LAG 18 2026-03-26

Lemma G connected solvable LAG, G not a torus.
$\exists$ closed normal $N \lhd G$: $N \cong \mathbb{G}_a$ and $N \subseteq Z(G_u)$.

Proof

G not torus $\Rightarrow$ $G_u \neq e$.

$G_u \lhd G$ normal $\Rightarrow$ $Z(G_u)^\circ \lhd G$ normal.

$G_u \neq e$ connected nilpotent $\Rightarrow$ $Z(G_u)^\circ \neq e$.

$p = \mathrm{char}(k)$.

$p > 0$: $L(G_u)$ unipotent $\Rightarrow$ $L(G_u)^{p^r} = e$ for some $r > 0$.

$\therefore$ $\exists$ $H \lhd G$ closed normal connected, $e \neq H \subseteq Z(G_u)$.

$p > 0 \Rightarrow H^p = e$.

H connected elementary unipotent.

$H \cong \mathbb{G}_a^m$ for some $m \geq 1$.

$m = 1$: Take $N = H$. Assume $m > 1$.

$G \circlearrowright H$, $g.h = ghg^{-1}$

$G \longrightarrow GL(k[H])$ locally rational. $(g.f)(h) = f(g^{-1}hg)$.

$\mathcal{A} \subseteq k[H]$ additive functions.

$G \longrightarrow GL(\mathcal{A})$ locally rational.

G_u acts trivially on H: $G \longrightarrow G/G_u \longrightarrow GL(\mathcal{A})$

G/G_u torus $\Rightarrow$ $\exists$ $0 \neq f \in \mathcal{A}$: $g.f \in kf$ $\forall g \in G$.

$f: H \longrightarrow \mathbb{G}_a$ group hom.

$H' = (\mathrm{Ker}\, f)^\circ \cong \mathbb{G}_a^{m-1}$.

$H' \lhd G$ normal: $g \in G$, $h \in \mathrm{Ker}(f)$ $\Rightarrow$

$f(ghg^{-1}) = (g^{-1}.f)(h) = 0$

Induction on m $\Rightarrow$ $\exists N$.

$\square$

<u>Note</u> $\mathbb{G}_m = k^\times \subseteq \mathbb{G}_a = k.$

<u>Standard action</u>: $\mathbb{G}_m \times \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad t.x = tx$

<u>Exer</u>

- $\mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$
- G LAG, $G \times \mathbb{G}_a \longrightarrow \mathbb{G}_a$ action by automorphisms.
  $\exists$ character $\alpha : G \longrightarrow \mathbb{G}_m : \quad g.x = \alpha(g)x$

<u>Note</u> $\mathrm{char}(k) = p > 0$:

$GL_2 \subsetneq \mathrm{Aut}(\mathbb{G}_a^2)$

Action by automorphisms:

$\mathbb{G}_a \times \mathbb{G}_a^2 \longrightarrow \mathbb{G}_a^2, \quad a.(x,y) = (x + ay^p, y)$

Not given by group hom. $\mathbb{G}_a \longrightarrow GL_2$.

<u>Note</u>:

G unipotent LAG, $\mathrm{char}(k) = 0 \Rightarrow G$ connected.

G/G° finite unipotent $\Rightarrow G/G^\circ = e$.

Thm G connected solvable LAG.

(1) $s \in G$ semi-simple $\Rightarrow$ $s \in$ max. torus of G.

(2) $s \in G$ semi-simple $\Rightarrow$ $Z_G(s)$ is connected.

(3) All max. tori in G are conjugate.

Proof

Assume first $\dim(G_u) = 1$.

Then $G_u \cong \mathbb{G}_a$. Fix isomorphism $\phi : \mathbb{G}_a \xrightarrow{\cong} G_u$.

$\psi : G \longrightarrow G/G_u$ projection.

$G/G_u \curvearrowright G_u$, $\psi(g).u = gug^{-1}$.

