fact</u>: ϕ finite $\Leftrightarrow$ ϕ is proper with finite fibers
$\Rightarrow$ ϕ is closed with finite fibers.

Y is normal if $k[Y]$ is normal.
non-singular $\Rightarrow$ normal.

<u>Note</u>: Assume X, Y irred. affine, Y normal,
$\phi : X \longrightarrow Y$ finite, birat.
Then $\phi : X \xrightarrow{\cong} Y$ isomorphism.
$k[Y] \subseteq k[X] \subseteq \overline{k[Y]} \subseteq k(Y) = k(X)$.

<u>Zariski's Main Theorem</u>

$\phi : X \longrightarrow Y$ morphism of irred. varieties.
Assume ϕ is bijective and birational, Y normal.
Then ϕ is an isomorphism.

Thm G alg. group. X, Y homogeneous G-varieties.

$\phi : X \longrightarrow Y$ equivariant. $r = \dim(X) - \dim(Y)$.

(a) $\forall Z$: $\phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ is open.

(b) $Y' \subseteq Y$ closed, irred., $X' \subseteq \phi^{-1}(Y')$ irred. comp.

$\Rightarrow \dim(X') = \dim(Y') + r$.

(c) ϕ isomorphism $\Leftrightarrow$ ϕ bijective and $\exists p \in X$:

$d\phi_p : T_p X \longrightarrow T_{\phi(p)} Y$ bijective.

Proof

WLOG G connected, X, Y irred.

(a) + (b) true for $\phi : U \longrightarrow Y$, $U \subseteq X$ dense open.

Translate.

(c): $d\phi_p$ surjective $\Rightarrow$ ϕ separable.

ϕ bijective $\Rightarrow$ ϕ birational.

$\phi : U \xrightarrow{\cong} \phi(U)$ for $U \subseteq X$ dense open.

Translate.

□

Cor $\phi : G \longrightarrow G'$ surjective hom. of alg. groups.

(a) $\dim(G) = \dim(G') + \dim \operatorname{Ker}(\phi)$.

(b) ϕ isomorphism $\Leftrightarrow$ ϕ and $d\phi$ are bijective.

Semi-simple automorphisms

G connected LAG. $\mathfrak{g} = L(G) = T_eG$.

$\sigma : G \xrightarrow{\cong} G$ automorphism.

$G_\sigma = \{x \in G \mid \sigma(x) = x\} \subseteq G$ closed subgroup.

$\mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\} \subseteq \mathfrak{g}$ Lie subalg.

Def $\chi : G \longrightarrow G$, $\chi(x) = \sigma(x)x^{-1}$

$G_\sigma = \chi^{-1}(e)$.

$$\chi : G \xrightarrow{(\sigma, i)} G \times G \xrightarrow{\mu} G$$

$$d\chi_e : T_eG \longrightarrow T_eG \oplus T_eG \longrightarrow T_eG$$

$$X \longmapsto (d\sigma(X), -X) \longmapsto d\sigma(X) - X.$$

$L(G_\sigma) \subseteq \mathrm{Ker}(d\chi_e) = \mathfrak{g}_\sigma$

$G \circlearrowright G$: $g.x = gx$ $\qquad$ $G \circlearrowright G$: $g \bullet x = \sigma(g)xg^{-1}$.

$\chi : (G, .) \longrightarrow (G, \bullet)$ equivariant morphism.

$\chi(G) = G \bullet e$ is an orbit for $\bullet$ action.

Note: $e \in \overline{\chi(G)}$ non-singular point.

$\chi : G \longrightarrow \overline{\chi(G)}$ separable $\Leftrightarrow$ $d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Lemma $L(G_\sigma) = \mathfrak{g}_\sigma \Leftrightarrow d\chi_e(\mathfrak{g}) = T_e\overline{\chi(G)}$

Proof

$$\dim d\chi_e(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{g}_\sigma$$

$$\leq \dim \mathfrak{g} - \dim L(G_\sigma) = \dim G - \dim G_\sigma = \dim \overline{\chi(G)}.$$

□

Def $\sigma : G \xrightarrow{\cong} G$ is semi-simple

$\Leftrightarrow$ $\sigma^* : k[G] \longrightarrow k[G]$ is semi-simple.

Lemma $\sigma : G \xrightarrow{\cong} G$ semi-simple

$\Leftrightarrow$ $\exists\ G \subseteq GL_n,\ s \in GL_n$ semi-simple:

$$\sigma(x) = s x s^{-1} \quad \forall x \in G.$$

Proof $(\Rightarrow)$:

$k[G] = k[f_1, .., f_n]$

$\exists\ \mathrm{Span}_k\{f_1, ..., f_n\} \subseteq V' \subseteq k[G]$:

$$\dim(V') < \infty, \quad \sigma^*(V') = V'.$$

$V = \sum_{x \in G} \rho(x).V' \subseteq k[G]$

$$\dim(V) < \infty, \quad \rho(x).V = V \quad \forall x \in G.$$

$\sigma^* \rho(x).f = \rho(\sigma^{-1}(x))\, \sigma^*.f \quad \forall f \in k[G]$:

$$(\sigma^* \rho(x).f)(y) = \rho(x).f(\sigma(y)) = f(\sigma(y)x) = f(\sigma(y\sigma^{-1}(x)))$$
$$= (\sigma^* f)(y \sigma^{-1}(x)) = (\rho(\sigma^{-1}(x))\, \sigma^*.f)(y).$$

$\sigma^*.\rho(x).V' = \rho(\sigma^{-1}(x)).V'$

$\therefore\ \sigma^*.V = V.$

$\rho : G \subseteq GL(V)$ closed.

$\sigma^* \rho(x) (\sigma^*)^{-1} = \rho(\sigma^{-1}(x))$

$s = (\sigma^*)^{-1} \in GL(V)$ semi-simple.

$\sigma(x) = s x s^{-1}.$

□

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G connected LAG.

Thm Let $\sigma : G \xrightarrow{\cong} G$ be semi-simple.

(1) $\chi(G) \subseteq G$ is closed.

(2) $d\chi_e : T_e G \twoheadrightarrow T_e \chi(G)$ is surjective.

Proof

WLOG $G \subseteq GL(V)$ closed, $s \in GL(V)$ ss.

$\sigma : GL(V) \longrightarrow GL(V)$, $\sigma(x) = sxs^{-1}$.

$\sigma : End(V) \longrightarrow End(V)$ linear extension.

$d\sigma = \sigma = Ad(s) \in GL(\mathfrak{gl}(V))$.

$G_\sigma = \{ x \in G \mid sxs^{-1} = x \}$

$\mathfrak{g}_\sigma = \{ X \in \mathfrak{g} \mid sXs^{-1} = X \} \subseteq \mathfrak{g} = L(G) \subseteq \mathfrak{gl}(V)$.

$\chi : GL(V) \longrightarrow GL(V)$, $\chi(x) = \sigma(x)x^{-1} = sxs^{-1}x^{-1}$.

Case $G = GL(V)$: $GL(V)_\sigma = \mathfrak{gl}(V)_\sigma \cap GL(V) \Rightarrow T_e(GL(V)_\sigma) = \mathfrak{gl}(V)_\sigma$

$\Rightarrow d\chi_e : T_e GL(V) \twoheadrightarrow T_e \overline{\chi(GL(V))}$ surjective.

Let $X \in T_e \overline{\chi(G)} \subseteq T_e \overline{\chi(GL(V))}$

$\exists\, Y \in \mathfrak{gl}(V) : \quad X = d\chi_e(Y) = d\sigma(Y) - Y$.

$\sigma(G) = G \Rightarrow d\sigma(\mathfrak{g}) \subseteq \mathfrak{g}$

s semi-simple $\Rightarrow d\sigma \in GL(\mathfrak{gl}(V))$ semi-simple

$\Rightarrow \exists\, \mathfrak{h} \subseteq \mathfrak{gl}(V)$ $d\sigma$-stable, $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{h}$.

$Y = Y' \oplus Y'' \in \mathfrak{g} \oplus \mathfrak{h}$

$X = d\sigma(Y') - Y' = d\chi_e(Y')$.

$\therefore\ d\chi_e : T_e G \twoheadrightarrow T_e \overline{\chi(G)}$ surjective.

Show: $\chi(G) \subseteq G$ closed.

