LAG 7 2026-02-10

Def G LAG.

$G \cong \mathbb{G}_a^n$: G is a vector group

$G \subseteq \mathbb{G}_a^n$ closed: G is an elementary unipotent group.

Theorem G LAG. TFAE:

(1) G is elementary unipotent.

(2) G is unipotent, abelian, and $pG = 0$.

(3) $G = \mathbb{G}_a^n \times F$, F finite elementary unipotent.

(4) $k[G]$ is generated by $\mathcal{A}(G)$ as k-algebra.

Note $p = 0 \Rightarrow F = 0$

$p > 0 \Rightarrow F = (\mathbb{Z}/p\mathbb{Z})^m$

Cor G connected LAG, $\dim(G) = 1$

$\Rightarrow G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.

Module structure on $\mathcal{A}(G)$

$\mathcal{A}(G) \subseteq k[G]$ vector subspace.

$p = 0$: $R = k$, $\mathcal{A}(G)$ is an R-module.

<u>Assume $p>0$:</u>

$\mathcal{A}(\mathbb{G}_a) = \mathrm{Span}_k\{T^{p^j} \mid j \geq 0\}$, $\dim \mathcal{A}(\mathbb{G}_a) = \infty$.

$f \in \mathcal{A}(G) \Rightarrow f^p \in \mathcal{A}(G)$.

$R = k[T]$ as <u>additive group</u>

$(aT^i)\cdot(bT^j) := a b^{p^j} T^{i+j}$.

$Tb = b^p T$.

<u>Properties:</u>

(1) R associative, non-commutative.

(2) R is "Euclidean": division algorithm works.

(3) All left/right ideals are principal.

(4) Any f.g. left R-module is direct sum of cyclic modules.

<u>R-module structure on $\mathcal{A}(G)$:</u>

$a.f = af$ for $a \in k$, $f \in \mathcal{A}(G)$.

$T.f = f^p$

$(aT^i) = a f^{p^i}$

<u>Exer</u>: $\mathcal{A}(\mathbb{G}_a^n) =$ free left R-module, basis $\{T_1, \dots, T_n\}$.

<u>Thm</u> G elementary unipotent.

(1) $\mathcal{A}(G)$ f.g. left R-module.

(2) G connected $\Leftrightarrow$ $\mathcal{A}(G)$ free left R-module.

Derivations

R com. ring. A com. R-algebra. M A-module.

R-derivation $D : A \longrightarrow M$:

(1) R-linear.

(2) $D(ab) = a.D(b) + b.D(a)$, $a, b \in A$.

<u>Note</u>: If D satisfies (2), then (1) $\Leftrightarrow$ $D(R) = 0$.

$\mathrm{Der}_R(A, M) = \{ D : A \longrightarrow M \text{ } R\text{-derivation} \}$

A-module: $(b.D + D')(a) = b.D(a) + D'(a)$.

<u>Example</u> $A = k[x_1, \ldots, x_n]$, $D : A \longrightarrow M$ any k-derivation.

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i).$$

$\mathrm{Der}_k(A, M) \cong M^{\oplus n}$ as A-module.

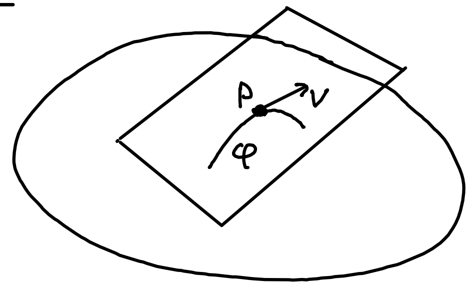
$\phi : A \longrightarrow B$ R-algebra hom, N B-module.

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_R(B, N) \xrightarrow{\phi_0} \mathrm{Der}_R(A, N).$$

$$D \longmapsto D \circ \phi$$

Tangent and cotangent vectors

X manifold, $p \in X$.



Tangent vector $v \in T_pX$:

Equiv. class of param. curves
$\varphi : \mathbb{R} \longrightarrow X$ with $\varphi(0) = p$.

Given C^∞ $f : X \longrightarrow \mathbb{R}$:

$$D_v(f) = \frac{d}{dt} f(\varphi(t))\Big|_{t=0}.$$

$C^\infty(X)$-module: $\mathbb{R}(p) = \mathbb{R}$, $f \cdot a = f(p)a$.

$D_v \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), \mathbb{R}(p)) =: T_pX$.

$df_p \in (T_pX)^*$ cotangent vector: $df_p(v) = D_v(f)$.

Local ring of variety

X irred. variety, $p \in X$.

$\mathcal{F} = \{(U, f) \mid p \in U \subseteq X \text{ open}, \ f : U \to k \text{ regular}\}$

Equiv. rel: $(U, f) \sim (U', f') \Leftrightarrow f|_{U \cap U'} = f'|_{U \cap U'}$

Local ring at p: $\mathcal{O}_{X,p} = \mathcal{F}/\sim \ = \{f \in k(X) \mid f \text{ def. at } p\}$

$\mathfrak{m}_x = \{f \in \mathcal{O}_{X,p} \mid f(p) = 0\} \subseteq \mathcal{O}_{X,p}$ unique max. ideal.

$k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \cong k$ is $\mathcal{O}_{X,p}$-module: $f \cdot a = f(p)a$

Example: X affine, $p \in X$.

$I(p) \subseteq k[X]$ max. ideal.

$$\mathcal{O}_{X,p} = k[X]_{I(p)} = (k[X] - I(p))^{-1} k[X].$$

this def. is valid when X not irred.

Zariski tangent space

$T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p))$ tangent space.

$T_p^*X = \mathfrak{m}_p/\mathfrak{m}_p^2$. cotangent space.

Note: $D \in T_pX$, $f \in \mathfrak{m}_p^2 \Rightarrow D(f) = 0$.

$g, h \in \mathfrak{m}_p \Rightarrow D(gh) = g(p)D(h) + h(p)D(g) = 0$.

Note:
$$\begin{aligned} \mathrm{Der}_k(\mathcal{O}_{X,p}, k(p)) &\xrightarrow{\cong} (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \\ D &\longmapsto [f + \mathfrak{m}_p^2 \mapsto D(f)] \\ [f \mapsto \bar{D}(f - f(p) + \mathfrak{m}_p^2)] &\longleftarrow\!\!\shortmid\ \bar{D} \end{aligned}$$

$\therefore$ Perfect pairing $T_p^*X \times T_pX \longrightarrow k$

$$(f + \mathfrak{m}_p^2, D) \longmapsto D(f)$$

Def $p \in X$ is a non-sing. point if $\dim_k(T_pX) = \dim(X)$.