$\exists$ character $\alpha : G/G_u \longrightarrow \mathbb{G}_m$:

$$\circledast \qquad g\phi(a)g^{-1} = \phi(\alpha(\psi(g))\, a) \qquad \text{for } g \in G,\ a \in \mathbb{G}_a.$$

Assume α trivial.

Then $G_u \subseteq Z(G)$ $\Rightarrow$ $G/Z(G)$ is commutative

$\Rightarrow$ G nilpotent $\Rightarrow$ $G \cong G_s \times G_u$.

WLOG: α not trivial.

Let $s \in G$ be semi-simple. $Z = Z_G(s)$.

(5.4.2): $L(Z) = \operatorname{Ker}(\operatorname{Ad}(s) - 1) \subseteq L(G)$.

$L(G) = (\operatorname{Ad}(s)-1)L(G) \oplus L(Z)$.

$\psi(sgs^{-1}) = \psi(g) \Rightarrow \psi \circ \operatorname{Int}(s) = \psi$

$\Rightarrow d\psi \circ \operatorname{Ad}(s) = d\psi \Rightarrow d\psi \circ (\operatorname{Ad}(s) - 1) = 0$

$\Rightarrow (\operatorname{Ad}(s)-1)L(G) \subseteq \operatorname{Ker}(d\psi) = L(G_u)$.

$\dim (\operatorname{Ad}(s)-1)L(G) \leq 1$

$\dim(Z) = \dim L(Z) \geq \dim(G) - 1$.

Assume $\alpha(\psi(s)) \neq 1$:

$Z \cap G_u = e$: $\quad \phi(a) = s\phi(a)s^{-1} = \phi(\alpha(\psi(s))a) \Leftrightarrow a = 0.$

$\therefore \dim(Z) = \dim(G) - 1.$

$Z^0 = Z^0/(Z^0)_u$ max. torus in G.

$G = Z^0 \ltimes G_u$

$Z = Z_G(s) = Z^0 \ltimes Z_{G_u}(s) = Z^0$

Conclude: $s \in G$ semi-simple, $\alpha(\psi(s)) \neq 1$

$\Rightarrow Z_G(s) \subseteq G$ max. torus.

Note: Any max. torus $T \subseteq G$ has this form:

Choose $t \in T$ s.t. $\alpha(\psi(t)) \neq 1$.

Then $T \subseteq Z_G(t) \subseteq G$, $\quad Z_G(t)$ max. torus.

Assume $\alpha(\psi(s)) = 1$:

$G_u \subseteq Z = Z_G(s)$.

$(Ad(s) - 1)\, L(G) \subseteq L(G_u) \subseteq L(Z) = \mathrm{Ker}(Ad(s) - 1)$

$Ad(s) - 1$ semi-simple $\Rightarrow Ad(s) - 1 = 0$.

$L(Z) = L(G) \Rightarrow Z_G(s) = G$ is connected.

Choose $t \in G$ s.t. $\alpha(\psi(t)) \neq 1$.

$s \in Z(G) \Rightarrow s \in Z_G(t) \subseteq G$ max. torus.

Let $T, T' \subseteq G$ be max. tori, $\quad T = Z_G(t)$, $\alpha(\psi(t)) \neq 1$.

$G = T' \ltimes G_u$. $\quad t = t'\phi(a)$, $\quad t' \in T'$, $a \in \mathbb{G}_a$.

$$\phi(b)\, t\, \phi(b)^{-1} = t\, t^{-1}\phi(b)\, t\, \phi(b)^{-1} = t'\phi(a)\, \phi(\alpha(\psi(t))^{-1} b)\, \phi(-b)$$

$$= t'\phi(a + (\alpha(\psi(t))^{-1} - 1)\, b)$$

$\exists\, b \in \mathbb{G}_a : \phi(b)\, t\, \phi(b)^{-1} = t'$.

$\Rightarrow \phi(b)\, T\, \phi(b)^{-1} = Z_G(t') = T'$.