Def $m(T) = \prod\limits_{\substack{a \text{ eigenval.} \\ \text{of } s^{-1}}} (T-a) \in k[T]$.

$$S = \left\{ \gamma \in GL(V) \;\middle|\; \begin{array}{l} (a)\ \gamma G \gamma^{-1} = G \\ (b)\ m(\gamma) = 0 \in End(V) \\ (c)\ ch.poly(Ad(\gamma)|\mathfrak{g}) = ch.poly(Ad(s^{-1})|\mathfrak{g}) \end{array} \right\}$$

$S \subseteq GL(V)$ closed, $s^{-1} \in S$, all elts. of S are semi-simple.

$$\gamma \in S: \quad \begin{aligned} G_\gamma &= \{x \in G \mid \gamma x \gamma^{-1} = x\} \\ \mathfrak{g}_\gamma &= \{X \in \mathfrak{g} \mid \gamma X \gamma^{-1} = X\} \end{aligned}$$

$$dim(G_\gamma) \underset{\substack{\uparrow \\ \sigma_\gamma : G \longrightarrow G \\ x \mapsto \gamma x \gamma^{-1}}}{=} dim(\mathfrak{g}_\gamma) \underset{\substack{\uparrow \\ (c)}}{=} dim(\mathfrak{g}_\sigma) = dim(G_\sigma)$$

$G \circlearrowright S$, $g.\gamma = g\gamma g^{-1}$.

$\phi_\gamma : G \longrightarrow G.\gamma$, $\phi_\gamma(g) = g\gamma g^{-1}$

$\phi_\gamma^{-1}(\gamma) = G_\gamma \Rightarrow dim(G.\gamma) = dim(G) - dim(G_\gamma)$

All orbits have same dimension

$\Rightarrow$ all orbits are closed.

$\therefore \chi(G) = s(G.s^{-1}) \subseteq G$ is closed.

$\square$

$Z_G(s) = \{x \in G \mid xs = sx\} \subseteq G$ centralizer of $s \in G$.

<u>Cor</u> $s \in G$ semi-simple.

(1) $C = \{xsx^{-1} \mid x \in G\} \subseteq G$ closed.

(2) $G \longrightarrow C$, $x \mapsto xsx^{-1}$ is separable.

(3) $\mathfrak{g} = (Ad(s)-1)\mathfrak{g} \oplus L(Z_G(s))$

<u>Proof</u>

$\sigma : G \longrightarrow G$, $\sigma(x) = s^{-1}xs$ semi-simple automorphism.

$\mathcal{X} : G \longrightarrow G$, $\mathcal{X}(x) = \sigma(x)x^{-1} = s^{-1}xsx^{-1}$.

$\mathcal{X}(G) \subseteq G$ closed, $\mathcal{X} : G \longrightarrow \mathcal{X}(G)$ separable.

$C = s\mathcal{X}(G)$ closed, $x \mapsto xsx^{-1} = s\mathcal{X}(x)$ separable.

$G_\sigma = \{s^{-1}xs = x\} = Z_G(s)$.

$$\begin{aligned}
L(Z_G(s)) &= L(G_\sigma) = \mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\} \\
&= \{X \in T_eG \mid s^{-1}Xs = X\} = \{X \in T_eG \mid sXs^{-1} = X\} \\
&= Ker(Ad(s)-1) \subseteq \mathfrak{g}.
\end{aligned}$$

s semi-simple $\Rightarrow$ $Ad(s)-1$ semi-simple

$\Rightarrow \mathfrak{g} = Im(Ad(s)-1) \oplus Ker(Ad(s)-1)$

□

Action by automorphisms

D diagonalizable LAG, G connected LAG.

$D \curvearrowright G$ by automorphisms:

- G D-variety.
- $G \xrightarrow{\cong} G$, $g \mapsto d.g$ group hom. $\forall d \in D$.

Differentiate: $T_eG \longrightarrow T_eG$, $X \mapsto d.X$

$\alpha : D \longrightarrow GL(k[G])$ locally rational rep.

$(\alpha(d).f)(x) = f(d^{-1}.x)$.

D diagonalizable $\Rightarrow$ $\alpha(d) : k[G] \longrightarrow k[G]$ semi-simple

$\Rightarrow$ $g \mapsto d.g$ semi-simple automorphism.

Def $Z_G(D) = \{ g \in G \mid d.g = g \;\; \forall d \in D \} = \bigcap_{d \in D} G_d$

$Z_{\mathfrak{g}}(D) = \{ X \in \mathfrak{g} \mid d.X = X \;\; \forall d \in D \} = \bigcap_{d \in D} \mathfrak{g}_d$

Note: $L(G_d) = \mathfrak{g}_d$, $L(Z_G(D)) \subseteq Z_{\mathfrak{g}}(D)$.

Cor $L(Z_G(D)) = Z_{\mathfrak{g}}(D)$

Proof

If $D \curvearrowright \mathfrak{g}$ trivial: $L(G_d) = \mathfrak{g}_d = \mathfrak{g} \Rightarrow G_d = G$.

$Z_G(D) = G$, $Z_{\mathfrak{g}}(D) = \mathfrak{g}$.

Otherwise choose $d \in D$ such that $\mathfrak{g}_d \subsetneq \mathfrak{g}$.

D commutative $\Rightarrow$ D acts on G_d, G_d°.

$Z_G(D) = Z_{G_d}(D) \supseteq Z_{G_d^\circ}(D)$, $Z_{\mathfrak{g}}(D) = Z_{\mathfrak{g}_d}(D)$.

Induction on $\dim(G)$ $\Rightarrow$

$\dim Z_{\mathfrak{g}_d}(D) = \dim Z_{G_d^\circ}(D) = \dim Z_{G_d}(D)$

□

$G_s = \{x \in G \mid x \text{ semi-simple}\}$

<u>Commutator</u>: $(x,y) = xyx^{-1}y^{-1}$.

$G \neq e$ <u>nilpotent</u> $\Leftrightarrow$ $Z(G) \neq e$ and $G/Z(G)$ nilpotent

$\Leftrightarrow \exists\, n \in \mathbb{N} : \forall x_1, \dots, x_n \in G : (x_1, (x_2, (\cdots (x_{n-1}, x_n) \cdots))) = e$.

<u>Cor</u> G connected nilpotent LAG

$\Rightarrow G_s \subseteq Z(G)$ subgroup.

<u>Proof</u>

$s \in G$ semi-simple.

$\sigma = \mathrm{Int}(s) : G \xrightarrow{\cong} G$

$\chi(x) = \sigma(x) x^{-1} = sxs^{-1}x^{-1} = (s,x)$.

$\chi^n(x) = (s,(s,(\cdots,(s,x)\cdots))) = e$.

$\chi^n(G) = e$.

$d\chi_e = Ad(s) - 1$.

$(Ad(s)-1)^n = (d\chi_e)^n = 0$.

s semi-simple $\Rightarrow Ad(s)-1$ ss.

$\therefore Ad(s) = 1$.

$L(G_\sigma) = \mathfrak{g}_\sigma = \mathrm{Ker}(Ad(s)-1) = \mathfrak{g}$

$\Rightarrow G_\sigma = G \Rightarrow \sigma$ trivial $\Rightarrow s \in Z(G)$.

product of commuting ss is ss $\Rightarrow G_s \subseteq Z(G)$ subgroup.

□

Ideal of a closed subgroup

G LAG, $H \subseteq G$ closed subgroup.
$I(H) \subseteq k[G]$ ideal of H.

Lemma $H = \{ g \in G \mid \rho(g).I(H) = I(H) \}$

Proof

$\subseteq$: $g \in H$, $f \in I(H)$, $h \in H \Rightarrow (\rho(g).f)(h) = f(hg) = 0$

$\supseteq$: $g \in \text{RHS}$, $f \in I(H) \Rightarrow f(g) = (\rho(g).f)(e) = 0$.

$\square$

Lemma $T_eH = \{ X \in T_eG \mid \overline{X}.I(H) \subseteq I(H) \}$

Proof

$\mathcal{D}_{G,H} = \{ D \in \mathrm{Der}_k(k[G], k[G]) \mid D.I(H) \subseteq I(H) \}$

$$\begin{tikzcd}
L(G) \cap \mathcal{D}_{G,H} \arrow[r, "\alpha_G"] \arrow[d, "\cong"'] & T_eG \\
L(H) \arrow[r, "\alpha_H", "\cong"'] & T_eH \arrow[u, "\subseteq"']
\end{tikzcd}$$

Let $X \in T_eG$.

$X \in T_eH \Leftrightarrow \overline{X} \in \mathcal{D}_{G,H}$.

$\square$

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G LAG, $H \subseteq G$ closed subgroup. $\mathfrak{g} = L(G)$, $\mathfrak{h} = L(H)$.

Lemma $\exists\ W \subseteq V \subseteq k[G]$:

(1) $\rho : G \longrightarrow GL(V)$ rational rep.

