LAG 5 2026-02-03

Prop G connected LAG, $\dim(G)=1$.

(1) G is commutative.

(2) $G=G_s$ or $G=G_u$.

(3) $G=G_u$ and $\mathrm{char}(k)=p>0 \Rightarrow g^p=e \quad \forall g\in G$.

Pf

(1) Assume G not commutative.

$(G,G)\neq e$ connected closed subgroup $\Rightarrow (G,G)=G$.

Let $g\in G\setminus Z(G)$.

$G=\overline{\{xgx^{-1}\mid x\in G\}}$

$G\subseteq GL_n$: $\chi_g(t)$ constant for $g\in G$.

$\Rightarrow \chi_g(t)=(t-1)^n \quad \forall g\in G$

$\Rightarrow G$ is unipotent $\Rightarrow$ solvable $\Rightarrow (G,G)\neq G$ $\lightning$

(2) clear.

(3) $G^{(p^h)}=\langle g^{p^h}\mid g\in G\rangle \subseteq G$ connected closed subgp.

$G\subseteq U_n$: $g^{p^h}=e$ for $h\geq n$.

Must have $G^{(p)}\neq G \Rightarrow G^{(p)}=e$.

□

Characters & cocharacters

$\mathbb{G}_m = k^\times$ mult. group.

Character: $\chi : G \longrightarrow \mathbb{G}_m$ alg. gp. hom.

$X^*(G) = \{\chi : G \longrightarrow \mathbb{G}_m\} \subseteq k[G]^\times$ (abelian) subgroup.

Dedekind: $X^*(G) \subseteq k[G]$ lin. independent.

Pf

Equation with n minimal: $\sum_{i=1}^{n} \lambda_i \chi_i(g) = 0$.

$$\Rightarrow \sum_{i=1}^{n} \lambda_i \chi_n(h) \chi_i(g) = 0$$

$$\sum_{i=1}^{n} \lambda_i \chi_i(h) \chi_i(g) = 0$$

$$\Rightarrow \sum_{i=1}^{n-1} \lambda_i (\chi_n(h) - \chi_i(h)) \chi_i(g) = 0$$

Choose $h \in G$ with $\chi_n(h) \neq \chi_1(h)$. ↯

□

Cocharacter: $\lambda : \mathbb{G}_m \longrightarrow G$ alg. group hom.

$X_*(G) = \{\lambda : \mathbb{G}_m \longrightarrow G\}$ <u>set</u> of cocharacters.

G commutative $\Rightarrow X_*(G)$ (abelian) group.

For $n \in \mathbb{Z}$: $(n.\lambda)(a) = \lambda(a)^n$. Prop. (1) $\Rightarrow n.\lambda \in X_*(G)$

$-\lambda = (-1).\lambda$.

Example: $X^*(\mathbb{G}_m) = X_*(\mathbb{G}_m) = \mathbb{Z}$.

$\chi : \mathbb{G}_m \longrightarrow \mathbb{G}_m$ alg. group hom.

$\chi(a) = a^n$ for some $n \in \mathbb{Z}$.

Note: $k[\mathbb{G}_m] = k[t, t^{-1}]$ has basis $\{t^n : n \in \mathbb{Z}\}$.

$D_n = (\mathbb{G}_m)^n$. $X^*(D_n) \cong \mathbb{Z}^n \cong X_*(D_n)$.

$k[D_n] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has basis $X^*(D_n)$.

<u>Def</u> G is <u>diagonalizable</u> $\Leftrightarrow$ $G \subseteq D_n$ closed.

G is a <u>torus</u> $\Leftrightarrow$ $G \cong D_n$

<u>Thm</u> G LAG. TFAE:

(1) G is diagonalizable.

(2) $X^*(G)$ is a basis of $k[G]$.

(3) $G \curvearrowright V$ rational rep. $\Rightarrow$ V direct sum of 1-dim. reps.

<u>Proof</u>

(1) $\Rightarrow$ (2): $G \subseteq D_n$. $k[D_n] \twoheadrightarrow k[G]$.

$k[G]$ spanned by image of $X^*(D_n)$.

$\therefore$ $k[G] = \mathrm{Span}(X^*(G))$.

(2) $\Rightarrow$ (3): $\phi : G \longrightarrow GL(V)$ rat. rep.

$\exists!\, A_\chi \in \mathrm{End}(V)$ for $\chi \in X^*(G)$:

$$\phi(g) = \sum_\chi \chi(g) A_\chi$$

($\phi : G \longrightarrow GL(V) \subseteq \mathrm{End}(V) = M_n$

$\phi(g) = (\phi_{ij}(g)) \in M_n$, $\phi_{ij} \in k[G] = \mathrm{Span}\, X^*(G)$.)

