LAG 17 2026-03-04

Cor G connected, $B \subseteq G$ Borel $\Rightarrow Z(G)^\circ \subseteq Z(B) \subseteq Z(G)$.

Proof

$Z(G)^\circ \subseteq G$ closed connected solvable

$\Rightarrow Z(G)^\circ \subseteq B'$, B' Borel

$\Rightarrow Z(G)^\circ \subseteq gBg^{-1}$, $g \in G$

$\Rightarrow Z(G)^\circ = g^{-1} Z(G)^\circ g \subseteq B$.

$\therefore Z(G)^\circ \subseteq Z(B)$.

Let $g \in Z(B)$.

Well defined morphism:

$$G/B \longrightarrow G, \quad x \longmapsto xgx^{-1}$$

G/B irred. and complete, G affine

$\Rightarrow$ morphism is constant.

$\therefore xgx^{-1} = g \ \forall x \in G \quad \Rightarrow \quad g \in Z(G)$.

□

Lemma $G \neq e$ connected nilpotent LAG $\Rightarrow Z(G)^\circ \neq e$.

Proof

$G_0 = G$, $G_{i+1} = (G, G_i)$ lower central series.

G nilpotent $\Leftrightarrow G_n = e$ for some n.

Choose $n \geq 1$ such that $G_{n-1} \neq e$, $G_n = e$.

$(G, G_{n-1}) = e \Rightarrow G_{n-1} \subseteq Z(G)$.

G_{n-1} connected $\Rightarrow G_{n-1} \subseteq Z(G)^\circ$.

□

<u>Cor</u> G LAG, $B \subseteq G$ Borel. B nilpotent $\Rightarrow B = G^\circ$

<u>Proof</u>

WLOG: G connected.

Assume $B \subsetneq G$.

G/B not affine $\Rightarrow B \neq e \Rightarrow e \neq Z(B)^\circ \subseteq Z(G)$.

$\therefore Z(B)^\circ \trianglelefteq G$ normal.

$B/Z(B)^\circ \subsetneq G/Z(B)^\circ$ proper nilpotent Borel subgp.

Induction on $\dim(G) \Rightarrow \lightning$

$\square$

<u>Cor</u> G connected nilpotent LAG.

(1) G_s and G_u are closed connected subgroups.

(2) $G_s \subseteq G$ is a central torus.

(3) $\mu : G_s \times G_u \xrightarrow{\cong} G$ iso. of alg. groups.

<u>Proof</u>

Already proved: $G_s \subseteq Z(G)$ abstract subgroup.

$G \subseteq GL(V)$ closed subgroup.

$V = \bigoplus_{\chi} V_\chi$, $\chi : G_s \longrightarrow \mathbb{G}_m$ char. of abstract groups.

$G.V_\chi = V_\chi \quad \forall \chi$.

$G \circlearrowright Fl(V_\chi)$, $Fl(V_\chi)^G \neq \emptyset$.

$\therefore \exists\ G \subseteq GL_n$ such that $G \subseteq B_n$ and $G_s = G \cap D_n$.

<u>Note</u>: $G_u = G \cap U_n$.

$G_s \times G_u \underset{\text{project}}{\overset{\mu}{\rightleftarrows}} G$ iso. of alg. groups (since $G_s \subseteq Z(G)$.)

$\square$

Note: G LAG. G diagonalizable $\Leftrightarrow$
G commutative & all elts. semi-simple.

Proof $\Leftarrow$: $G \subseteq GL_n$ closed subgroup.
$(2.4.2) \Rightarrow \exists x \in GL_n : xGx^{-1} \subseteq D_n$.

Cor G connected solvable LAG.
(1) $(G,G) \subseteq G$ closed connected unipotent normal subgp.
(2) $G_u \subseteq G$ closed connected unipotent normal subgp.
(3) G/G_u is a torus.

Proof
$(G,G) \subseteq G$ closed connected normal.
WLOG: $G \subseteq B_n \subseteq GL_n$
$(G,G) \subseteq (B_n, B_n) = U_n \Rightarrow (G,G)$ unipotent.
$G_u = G \cap U_n \Rightarrow G_u \subseteq G$ closed unipotent normal.
$G/G_u \xrightarrow{\phi} B_n/U_n \cong D_n$ injective $\Rightarrow G/G_u$ commutative.
All $x \in G/G_u$ semi-simple:
$$\phi(x_u) = \phi(x)_u = e \Rightarrow x_u = e \Rightarrow x = x_s.$$
G/G_u connected $\Rightarrow G/G_u$ torus (by Note).

G_u connected:
$G_u^\circ \triangleleft G$ normal, $G_u/G_u^\circ \xrightarrow{\cong} (G/G_u^\circ)_u$ iso. of finite groups.
Show: G connected solvable, G_u finite $\Rightarrow G_u = e$.
$G_u \triangleleft G$ normal & finite $\Rightarrow G_u \subseteq Z(G)$.
($y \in G_u$. $G \longrightarrow G_u$, $x \longmapsto xyx^{-1}$ must be constant.)
G/G_u commutative $\Rightarrow G/Z(G)$ commutative
$\Rightarrow G$ nilpotent $\Rightarrow G_u$ connected.
$\square$

Def G connected solvable LAG. A *maximal torus* of G is a subtorus $T \subseteq G$ with $\dim(T) = \dim(G/G_u)$.

Lemma G connected solvable LAG, $T \subseteq G$ max torus. Then $\mu : T \times G_u \xrightarrow{\cong} G$ isomorphism of varieties.

Proof

$T \times G_u \circlearrowright G$, $(t,u).x = txu^{-1}$.

Isotropy group of $e \in G$: $(T \times G_u)_e = T \cap G_u = e$.

$\pi : T \times G_u \longrightarrow G$, $\pi(t,u) = tu^{-1}$

bijective equiv. morphism of hom. varieties.

$d\pi_{(e,e)} : L(T) \oplus L(G_u) \longrightarrow L(G)$, $(X,Y) \longmapsto X - Y$

X semi-simple and Y nilpotent (in $\mathrm{End}_k(k[G])$).

$\Rightarrow d\pi_{(e,e)}$ injective $\Rightarrow d\pi_{(e,e)}$ bijective.

$\square$

Cor G connected solvable, $T \subseteq G$ max. torus $\Rightarrow T \xrightarrow{\cong} G/G_u$ isomorphism.