(2) $H = \{ g \in G \mid \rho(g).W = W \}$

(3) $\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\rho(X).W \subseteq W \}$

Proof

$I(H) = \langle f_1, \dots, f_r \rangle \subseteq k[G]$.

$\exists\ \mathrm{Span}_k\{f_1, \dots, f_r\} \subseteq V \subset k[G]$:

$\rho : G \longrightarrow GL(V)$ rational rep.

$W = V \cap I(H)$.

$g \in G$: $g \in H \Leftrightarrow \rho(g).I(H) = I(H) \Leftrightarrow \rho(g).W = W$.

$X \in \mathfrak{g}$: $X \in \mathfrak{h} \Leftrightarrow \bar{X}.I(H) \subseteq I(H) \Leftrightarrow \bar{X}.W \subseteq W$.

$\square$

Thm $\exists$ rational rep. $\phi: G \longrightarrow GL(U)$, $0 \neq u \in U$:
$H = \{g \in G \mid \phi(g).u \in ku\}$ and
$\mathfrak{h} = \{X \in \mathfrak{g} \mid d\phi(X).u \in ku\}$.

Proof

Let $W \subseteq V \subseteq k[G]$ be as in lemma, $d = \dim(W)$.

$U = \Lambda^d V$, $0 \neq u \in \Lambda^d W \subseteq U$.

$W = \{v \in V \mid v \wedge u = 0 \in \Lambda^{d+1} V\}$ determined by u.

$\phi = \Lambda^d \rho : G \longrightarrow GL(U)$.

$x \in G$: $x \in H \Leftrightarrow \rho(x).W = W \Leftrightarrow \phi(x).u \in ku$.

$u = w_1 \wedge \cdots \wedge w_d$, $\{w_1, \dots, w_d\}$ basis of W.

$$d\phi(X).u = \sum_{i=1}^{d} w_1 \wedge \cdots \wedge d\rho(X).w_i \wedge \cdots \wedge w_d$$

If $d\rho(X).W \nsubseteq W$: $d\rho(X).w_j = w + v$, $w \in W$, $v \notin W$.

$d\phi(X).u$ "contains" $w_1 \wedge \cdots \wedge w_{j-1} \wedge v \wedge w_{j+1} \wedge \cdots \wedge w_d$.

$X \in \mathfrak{h} \Leftrightarrow d\rho(X).W \subseteq W \Leftrightarrow d\phi(X).u \in ku$.

□

Def $\phi: X \longrightarrow Y$ morphism of varieties.
ϕ is <u>separable</u> if $\forall$ conn. comp. $X' \subseteq X$:
X' is irred., $\overline{\phi(X')}$ is conn. comp. of Y,
$k(\overline{\phi(X')}) \subseteq k(X')$ separably generated.

Cor $\exists$ quasi-projective hom. G-variety X, $x \in X$:

(1) $H = G_x = \{ g \in G \mid g.x = x \}$

(2) $\psi : G \longrightarrow X$, $g \mapsto g.x$ separable.

Proof

Let $\phi : G \longrightarrow GL(U)$, $0 \neq u \in U$ be as in Theorem.

$\mathbb{P}(U) = \{ [v] = kv \mid 0 \neq v \in U \}$

$G \circlearrowright \mathbb{P}(U)$, $g.[v] = [\phi(g).v]$

$x = [u]$, $X = G.x \subseteq \mathbb{P}(U)$. $H = G_x$ is clear.

$$\begin{tikzcd}
\psi : G \arrow[r, "\phi"] & GL(U) \arrow[r, "A \mapsto A.u"] \arrow[d, phantom, "\cap"] & U \setminus \{0\} \arrow[r, "\pi"] \arrow[d, phantom, "\cap"] & \mathbb{P}(U) \\
 & \mathrm{End}(U) \arrow[r, "\text{linear}"] & U &
\end{tikzcd}$$

$$\begin{tikzcd}
d\psi_e : T_eG \arrow[r, "d\phi"] & \mathrm{End}(U) \arrow[r, "A \mapsto A.u"] & U \arrow[r] & U/ku. \\
X \arrow[r, mapsto] & d\phi(X) \arrow[rr, mapsto] & & d\phi(X).u + ku
\end{tikzcd}$$

$\mathrm{Ker}(d\psi_e) = \mathfrak{h} \Rightarrow$

$\dim d\psi_e(\mathfrak{g}) = \dim G - \dim H = \dim X$.

$d\psi_e : T_eG \twoheadrightarrow T_xX$ surjective.

$\therefore$ $\psi : G \longrightarrow X$ separable.

$\square$

Lemma $h : X \longrightarrow Y$ surjective open map of top. spaces. $Y' \subseteq Y$ subset. $h^{-1}(Y') \subseteq X$ closed $\Rightarrow$ $Y' \subseteq Y$ closed.

Proof: $Y - Y' = h(X - h^{-1}(Y'))$ is open. $\square$

Lemma $F \subseteq E$ separably gen. extension.
$a \in E$ alg. over F $\Rightarrow$ a separable over F.

Proof

Choose tr. basis $\{b_1, \dots, b_n\}$ of E/F s.t.
E/E' separable, $E' = F(b_1, \dots, b_n)$.

$p(T) \in E'[T]$ min. poly of a / E'.
Then $p(T)$ has distinct roots.
$q(T) \in F[T]$ min. poly of a / F.
$p(T) \mid q(T)$ in $E'[T]$ $\Rightarrow$ $p(T) \in \overline{F}[T] \cap E'[T] = F[T]$.
$\therefore$ $q(T) = p(T)$ has distinct roots.
$\square$

Prop X hom. G-variety, $x \in X$.
Assume $\psi: G \longrightarrow X$, $\psi(g) = g.x$ separable.
$U \subseteq X$ open, $f: U \longrightarrow k$ any function.
Then $f \in \mathcal{O}_X(U) \Leftrightarrow f\psi \in \mathcal{O}_G(\psi^{-1}(U))$.

Proof of $\Leftarrow$:
WLOG G connected.

$\Gamma = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{A}^1$ subset.

$\psi: G \longrightarrow X$ equivariant of hom. G-varieties
$\Rightarrow \psi \times 1: G \times \mathbb{A}^1 \longrightarrow U \times \mathbb{A}^1$ is open.

$f\psi$ regular fcn $\Rightarrow$
$(\psi \times 1)^{-1}(\Gamma) = \{(g, f\psi(g)) \mid g \in \psi^{-1}(U)\} \subseteq \psi^{-1}(U) \times \mathbb{A}^1$ closed
$\Rightarrow \Gamma \subseteq U \times \mathbb{A}^1$ closed. (Lemma)
$\therefore \Gamma$ is a variety.

$$G \xrightarrow{(\psi, f\psi)} \Gamma \xrightarrow{pr_1} U$$

$$k(G) \supseteq k(\Gamma) \supseteq k(X)$$

$k(G)/k(X)$ separably gen. $\Rightarrow k(\Gamma)/k(X)$ separable.
(Lemma)

$\Gamma \longrightarrow U$ bijective & separable $\Rightarrow$ birational.

U non-singular.
Zariski's Main Thm. $\Rightarrow pr_1: \Gamma \xrightarrow{\cong} U$ iso.
$\therefore f: U \xrightarrow{\cong} \Gamma \xrightarrow{pr_2} \mathbb{A}^1$ regular.
$\square$

Quotients

X SWF. $\sim$ equiv. rel. on X.

$\pi : X \longrightarrow X'$ morphism.

Def π respects $\sim$ if $x_1 \sim x_2 \Rightarrow \pi(x_1) = \pi(x_2)$.

π is a universal morphism respecting $\sim$ if

$\forall$ morphism of SWF $f : X \longrightarrow Y$ respecting $\sim$

$\exists !$ morphism $\tilde{f} : X' \longrightarrow Y$ s.t. $f = \tilde{f}\pi$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\
& X' \arrow[ur, "\exists !", "\tilde{f}"'] &
\end{tikzcd}$$

Construction

$X' = X/\sim$ as set. $\pi : X \longrightarrow X/\sim$

$U \subseteq X'$ open $\Leftrightarrow$ $\pi^{-1}(U) \subseteq X$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow$ $f\pi : \pi^{-1}(U) \longrightarrow k$ regular.

Exer $\pi : X \longrightarrow X/\sim$ univ. morphism respecting $\sim$.

X SWF, $X \circlearrowleft G$ right action.

$X/G = X/\sim$, $x_1 \sim x_2 \Leftrightarrow x_1 . G = x_2 . G$.

Example $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$.

Example $\mathbb{A}^1/\mathbb{G}_m = \{0, *\}$ SWF.

$\{0\}$ closed, $\{*\}$ open. $\mathcal{O}(\mathbb{A}^1/\mathbb{G}_m) = k$.

