LAG 3 2026-01-27

G alg. group.

<u>G-variety</u>: Variety X with action (morphism) $a: G \times X \to X$.

$a(g,x) = g.x$.

<u>Orbit of x</u>: $G.x$

X <u>homogeneous</u> $\Leftrightarrow$ $X = G.x_0$.

<u>G-equivariant morphism</u>: $\phi : X \longrightarrow Y$, $\phi(g.x) = g.\phi(x)$.

<u>Rational representation</u>:

Alg. group hom. $r : G \longrightarrow GL(V)$, $\dim(V) < \infty$.

$G \circlearrowright V$: $g.v = r(g)(v)$.

<u>Lemma</u> X G-variety.

(1) $G.x \subseteq \overline{G.x}$ is open $\forall x \in X$.

(2) $\exists$ closed orbits in X.

<u>Pf</u> (1) $G \longrightarrow X$, $g \mapsto g.x$.

Image $G.x \supseteq$ dense open $U \subseteq \overline{G.x}$.

$G.x = \bigcup_{g \in G} g.U \subseteq \overline{G.x}$ open.

(2) $\overline{G.x} - G.x$ = union of orbits.

Choose $x \in X$ with $\dim G.x$ minimal.

$\square$

Now: G LAG, X affine G-variety.

$a : G \times X \longrightarrow X$

$a^* : k[X] \longrightarrow k[G] \otimes k[X]$. $\quad a^*(f)(g, x) = f(g.x)$.

Def: $s : G \longrightarrow GL(k[X])$ rep. of abstract gps.

$$(s(g)f)(x) = f(g^{-1}.x)$$

Relation: let $f \in k[X]$.

$$a^*(f) = \sum_{i=1}^{n} u_i \otimes f_i \in k[G] \otimes k[X].$$

$$(s(g)f)x = f(g^{-1}.x) = a^*(f)(g^{-1}, x) = \sum u_i(g^{-1}) f_i(x)$$

$$\Rightarrow s(g)f = \sum u_i(g^{-1}) f_i$$

Assume $V \subseteq k[X]$, $\dim(V) < \infty$.

Lemma $\exists V \subseteq W \subseteq k[X]$: $\dim(W) < \infty$, $s(g)(W) \subseteq W$ $\forall g \in G$.

Pf WLOG $V = \mathrm{Span}\{f\}$.

Relation $\Rightarrow$ $s(g)f \in \mathrm{Span}\{f_1, \dots, f_n\}$ $\forall g \in G$.

$\Rightarrow \dim\left(W = \mathrm{Span}\{s(g)f \mid g \in G\}\right) < \infty$.

$\square$

Lemma $V \subseteq k[X]$ $s(G)$-stable $\Leftrightarrow$ $a^*(V) \subseteq k[G] \otimes V$.

$\Rightarrow$ V rational rep. of G, $s : G \times V \longrightarrow V$.

- similar.

Action by translation

$G \circlearrowright G$, $\quad g.x = gx$ (left), $\quad g.x = xg^{-1}$ (right)

$\lambda : G \longrightarrow GL(k[G])$, $\quad (\lambda(g)f)(x) = f(g^{-1}x)$

$\rho : G \longrightarrow GL(k[G])$, $\quad (\rho(g)f)(x) = f(xg)$.

<u>Note</u>: λ and ρ are faithful (= injective):

$\lambda(g)$ determines $G \longrightarrow G$, $x \mapsto g^{-1}x$

determines $g \in G$.

<u>Thm</u> $G \cong$ closed subgroup $\subseteq GL_n$.

<u>Pf</u> Choose $f_1, \dots, f_n \in k[G]$:

- $k[G] = k[f_1, \dots, f_n]$
- $V = \operatorname{Span}\{f_1, \dots, f_n\}$ is $\rho(G)$-stable
- $\{f_1, \dots, f_n\}$ lin. indep.

$\exists\ m_{ij} \in k[G]$, $\quad 1 \leq i,j \leq n$ such that

$$\rho(g) f_j = \sum_{i=1}^{n} m_{ij}(g) f_i$$

Check: $a : G \times G \longrightarrow G$, $\quad a(g,x) = xg$.

$a^*(f_j) = \sum_i m_{ij} \otimes f_i$ for some $m_{ij} \in k[G]$.

$(\rho(g) f_j)(x) = f_j(xg) = a^*(f_j)(g,x) = \sum_i m_{ij}(g) f_i(x)$.

$\phi : G \longrightarrow GL_n \ , \ \phi(g) = (m_{ij}(g))_{i,j}$ alg. gp. hom.

$k[GL_n] = k[T_{ij}, det^{-1}]$.

$$\phi^* : k[T_{ij}, det^{-1}] \longrightarrow k[G]$$
$$T_{ij} \longmapsto m_{ij}$$
$$det^{-1} \longmapsto det(m_{ij})^{-1}.$$

Surjective:

$$f_j(g) = f_j(eg) = (\rho(g) f_j)(e) = \sum_i m_{ij}(g) f_i(e)$$
$$f_j = \sum_i f_i(e) m_{ij} \in Im(\phi^*).$$

$\therefore \ k[G] = k[GL_n]/I \ , \ I = ker(\phi^*)$

$\Updownarrow \ G \cong Z(I) \subseteq GL_n$.

$\square$

Jordan decomposition

V vector space, $\dim(V)<\infty$.
$a \in \mathrm{End}_k(V)$.

a **semi-simple**: $\exists$ basis of eigenvectors.

a **nilpotent**: $a^n = 0$ for some $n \geq 0$.

a **unipotent**: $a-1$ nilpotent.

Note: $\mathrm{char}(k) = p > 0$: a unipotent $\Leftrightarrow a^{p^s} = 1$, $s \in \mathbb{N}$.

$M_n = \mathrm{Mat}(n \times n) = \mathrm{End}(k^n)$.

Lemma $S \subseteq M_n$ set of pairwise commuting matrices.
(1) $\exists\, x \in GL_n$: xSx^{-1} upper Δ.
(2) All elts. of S semi-simple $\Rightarrow xSx^{-1}$ diagonal.

Proof Simultaneous Jordan decomp (almost)! $\square$

Lemma

(1) $a, b \in \mathrm{End}(V)$, $ab = ba$.
a, b both ss/nilpot/unipot $\Rightarrow$ so is ab.

(2) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot/unipot
$\Rightarrow$ so is $a \oplus b \in \mathrm{End}(V \oplus W)$ and $a \otimes b \in \mathrm{End}(V \otimes W)$.

(3) $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$ both ss/nilpot
$\Rightarrow$ so is $a \otimes 1 + 1 \otimes b \in \mathrm{End}(V \otimes W)$.

Note: $a \otimes 1 + 1 \otimes b - 2(1 \otimes 1)$ is nilpotent!