<u>Note</u>: $A_\chi \neq 0$ for finitely many χ.

$$1_V = \phi(e) = \sum_{\chi} A_\chi$$

$g, h \in G$: $$\sum_{\chi} \chi(g)\chi(h) A_\chi = \phi(gh) = \phi(g)\phi(h)$$
$$= \sum_{\chi,\psi} \chi(g)\psi(h) A_\chi A_\psi$$

$X^*(G\times G)$ linearly indep. $\Rightarrow A_\chi A_\psi = \delta_{\chi,\psi} A_\chi$

$\therefore$ $V = \bigoplus_{\chi} A_\chi(V)$.

<u>Note</u>: $\phi(g).v = \chi(g)v$ for $v \in A_\chi(V)$.

(3) $\Rightarrow$ (1): $G \subseteq GL(V) = GL_n$ closed. Clear.

□

<u>Cor</u> Assume G is diagonalizable.

(1) $X^*(G)$ f.g. abelian group.

(2) $k[G] = k[X^*(G)]$ group algebra.

(3) $\mathrm{char}(k) = p > 0 \Rightarrow X^*(G)$ has no p-torsion.

<u>Pf</u>

(1) $G \subseteq D_n$ closed $\Rightarrow \mathbb{Z}^n = X^*(D_n) \twoheadrightarrow X^*(G)$.

(3) $\chi^p = 1 \Rightarrow \chi(g)^p = 1 \in k \ \forall g \Rightarrow \chi = 1$.

□

Diagonalizable LAG $\longleftrightarrow$ f.g. abelian group

M f.g. abelian group.

$k[M]$ = k-vector space with basis $\{e(m) : m \in M\}$.

$$e(m)\,e(n) = e(m+n).$$

Assume M has no p-torsion.

$\Leftrightarrow$ $k[M]$ reduced f.g. k-alg.

$\mathcal{G}(M) = \operatorname{Spec}(k[M])$ affine variety.

$$\Delta : k[M] \longrightarrow k[M] \otimes k[M], \quad \Delta(e(m)) = e(m) \otimes e(m).$$

$$\iota : k[M] \longrightarrow k[M] \quad\quad \iota(e(m)) = e(-m).$$

$$\varepsilon : k[M] \longrightarrow k \quad\quad \varepsilon(e(m)) = 1.$$

Prop

(1) $\mathcal{G}(M)$ is diagonalizable LAG.

(2) $X^*(\mathcal{G}(M)) = M$

(3) G diagonalizable LAG $\Rightarrow$ $\mathcal{G}(X^*(G)) = G$.

Note: M_1, M_2 f.g. abelian groups.

$$k[M_1 \oplus M_2] = k[M_1] \otimes_k k[M_2]$$

$$\mathcal{G}(M_1 \oplus M_2) = \mathcal{G}(M_1) \times \mathcal{G}(M_2).$$

Exer: M finite $\Rightarrow$ $\mathcal{G}(M) \cong M$.

Cor G diagonalizable LAG.

(1) $G \cong D_n \times F$, F finite abelian w/o p-torsion.

(2) G torus $\Leftrightarrow$ G connected $\Leftrightarrow$ $X^*(G)$ free abelian.

Prop (Rigidity)

G, H diagonalizable LAGs. V connected affine var.

$\phi : V \times G \longrightarrow H$ morphism.

Assume $g \longmapsto \phi(v,g)$ is alg. gp. hom. $\forall v \in V$.

Then $\phi(v,g)$ is independent of v.

Proof

Let $\psi \in X^*(H) \subseteq k[H]$.

$$\phi^*(\psi) = \sum_{\chi \in X^*(G)} f_{\chi,\psi} \otimes \chi \in k[V] \otimes k[G].$$

$$\psi(\phi(v,g)) = \sum_{\chi} f_{\chi,\psi}(v)\, \chi(g)$$

$v \in V$ fixed: LHS $\in X^*(G)$.

$$\Rightarrow \quad f_{\chi,\psi}(v) = \begin{cases} 1 & \text{if } \chi = \text{LHS} \\ 0 & \text{else.} \end{cases}$$

V connected $\Rightarrow$ $f_{\chi,\psi}$ constant.

□

G alg. group, $H \subseteq G$ closed subgroup.

$$Z_G(H) = \{ g \in G \mid gh = hg \ \forall h \in H \}$$

$$N_G(H) = \{ g \in G \mid gHg^{-1} = H \}$$

<u>Exer</u> $G = GL_n$

$Z_G(D_n) = D_n$

$N_G(D_n) = S_n D_n$, $S_n \subseteq G$ perm. matrices.

$N_G(D_n)/Z_G(D_n) = S_n$ Weyl group of GL_n.

<u>Cor</u> G LAG, $H \subseteq G$ diagonalizable closed subgroup.

Then $N_G(H)^\circ = Z_G(H)^\circ$ and $N_G(H)/Z_G(H)$ is finite.

<u>Proof</u>

The morphism

$$N_G(H)^\circ \times H \longrightarrow H, \quad (g, h) \longmapsto ghg^{-1}$$

is independent of g.

$\Rightarrow \ ghg^{-1} = h \quad \forall \ g \in N_G(H), \ h \in H$

$\Rightarrow \ N_G(H)^\circ \subseteq Z_G(H).$

□