Def G alg. group, X G-variety.

The quotient X/G is separable if

(1) X/G alg. variety

(2) $\pi : X \longrightarrow X/G$ is separable.

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Thm G LAG, $H \subseteq G$ closed subgroup.

(1) G/H is quasi-projective.

(2) $G \longrightarrow G/H$ is separable.

(3) $\dim(G/H) = \dim(G) - \dim(H)$.

Proof

Let (X, x) be as in Corollary:

- X quasi-projective homogeneous G-variety.
- $G_x = H$
- $\psi : G \longrightarrow X$, $g \longmapsto g.x$ is separable.

$H = G_x \Rightarrow X = G/H$ as set.

ψ open: $U \subseteq X$ open $\Leftrightarrow$ $\psi^{-1}(U) \subseteq G$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow$ $f\psi : \psi^{-1}(U) \longrightarrow k$ regular.

$\therefore$ $X = G/H$ as SWF.

$\square$

Notation:

$G/H = \{g.H \mid g \in G\}$, $\pi : G \longrightarrow G/H$, $\pi(g) = g.H$.

Fiber bundles

G LAG, $H \subseteq G$ closed subgp, $H \circlearrowright X$.

$G \times X \circlearrowleft H$, $(g,x).h = (gh, h^{-1}.x)$.

$$G \times^H X = (G \times X)/H \quad \text{SWF}$$

$$= \{[g,x] \mid g \in G,\ x \in X\} / \{[gh,x] = [g,h.x]\ \forall h \in H\}$$

$p : G \times^H X \longrightarrow G/H$ morphism

$[g,x] \longmapsto g.H$

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\ G \times^H X \arrow[r, dashed, "\exists !", "p"'] & G/H \end{tikzcd}$$

Note: $y = g.H \in G/H$

$\Rightarrow X \longrightarrow p^{-1}(y)$, $x \longmapsto [g,x]$ bijective morphism.

Local section of π:

$$\begin{tikzcd} G \arrow[r, "\pi"] & G/H \\ & U \neq \emptyset \text{ open} \arrow[u, phantom, sloped, "\subseteq"] \arrow[ul, "\sigma"] \end{tikzcd}$$

$$\pi \sigma = 1_U.$$

Note: $\exists$ local section $\Rightarrow$ G/H "covered" by sections.

$\sigma^g : g.U \longrightarrow G$, $\sigma^g(y) = g\sigma(g^{-1}.y)$.

$\pi \sigma^g = 1_{g.U}$

Example G/H projective $\Rightarrow$ $\exists$ local sections.

Prop Assume $\pi : G \longrightarrow G/H$ has local sections. Then $G \times^H X$ is a variety and $p : G \times^H X \longrightarrow G/H$ is locally trivial with fiber X.

$$\begin{tikzcd}
G \times^H X \arrow[r, "p"] & G/H \\
p^{-1}(U) \cong U \times X \arrow[u, phantom, "\cup" description] \arrow[r, "pr_1"] & U \text{ open} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

Proof

Assume $\sigma : U \longrightarrow G$ local section of π.

$$\begin{tikzcd}
U \times X \arrow[r, "\cong"] & p^{-1}(U) \\
(y, x) \arrow[r, mapsto] & {[\sigma(y), x]} \\
\left(\pi(g), \underbrace{\sigma(\pi(g))^{-1} g}_{\in H} . x\right) & {[g, x]} \arrow[l, mapsto]
\end{tikzcd}$$

□

Cor π has local sections ⇔ π locally trivial (Fiber H)

Proof: $G = G \times^H H \xrightarrow{\pi = p} G/H$. □

Example $H \longrightarrow GL(V)$ rational rep. ⇒

$G \times^H V \longrightarrow G/H$ vector bundle.

Thm $\phi : X \longrightarrow Y$ morphism of irred. vars.

ϕ is separable (and dominant) $\Leftrightarrow$

$\exists$ dense open $U \subseteq X$: $\forall x \in U$: $d\phi_x : T_x X \twoheadrightarrow T_{\phi(x)} Y$ surj.

Cor $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$, X, Y, Z irred.

ϕ and ψ separable $\Rightarrow$ $\psi\phi$ separable $\Rightarrow$ ψ separable.

Cor $\phi : X \longrightarrow X'$, $\psi : Y \longrightarrow Y'$, $\phi \times \psi : X \times Y \longrightarrow X' \times Y'$.

ϕ and ψ separable $\Leftrightarrow$ $\phi \times \psi$ separable.

Quotients of products

X, Y irred. varieties, $\sim_X$, $\sim_Y$ equiv rels.

$\sim \; = (\sim_X, \sim_Y)$ on $X \times Y$.

Bijective morphism of SWF:

$$(X \times Y)/\sim \longrightarrow (X/\sim_X) \times (Y/\sim_Y)$$

Prop Assume $X/\sim_X$, $Y/\sim_Y$, $(X\times Y)/\sim$ are varieties,
$X/\sim_X$ and $Y/\sim_Y$ are normal,
and $X \longrightarrow X/\sim_X$ and $Y \longrightarrow Y/\sim_Y$ separable.
Then $(X \times Y)/\sim \; \cong (X/\sim_X) \times (Y/\sim_Y)$.

Proof

$$\begin{tikzcd}
X \times Y \arrow[d] \arrow[dr, "\text{separable}"] & \\
(X \times Y)/\sim \arrow[r, "\substack{\text{bijective} \\ \text{separable,} \\ \text{normal target.}}"'] & (X/\sim_X) \times (Y/\sim_Y)
\end{tikzcd}$$

Zariski $\Rightarrow$ iso.

□

Prop G LAG, $H \subseteq G$ closed normal subgroup.
Then G/H is a LAG.

Proof

$$\begin{tikzcd}
G \times G \arrow[r, "(x,y) \mapsto xy^{-1}"] \arrow[d] & G \arrow[d] & \\
G \times G / H \times H \arrow[r, dashed, "\exists!"] \arrow[d, equal] & G/H & xy^{-1}.H \\
G/H \times G/H \arrow[ur] & (x.H, y.H) \arrow[ur, mapsto] &
\end{tikzcd}$$

$\therefore$ G/H alg. group.

Choose $\phi : G \longrightarrow GL(V)$, $0 \neq v \in V$:

$H = \{ g \in G \mid \phi(g).v \in kv \}$

$\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\phi(X).v \in kv \}$

Given character $\chi : H \longrightarrow \mathbb{G}_m$:

$V_\chi = \{ u \in V \mid \phi(h).u = \chi(h) u \ \forall h \in H \}$

$g \in G \Rightarrow g.V_\chi = V_{g.\chi}$ where $(g.\chi)(h) = \chi(g^{-1}hg)$:

$g.u \in g.V_\chi$: $h.(g.u) = gg^{-1}hg.u = \chi(g^{-1}hg)\, g.u$

Note: $v \in V_\chi$ for some χ.

Note: $\sum V_\chi = \bigoplus V_\chi$.

WLOG: $V = \bigoplus_\chi V_\chi$.

Def: $W = \bigoplus_\chi \mathrm{End}_k(V_\chi) \subseteq \mathrm{End}_k(V)$.

Note: Given $\alpha : V \longrightarrow V$ linear:

$\alpha f = f \alpha \ \forall f \in W \Leftrightarrow$

$\alpha : V_\chi \longrightarrow V_\chi$ mult. by scalar $\forall \chi$

Def: $\psi : G \longrightarrow GL(W)$, $\psi(g).f = \phi(g) f \phi(g)^{-1}$, $f \in W$.

$\mathrm{Ker}(\psi) = H$: $\psi(g) = 1_W \Leftrightarrow \phi(g) f = f \phi(g) \ \forall f \in W$

$\Rightarrow \phi(g).v \in kv \Rightarrow g \in H$

$$\begin{tikzcd}
G \arrow[rr, "\psi"] \arrow[dr, Rightarrow, "\pi"'] & & GL(W) \\
& G/H \arrow[ur, dashed, "\exists!", "\lambda"'] &
\end{tikzcd}$$

λ injective hom. of alg. groups.

$\lambda(G/H) = \psi(G) \subseteq GL(W)$ closed subgroup.

$$\begin{tikzcd}
T_eG \arrow[rr, "d\psi"] \arrow[dr, "d\pi_e"'] & & T_e(\psi(G)) \\
& T_e(G/H) \arrow[ur, "d\lambda_e"'] &
\end{tikzcd}$$

Exer: $X \in \mathfrak{g}$, $f \in W \Rightarrow d\psi(X).f = d\phi(X) f - f\, d\phi(X)$

$\mathrm{Ker}(d\psi) \subseteq T_eH$:

$X \in \mathrm{Ker}(d\psi) \Leftrightarrow d\phi(X) f = f\, d\phi(X) \ \forall f \in W$

$\Rightarrow d\phi(X).v \in kv \Rightarrow X \in T_eH$

$\dim d\psi(T_eG) = \dim G - \dim \mathrm{Ker}(d\psi)$

$\geq \dim G - \dim H = \dim G/H = \dim \psi(G)$.