Assume $\dim(G_u) \geq 2$:

Choose $N \triangleleft G$ closed normal, $N \cong \mathbb{G}_a$, $N \subseteq Z(G_u)$.

$\overline{G} = G/N$. $\dim(G/G_u) = \dim(\overline{G}/\overline{G}_u)$

$s \in G$ semi-simple. $\bar{s}$ = image in $\overline{G}$.

Induction on $\dim(G)$ $\Rightarrow$ $\exists$ max. torus $\overline{T} \subseteq \overline{G}$, $\bar{s} \in \overline{T}$.

$$\begin{array}{ccc} G & \longrightarrow & \overline{G} \\ \cup\!\!| & & \cup\!\!| \\ H & \longrightarrow & \overline{T} \end{array}$$

H = inverse image in G.

H connected solvable, $H_u = N$, $s \in H$.

$\exists$ max. torus $T \subseteq H$, $s \in T$.

$T \subseteq G$ also max. torus.

Let $T, T' \subseteq G$ be max. tori.

$\overline{T}, \overline{T}' \subseteq \overline{G}$ max. tori.

$\exists\, g \in G$: $\overline{gTg^{-1}} = \overline{T}'$ $\Rightarrow$ $(gTg^{-1})N = T'N$.

$T'N$ connected solvable, $(T'N)_u = N$

$\Rightarrow$ gTg^{-1} and T' are conjugate in $T'N$.

<u>Show</u>: $Z_G(s)$ is connected.

Choose max. torus $T \subseteq G$ with $s \in T$.

$G = T \ltimes G_u$ $\Rightarrow$ $Z_G(s) = T \ltimes Z_{G_u}(s)$.

<u>Enough</u>: $Z_{G_u}(s)$ is connected.

<u>Note</u>: Clear if $\operatorname{char}(k) = 0$ since $Z_{G_u}(s)$ is unipotent.

WLOG: $s \notin Z(G)$ $\Rightarrow$ $G_u \not\subseteq Z_G(s)$.

$G_1 = \{g \in G \mid sgs^{-1}g^{-1} \in N\} \subseteq G$ closed subgroup.

$Z_G(s) \subseteq G_1$ and $G_1/N = Z_{\bar{G}}(\bar{s})$.

Induction on $\dim(G) \Rightarrow Z_{\bar{G}}(\bar{s})$ connected.

G_1/N and N connected $\Rightarrow G_1$ connected.

If $G_1 \neq G$ then $Z_G(s) = Z_{G_1}(s)$ is connected by induction on $\dim(G)$.

WLOG: $G_1 = G$.

$\{sus^{-1}u^{-1} \mid u \in G_u\} \subseteq N$.

LHS is closed (5.4.4 (i)) and connected.

LHS $\neq e$ since $G_u \nsubseteq Z_G(s)$.

$\therefore\ N = \{sus^{-1}u^{-1} \mid u \in G_u\}$.

<u>Enough</u>: $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is bijective.

Assume $z \in Z_{G_u}(s)$, $x, y \in N$, $zx = y \in G_u$.

$N \subseteq Z(G_u) \Rightarrow zx = xz$; $z \in Z_G(s) \Rightarrow zs = sz$.

Write $x = usu^{-1}s^{-1}$, $y = vsv^{-1}s^{-1}$, $u, v \in G_u$.

$$\underset{\text{unipotent}}{z}\ usu^{-1} = vsv^{-1} \qquad \Rightarrow z = e,\ x = y.$$

semi-simple

This shows $\mu : Z_{G_u}(s) \times N \longrightarrow G_u$ is injective.

$Z_{G_u}(s) =$ Fiber of morphism $G_u \longrightarrow N$.

$\dim Z_{G_u}(s) \geq \dim(G_u) - 1$.

$\therefore\ Z_{G_u}(s) \times N \longrightarrow G_u$ surjective.

$\square$