$\lambda : G/H \longrightarrow \psi(G)$ bijective separable group hom.

$\Rightarrow$ isomorphism.

$\square$

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Def X variety. X is complete if $\forall$ varieties Z:

$\pi_2 : X \times Z \longrightarrow Z$ is a closed map.

Example: $\mathbb{A}^1$ is not complete. $\pi_2 : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$.

$\pi_2(Z(xy-1)) = \mathbb{A}^1 \smallsetminus \{0\}$ not closed.

Properties: Assume X complete.

(1) $Y \subseteq X$ closed $\Rightarrow$ Y is complete.

(2) Y complete $\Rightarrow$ $X \times Y$ is complete.

(3) $\phi : X \longrightarrow Y$ morphism $\Rightarrow$ $\phi(X)$ closed & complete.

Pf: $X \xrightarrow{\gamma} X \times Y \xrightarrow{\pi_Y} Y$, $\gamma(X) \subseteq X \times Y$ closed.

(4) X connected $\Rightarrow$ $\mathcal{O}_X(X) = k$.

Pf: $f : X \longrightarrow \mathbb{A}^1$, $f(X)$ closed, complete, connected.

(5) X affine $\Rightarrow$ X is finite.

Thm $\mathbb{P}^n$ is complete.

Thm X complete, C non-singular curve, $P \in C$.

Any morphism $\phi : C \smallsetminus \{P\} \longrightarrow X$ extends to $\phi : C \longrightarrow X$.

Lemma $\phi : X \times Y \longrightarrow Z$ morphism, X, Y, Z irred., X complete. For $y \in Y$ set $\phi_y(x) = \phi(x, y)$.

Assume $\exists\, a \in Y$: $\phi_a : X \longrightarrow Z$ constant.

Then ϕ_y constant $\forall y \in Y$.

Proof

$\Gamma = \{(x, y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq X \times Y \times Z$ closed & irred.
(graph)

$C = \{(y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq Y \times Z$ closed & irred.
(image of Γ, X complete)

$\therefore$ C irred. variety.

$\pi_Y : C \longrightarrow Y$ surjective morphism.

ϕ_a constant $\Leftrightarrow$ $\pi_Y^{-1}(a) =$ point.

$\therefore$ $\dim(C) = \dim(Y)$.

$x \in X$: $C_x = \{(y, \phi(x,y)) \mid y \in Y\} \subseteq C$ closed & irred.
(graph)

$C_x \cong Y \Rightarrow \dim(C_x) = \dim(C) \Rightarrow C_x = C$.

$\pi_Y : C \xrightarrow{\cong} Y$ isomorphism.

$\pi_Y^{-1}(y) =$ point $\Rightarrow$ ϕ_y constant $\forall y \in Y$.

□

Cor G complete alg. group $\Rightarrow$ G is commutative.

Proof $\phi : G \times G \longrightarrow G$, $\phi(x, y) = x y x^{-1}$.

ϕ_e constant $\Rightarrow$ ϕ_y const. $\forall y \in G$.

□

Exer $\mathbb{P}^n$ is not an alg. group for $n \geq 1$.

Lemma $\phi : X \longrightarrow Y$ bijective equivariant morphism of homogeneous G-varieties.
Then X complete $\Leftrightarrow$ Y complete.

Proof: $\phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ homeomorphism $\forall Z$. $\square$

Def G LAG, $P \subseteq G$ closed subgroup.
P is *parabolic* if G/P is complete.

Def $P \subseteq G$ subgroup, Z set. $A \subseteq G \times Z$ is *P-stable* if $(g, z) \in A$, $p \in P$ $\Rightarrow$ $(gp, z) \in A$.

Lemma $P \subseteq G$ is parabolic $\Leftrightarrow$
$\forall$ var. Z $\forall$ $A \subseteq G \times Z$ closed P-stable: $\pi_Z(A) \subseteq Z$ closed.

Proof

$$\begin{tikzcd}
G \times Z \arrow[r] \arrow[d, "\pi"'] & Z \\
G/P \times Z \arrow[ur] &
\end{tikzcd}$$

$A \subseteq G/P \times Z$ closed

$\updownarrow$

$\pi^{-1}(A) \subseteq G \times Z$ closed, P-stable.

Same image in Z.

$\square$

Lemma G LAG, $Q \subseteq P \subseteq G$ closed subgroups.
$Q \subseteq G$ parabolic $\Leftrightarrow$ $Q \subseteq P$ and $P \subseteq G$ parabolic.

Proof

$\Rightarrow$: $P/Q \subseteq G/Q$ closed and $G/Q \twoheadrightarrow G/P$.

$\Leftarrow$:
$$\begin{tikzcd} P \times G \times Z \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\pi"'] & G \times Z \arrow[r, "\pi_Z"] & Z \end{tikzcd}$$

$$\alpha(p, g, z) = (gp, z), \quad \pi(p, g, z) = (g, z).$$

Let $A \subseteq G \times Z$ be closed, Q-stable.

$\alpha^{-1}(A) \subseteq P \times G \times Z$ closed, Q-stable.

$Q \subseteq P$ parab. $\Rightarrow$ $\pi(\alpha^{-1}(A)) \subseteq G \times Z$ closed, P-stable.

$P \subseteq G$ parab. $\Rightarrow$ $\pi_Z(A) = \pi_Z(\pi(\alpha^{-1}(A))) \subseteq Z$ closed.

$\square$

Cor $P \subseteq G$ parabolic $\Leftrightarrow$ $P^\circ \subseteq G^\circ$ parabolic.

Proof

$$\begin{array}{ccc} P & \subset & G \\ \cup & & \cup \\ P^\circ & \subset & G^\circ \end{array}$$

Note:
$G^\circ \subseteq G$ parabolic.

$\square$

Thm G connected LAG. TFAE:

(a) G has no proper parabolic subgroups.

(b) $G \circlearrowright X$, X complete $\Rightarrow X^G \neq \emptyset$.

(c) $G \subseteq GL_n \Rightarrow \exists\, x \in GL_n : xGx^{-1} \subseteq B_n$ (upper Δ)

(d) G is solvable.

Proof

(a)$\Rightarrow$(b): Choose closed orbit $\Omega \subseteq X$.

$x \in \Omega$, $G_x \subseteq G$ isotropy group.

$$G/G_x \longrightarrow \Omega, \quad g.G_x \longmapsto g.x$$

bijective morphism of hom. G-varieties.

X complete $\Rightarrow \Omega$ complete $\Rightarrow G/G_x$ complete

$\Rightarrow G_x \subseteq G$ parabolic $\Rightarrow G_x = G \Rightarrow x \in X^G$.

(b)$\Rightarrow$(c): $G \subseteq GL(V)$ closed subgroup.

$$Fl(V) = \{ V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V) \mid \dim(V_i) = i \}$$

Exer: $Fl(V)$ projective variety.

(b) $\Rightarrow \exists$ G-stable flag $V_\bullet \subseteq V \Rightarrow$ (c).

(c)$\Rightarrow$(d): B_n is solvable.

(d)$\Rightarrow$(a): Choose minimal parabolic $P \subseteq G$.

$P \subsetneq G \Rightarrow G/P$ not affine $\Rightarrow P \subseteq G$ not normal $\Rightarrow (G,G) \not\subseteq P$.

$$(G,G)/(G,G)\cap P \longrightarrow (G,G)P/P$$

bijective equiv. morphism of homogeneous (G,G)-varieties.

$P \subsetneq (G,G)P$ parab. $\Rightarrow (G,G) \cap P \subsetneq (G,G)$ parab.

Induction on $\dim(G) \Rightarrow \lightning$

$\square$

Lemma $H \subseteq G$ connected solvable, $P \subseteq G$ parabolic.

$\exists g \in G : gHg^{-1} \subseteq P$.

Proof

$H \circlearrowright G/P$. Let $g.P \in (G/P)^H$ be a fixed point.

$\forall h \in H : hg.P = g.P \Rightarrow g^{-1}Hg \subseteq P$.

$\square$

Def G LAG. A Borel subgroup of G is a maximal closed connected solvable subgroup.

Thm G LAG, $B \subseteq G$ closed subgroup. TFAE:

(1) $B \subseteq G$ Borel

(2) $B \subseteq G$ min. parabolic.

(3) $B \subseteq G$ connected solvable parabolic.

Proof

(3) $\Rightarrow$ (1) + (2): Lemma.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3):

Choose $B \subseteq G$ Borel, $P \subseteq G$ min. parabolic.

WLOG $B \subseteq P$.

$P = P^\circ$ is connected, contains no proper parabolic.

$\Rightarrow$ P closed connected solvable.

$\therefore$ $B = P$ satisfy (3).

$\square$

Cor All Borel subgroups are conjugate.

Cor $\phi : G \twoheadrightarrow G'$ surjective homomorphism of LAGs.

$P \subseteq G$ Borel/parabolic $\Rightarrow$ $\phi(P) \subseteq G'$ Borel/parabolic.

Proof

$G/P \twoheadrightarrow G'/\phi(P)$. $P \subseteq G$ parab. $\Rightarrow$ $\phi(P) \subseteq G'$ parab.

P Borel $\Rightarrow$ $\phi(P)$ connected solvable parabolic.

$\square$

Cor G connected, $B \subseteq G$ Borel $\Rightarrow Z(G)^\circ \subseteq Z(B) \subseteq Z(G)$.

Proof

$Z(G)^\circ \subseteq G$ closed connected solvable

$\Rightarrow Z(G)^\circ \subseteq B'$, B' Borel

$\Rightarrow Z(G)^\circ \subseteq gBg^{-1}$, $g \in G$

$\Rightarrow Z(G)^\circ = g^{-1} Z(G)^\circ g \subseteq B$.

$\therefore Z(G)^\circ \subseteq Z(B)$.

Let $g \in Z(B)$.

Well defined morphism:

$$G/B \longrightarrow G, \quad x \longmapsto xgx^{-1}$$

G/B irred. and complete, G affine

$\Rightarrow$ morphism is constant.

$\therefore xgx^{-1} = g \;\; \forall x \in G \quad \Rightarrow \quad g \in Z(G)$.

□

Lemma $G \neq e$ connected nilpotent LAG $\Rightarrow Z(G)^\circ \neq e$.

Proof

$G_0 = G$, $G_{i+1} = (G, G_i)$ lower central series.

G nilpotent $\Leftrightarrow G_n = e$ for some n.

Choose $n \geq 1$ such that $G_{n-1} \neq e$, $G_n = e$.

$(G, G_{n-1}) = e \Rightarrow G_{n-1} \subseteq Z(G)$.

G_{n-1} connected $\Rightarrow G_{n-1} \subseteq Z(G)^\circ$.

□

Cor G LAG, $B \subseteq G$ Borel. B nilpotent $\Rightarrow B = G^\circ$

Proof

WLOG: G connected.

Assume $B \subsetneq G$.

G/B not affine $\Rightarrow B \neq e \Rightarrow e \neq Z(B)^\circ \subseteq Z(G)$.

$\therefore Z(B)^\circ \trianglelefteq G$ normal.

$B/Z(B)^\circ \subsetneq G/Z(B)^\circ$ proper nilpotent Borel subgp.

Induction on $\dim(G) \Rightarrow \lightning$

$\square$

Cor G connected nilpotent LAG.

(1) G_s and G_u are closed connected subgroups.

(2) $G_s \subseteq G$ is a central torus.

(3) $\mu : G_s \times G_u \xrightarrow{\cong} G$ iso. of alg. groups.

Proof

Already proved: $G_s \subseteq Z(G)$ abstract subgroup.

$G \subseteq GL(V)$ closed subgroup.

$V = \bigoplus_{\chi} V_\chi$, $\chi : G_s \longrightarrow \mathbb{G}_m$ char. of abstract groups.

$G.V_\chi = V_\chi \quad \forall \chi$.

$G \circlearrowright Fl(V_\chi)$, $Fl(V_\chi)^G \neq \emptyset$.

$\therefore \exists\, G \subseteq GL_n$ such that $G \subseteq B_n$ and $G_s = G \cap D_n$.

Note: $G_u = G \cap U_n$.

$$G_s \times G_u \underset{\text{project}}{\overset{\mu}{\rightleftarrows}} G$$ iso. of alg. groups (since $G_s \subseteq Z(G)$.)

$\square$

Note: G LAG. G diagonalizable $\Leftrightarrow$
G commutative & all elts. semi-simple.

Proof $\Leftarrow$: $G \subseteq GL_n$ closed subgroup.
(2.4.2) $\Rightarrow \exists x \in GL_n : xGx^{-1} \subseteq D_n$.

Cor G connected solvable LAG.

(1) $(G,G) \subseteq G$ closed connected unipotent normal subgp.

(2) $G_u \subseteq G$ closed connected unipotent normal subgp.

(3) G/G_u is a torus.

Proof

$(G,G) \subseteq G$ closed connected normal.

WLOG: $G \subseteq B_n \subseteq GL_n$

$(G,G) \subseteq (B_n, B_n) = U_n \Rightarrow (G,G)$ unipotent.

$G_u = G \cap U_n \Rightarrow G_u \subseteq G$ closed unipotent normal.

$G/G_u \xrightarrow{\phi} B_n/U_n \cong D_n$ injective $\Rightarrow G/G_u$ commutative.

All $x \in G/G_u$ semi-simple:
$$\phi(x_u) = \phi(x)_u = e \Rightarrow x_u = e \Rightarrow x = x_s.$$

G/G_u connected $\Rightarrow G/G_u$ torus (by Note).

G_u connected:

$G_u^\circ \triangleleft G$ normal, $G_u/G_u^\circ \xrightarrow{\cong} (G/G_u^\circ)_u$ iso. of finite groups.

Show: G connected solvable, G_u finite $\Rightarrow G_u = e$.

$G_u \triangleleft G$ normal & finite $\Rightarrow G_u \subseteq Z(G)$.

($y \in G_u$. $G \longrightarrow G_u$, $x \longmapsto xyx^{-1}$ must be constant.)

G/G_u commutative $\Rightarrow G/Z(G)$ commutative
$\Rightarrow G$ nilpotent $\Rightarrow G_u$ connected.

□

Def G connected solvable LAG. A maximal torus of G is a subtorus $T \subseteq G$ with $\dim(T) = \dim(G/G_u)$.

Lemma G connected solvable LAG, $T \subseteq G$ max torus. Then $\mu : T \times G_u \xrightarrow{\cong} G$ isomorphism of varieties.

Proof

$T \times G_u \circlearrowright G, \quad (t,u).x = txu^{-1}$.

Isotropy group of $e \in G$: $(T \times G_u)_e = T \cap G_u = e$.

$\pi : T \times G_u \longrightarrow G, \quad \pi(t,u) = tu^{-1}$

bijective equiv. morphism of hom. varieties.

$d\pi_{(e,e)} : L(T) \oplus L(G_u) \longrightarrow L(G), \quad (X,Y) \longmapsto X - Y$

X semi-simple and Y nilpotent (in $\mathrm{End}_k(k[G])$).

$\Rightarrow d\pi_{(e,e)}$ injective $\Rightarrow d\pi_{(e,e)}$ bijective.

□

Cor G connected solvable, $T \subseteq G$ max. torus

$\Rightarrow T \xrightarrow{\cong} G/G_u$ isomorphism.

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Lemma G connected solvable LAG, G not a torus.
$\exists$ closed normal $N \lhd G$: $N \cong \mathbb{G}_a$ and $N \subseteq Z(G_u)$.

Proof

G not torus $\Rightarrow$ $G_u \neq e$.

$G_u \lhd G$ normal $\Rightarrow$ $Z(G_u)^\circ \lhd G$ normal.

$G_u \neq e$ connected nilpotent $\Rightarrow$ $Z(G_u)^\circ \neq e$.

$p = \mathrm{char}(k)$.

$p > 0$: $L(G_u)$ unipotent $\Rightarrow$ $L(G_u)^{p^r} = e$ for some $r > 0$.

$\therefore$ $\exists$ $H \lhd G$ closed normal connected, $e \neq H \subseteq Z(G_u)$.
$p > 0 \Rightarrow H^p = e$.

H connected elementary unipotent.

$H \cong \mathbb{G}_a^m$ for some $m \geq 1$.

$m = 1$: Take $N = H$. Assume $m > 1$.

$G \circlearrowright H$, $g.h = ghg^{-1}$

$G \longrightarrow GL(k[H])$ locally rational. $(g.f)(h) = f(g^{-1}hg)$.

$\mathcal{A} \subseteq k[H]$ additive functions.

$G \longrightarrow GL(\mathcal{A})$ locally rational.

G_u acts trivially on H: $G \longrightarrow G/G_u \longrightarrow GL(\mathcal{A})$

G/G_u torus $\Rightarrow$ $\exists$ $0 \neq f \in \mathcal{A}$: $g.f \in kf$ $\forall g \in G$.

$f: H \longrightarrow \mathbb{G}_a$ group hom.

$H' = (\mathrm{Ker}\, f)^\circ \cong \mathbb{G}_a^{m-1}$.

$H' \lhd G$ normal: $g \in G$, $h \in \mathrm{Ker}(f) \Rightarrow$
$f(ghg^{-1}) = (g^{-1}.f)(h) = 0$

Induction on m $\Rightarrow$ $\exists N$.

$\square$

Note $\mathbb{G}_m = k^\times \subseteq \mathbb{G}_a = k$.

Standard action: $\mathbb{G}_m \times \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad t.x = tx$

Exer

- $\mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$
- G LAG, $G \times \mathbb{G}_a \longrightarrow \mathbb{G}_a$ action by automorphisms.
  $\exists$ character $\alpha : G \longrightarrow \mathbb{G}_m : \quad g \cdot x = \alpha(g) x$

Note $\mathrm{char}(k) = p > 0$:

$GL_2 \subsetneq \mathrm{Aut}(\mathbb{G}_a^2)$

Action by automorphisms:

$$\mathbb{G}_a \times \mathbb{G}_a^2 \longrightarrow \mathbb{G}_a^2, \quad a.(x,y) = (x + ay^p, y)$$

Not given by group hom. $\mathbb{G}_a \longrightarrow GL_2$.

Note:

G unipotent LAG, $\mathrm{char}(k) = 0 \Rightarrow G$ connected.

G/G^0 finite unipotent $\Rightarrow G/G^0 = e$.

Thm G connected solvable LAG.

(1) $s \in G$ semi-simple $\Rightarrow$ $s \in$ max. torus of G.

(2) $s \in G$ semi-simple $\Rightarrow$ $Z_G(s)$ is connected.

(3) All max. tori in G are conjugate.

Proof

Assume first $\dim(G_u) = 1$.

Then $G_u \cong \mathbb{G}_a$. Fix isomorphism $\phi : \mathbb{G}_a \xrightarrow{\cong} G_u$.

$\psi : G \longrightarrow G/G_u$ projection.

$G/G_u \circlearrowright G_u$, $\psi(g).u = gug^{-1}$.

$\exists$ character $\alpha : G/G_u \longrightarrow \mathbb{G}_m$:

$$\circledast \quad g\phi(a)g^{-1} = \phi(\alpha(\psi(g))\, a) \qquad \text{for } g \in G,\ a \in \mathbb{G}_a.$$

Assume α trivial.

Then $G_u \subseteq Z(G)$ $\Rightarrow$ $G/Z(G)$ is commutative

$\Rightarrow$ G nilpotent $\Rightarrow$ $G \cong G_s \times G_u$.

WLOG: α not trivial.

Let $s \in G$ be semi-simple. $Z = Z_G(s)$.

(5.4.2): $L(Z) = \operatorname{Ker}(\operatorname{Ad}(s) - 1) \subseteq L(G)$.

$L(G) = (\operatorname{Ad}(s) - 1) L(G) \oplus L(Z)$.

$\psi(sgs^{-1}) = \psi(g) \Rightarrow \psi \circ \operatorname{Int}(s) = \psi$

$\Rightarrow d\psi \circ \operatorname{Ad}(s) = d\psi \Rightarrow d\psi \circ (\operatorname{Ad}(s) - 1) = 0$

$\Rightarrow (\operatorname{Ad}(s) - 1) L(G) \subseteq \operatorname{Ker}(d\psi) = L(G_u)$.

$\dim(\operatorname{Ad}(s) - 1) L(G) \leq 1$

$\dim(Z) = \dim L(Z) \geq \dim(G) - 1$.

Assume $\alpha(\psi(s)) \neq 1$:

$Z \cap G_u = e$: $\phi(a) = s\phi(a)s^{-1} = \phi(\alpha(\psi(s))a) \Leftrightarrow a = 0.$

$\therefore \dim(Z) = \dim(G) - 1.$

$Z^\circ = Z^\circ/(Z^\circ)_u$ max. torus in G.

$G = Z^\circ \ltimes G_u$

$Z = Z_G(s) = Z^\circ \ltimes Z_{G_u}(s) = Z^\circ$

Conclude: $s \in G$ semi-simple, $\alpha(\psi(s)) \neq 1$

$\Rightarrow Z_G(s) \subseteq G$ max. torus.

Note: Any max. torus $T \subseteq G$ has this form:

Choose $t \in T$ s.t. $\alpha(\psi(t)) \neq 1$.

Then $T \subseteq Z_G(t) \subseteq G$, $Z_G(t)$ max. torus.

Assume $\alpha(\psi(s)) = 1$:

$G_u \subseteq Z = Z_G(s)$.

$(\mathrm{Ad}(s)-1)L(G) \subseteq L(G_u) \subseteq L(Z) = \mathrm{Ker}(\mathrm{Ad}(s)-1)$

$\mathrm{Ad}(s)-1$ semi-simple $\Rightarrow \mathrm{Ad}(s)-1 = 0$.

$L(Z) = L(G) \Rightarrow Z_G(s) = G$ is connected.

Choose $t \in G$ s.t. $\alpha(\psi(t)) \neq 1$.

$s \in Z(G) \Rightarrow s \in Z_G(t) \subseteq G$ max. torus.

Let $T, T' \subseteq G$ be max. tori, $T = Z_G(t)$, $\alpha(\psi(t)) \neq 1$.

$G = T' \ltimes G_u$. $t = t'\phi(a)$, $t' \in T'$, $a \in \mathbb{G}_a$.

$$\phi(b)t\phi(b)^{-1} = t t^{-1}\phi(b)t\phi(b)^{-1} = t'\phi(a)\phi(\alpha(\psi(t))^{-1}b)\phi(-b)$$
$$= t'\phi(a + (\alpha(\psi(t))^{-1}-1)b)$$

$\exists\, b \in \mathbb{G}_a$: $\phi(b)t\phi(b)^{-1} = t'$.

$\Rightarrow \phi(b)T\phi(b)^{-1} = Z_G(t') = T'$.

Assume $\dim(G_u) \geq 2$:

Choose $N \lhd G$ closed normal, $N \cong \mathbb{G}_a$, $N \subseteq Z(G_u)$.

$\bar{G} = G/N$. $\dim(G/G_u) = \dim(\bar{G}/\bar{G}_u)$

$s \in G$ semi-simple. $\bar{s}$ = image in $\bar{G}$.

Induction on $\dim(G)$ $\Rightarrow$ $\exists$ max. torus $\bar{T} \subseteq \bar{G}$, $\bar{s} \in \bar{T}$.

$$\begin{tikzcd} G \arrow[r] & \bar{G} \\ H \arrow[r] \arrow[u, phantom, "\cup"] & \bar{T} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

H = inverse image in G.

H connected solvable, $H_u = N$, $s \in H$.

$\exists$ max. torus $T \subseteq H$, $s \in T$.

$T \subseteq G$ also max. torus.

Let $T, T' \subseteq G$ be max. tori.

$\bar{T}, \bar{T}' \subseteq \bar{G}$ max. tori.

$\exists\, g \in G$: $\overline{gTg^{-1}} = \bar{T}'$ $\Rightarrow$ $(gTg^{-1})N = T'N$.

$T'N$ connected solvable, $(T'N)_u = N$

$\Rightarrow$ gTg^{-1} and T' are conjugate in $T'N$.

Show: $Z_G(s)$ is connected.

Choose max. torus $T \subseteq G$ with $s \in T$.

$G = T \ltimes G_u$ $\Rightarrow$ $Z_G(s) = T \ltimes Z_{G_u}(s)$.

Enough: $Z_{G_u}(s)$ is connected.

Note: Clear if $\mathrm{char}(k) = 0$ since $Z_{G_u}(s)$ is unipotent.

WLOG: $s \notin Z(G)$ $\Rightarrow$ $G_u \not\subseteq Z_G(s)$.

$G_1 = \{g \in G \mid sgs^{-1}g^{-1} \in N\} \subseteq G$ closed subgroup.

$Z_G(s) \subseteq G_1$ and $G_1/N = Z_{\bar{G}}(\bar{s})$.

Induction on $\dim(G) \Rightarrow Z_{\bar{G}}(\bar{s})$ connected.

G_1/N and N connected $\Rightarrow G_1$ connected.

If $G_1 \neq G$ then $Z_G(s) = Z_{G_1}(s)$ is connected
by induction on $\dim(G)$.

WLOG: $G_1 = G$.

$\{sus^{-1}u^{-1} \mid u \in G_u\} \subseteq N$.

LHS is closed (5.4.4 (i)) and connected.

LHS $\neq e$ since $G_u \not\subseteq Z_G(s)$.

$\therefore\ N = \{sus^{-1}u^{-1} \mid u \in G_u\}$.

<u>Enough</u>: $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is bijective.

Assume $z \in Z_{G_u}(s)$, $x, y \in N$, $zx = y \in G_u$.

$N \subseteq Z(G_u) \Rightarrow zx = xz$; $z \in Z_G(s) \Rightarrow zs = sz$.

Write $x = usu^{-1}s^{-1}$, $y = vsv^{-1}s^{-1}$, $u, v \in G_u$.

$$\underset{\substack{\uparrow\\ \text{unipotent}}}{z}\ \underset{\nwarrow}{usu^{-1}} = \underset{\substack{\uparrow\\ \text{semi-simple}}}{vsv^{-1}} \qquad \Rightarrow z = e,\ x = y.$$

This shows $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is injective.

$Z_{G_u}(s) =$ Fiber of morphism $G_u \longrightarrow N$.

$\dim Z_{G_u}(s) \geq \dim(G_u) - 1$.

$\therefore\ Z_{G_u}(s) \times N \longrightarrow G_u$ surjective.

$\square$

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Cor G connected solvable.
$H \subseteq G$ (any) subgroup whose elts. are semi-simple.

(1) $H \subseteq T$ for some max. torus $T \subseteq G$.

(2) $Z_G(H) = N_G(H)$ is connected.

Proof

$H \longrightarrow G/G_u$ injective $\Rightarrow$ H commutative.

If $H \subseteq Z(G)$ then clear (since all max. tori conjugate).

Assume $s \in H$, $s \notin Z(G)$.

$Z_G(s)$ is connected, $H \subseteq Z_G(s) \subsetneq G$.

Induction on $\dim(G) \Rightarrow$

$\exists$ max. torus $T \subseteq Z_G(s)$ with $H \subseteq T$.

$Z_G(H) = Z_{Z_G(s)}(H)$ is connected.

Let $x \in N_G(H)$, $h \in H$.

$xhx^{-1}h^{-1} \in H \cap (G,G) \subseteq H \cap G_u = e$

$\therefore x \in Z_G(H)$.

□

G LAG.

Max. torus in G: subtorus not contained in strictly larger subtorus.

Cor $\Rightarrow$ same def. when G is connected solvable.

Thm G LAG. All max. tori in G are conjugate.

Proof

$T, T' \subseteq G$ max. tori.

Choose Borel subgps. $B, B' \subseteq G$ with $T \subseteq B$, $T' \subseteq B'$.

$\exists\, g \in G: \quad g B' g^{-1} = B.$

$T,\ g T' g^{-1} \subseteq B$ max. tori.

$\exists\, b \in B: \quad b g T' g^{-1} b^{-1} = T.$

$\square$

Lemma $T \subseteq G$ max. torus, $B, B' \subseteq G$ Borel subgps, $T \subseteq B \cap B'$. Then $\exists\, n \in N_G(T): B' = n B n^{-1}.$

Proof

$\exists\, g \in G: \ B' = g B g^{-1}.$

$g^{-1} T g,\ T \subseteq B$ max. tori.

$\exists\, b \in B: \ b^{-1} g^{-1} T g b = T.$

$n = gb \in N_G(T), \quad n B n^{-1} = B'.$

$\square$

Cartan subgroup of LAG G:

$C = Z_G(T)^\circ$ where $T \subseteq G$ max. torus.

Properties

(1) C is nilpotent:

$B \subseteq C$ Borel subgp., $T \subseteq B$.

$B = T \ltimes B_u = T \times B_u$ since $T \subseteq Z(B)$.

B nilpotent $\Rightarrow$ $B = C^\circ = C$.

(2) $T = C_s$ is the only max. torus of C.

(3) $N_G(T) = N_G(C)$.

(4) $N_G(C)/C$ is finite.

Since $N_G(T)/Z_G(T)$ and $Z_G(T)/C$ are finite.

(5) $T \subseteq B \subseteq G$, B Borel $\Rightarrow$ $C \subseteq B$:

$\exists$ Borel subgp. $B' \subseteq G$ with $C \subseteq B'$.

$\exists\, n \in N_G(T)$: $B = nB'n^{-1}$.

$C = nCn^{-1} \subseteq B$.

Lemma G LAG, $S \subseteq G$ subtorus. $\exists\, s \in S$: $Z_G(s) = Z_G(S)$.

Proof

$G \subseteq GL(V)$, $V = \bigoplus V_\chi$, $\chi : S \longrightarrow \mathbb{G}_m$.

Choose $s \in S$ such that

$$0 \neq V_\chi \neq V_{\chi'} \neq 0 \Rightarrow \chi(s) \neq \chi'(s)$$

$Z_G(s) = \{g \in G \mid \forall \chi : g.V_\chi = V_\chi\} = Z_G(S)$.

□

Lemma G LAG, $T \subseteq G$ max. torus, $C = Z_G(T)^\circ$.

$\exists t \in T \ \forall g \in G: \ t \in gCg^{-1} \Rightarrow gCg^{-1} = C$.

Proof

Choose $t \in T$ such that $Z_G(t) = Z_G(T)$.

$t \in gCg^{-1} \Rightarrow g^{-1}tg \in C_s = T$.

$C = Z_G(T)^\circ \subseteq Z_G(g^{-1}tg)^\circ = g^{-1}Cg$.

□

Lemma G LAG, $H \subseteq G$ closed subgp., $X = \bigcup_{g \in G} gHg^{-1} \subseteq G$.

(1) X contains dense open $\subseteq \overline{X}$.

(2) H parabolic $\Rightarrow X = \overline{X} \subseteq G$ is closed.

(3) Assume $N_G(H)/H$ is finite and $\exists h \in H$:
h is in finitely many conjugates of H.
Then $\dim(\overline{X}) = \dim(G)$.

Proof

$A = \{(g,x) \in G \times G \mid x \in gHg^{-1}\} \subseteq G \times G$ closed.

A is H-stable: $(g,x) \in A$, $h \in H \Rightarrow (gh,x) \in A$.

$X = \pi_2(A)$, $\pi_2 : G \times G \longrightarrow G$. (1) + (2) follow from this.

$\dim(A) = \dim(G \times H)$: $G \times H \xrightarrow{\cong} A$, $(g,h) \longmapsto (g, ghg^{-1})$

Assume $N_G(H)/H$ and $\mathcal{H} = \{gHg^{-1} \mid g \in G, t \in gHg^{-1}\}$ finite.

$\mathcal{H} = \{y_1 H y_1^{-1}, \dots, y_n H y_n^{-1}\}$, $y_1, \dots, y_n \in G$.

$$\pi_2^{-1}(h) \cong \{g \in G \mid h \in gHg^{-1}\} = \bigcup_{i=1}^{n} \{g \in G \mid gHg^{-1} = y_i H y_i^{-1}\}$$
$$= \bigcup_{i=1}^{n} y_i N_G(H).$$

$\Rightarrow \dim \pi_2^{-1}(h) = \dim(H)$

$\Rightarrow \dim \pi_2(A) \geq \dim(A) - \dim(H) = \dim(G)$.

□

Thm G connected LAG.

(1) $\forall g \in G \ \exists B \subseteq G$ Borel : $g \in B$.

(2) $\forall s \in G_s \ \exists T \subseteq G$ max. torus : $s \in T$.

(3) The union of all Cartan subgps contains dense open $\subseteq G$.

Proof

$T \subseteq G$ max. torus, $C = Z_G(T)^0$.

$N_G(C)/C$ is finite.

$\exists t \in T$: t is in finitely many conjugates of C.

Lemma $\Rightarrow \bigcup_{g \in G} gCg^{-1}$ contains dense open $\subseteq G$.

$\Rightarrow \bigcup_{g \in G} gBg^{-1} = G$ (since larger and closed.)

Let $s \in G$ be semi-simple.

$\exists B \subseteq G$ Borel, $s \in B$.

$\exists T \subseteq B$ max. torus, $s \in T$.

□

Cor G connected LAG, $B \subseteq G$ Borel $\Rightarrow Z(B) = Z(G)$.

Proof

Already proved: $Z(B) \subseteq Z(G)$.

Let $z \in Z(G)$.

$\exists B' \subseteq G$ Borel, $z \in B'$.

$\exists g \in G$: $B = gB'g^{-1}$

$z = gzg^{-1} \in B$.

□