LAG 16 2026-03-12

Def X variety. X is <u>complete</u> if $\forall$ varieties Z:

$$\pi_Z : X \times Z \longrightarrow Z \text{ is a closed map.}$$

Example: $\mathbb{A}^1$ is not complete. $\pi_2 : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$.

$\pi_2(Z(xy-1)) = \mathbb{A}^1 \setminus \{0\}$ not closed.

Properties: Assume X complete.

(1) $Y \subseteq X$ closed $\Rightarrow$ Y is complete.

(2) Y complete $\Rightarrow$ $X \times Y$ is complete.

(3) $\phi : X \longrightarrow Y$ morphism $\Rightarrow$ $\phi(X)$ closed & complete.

<u>Pf</u>: $X \xrightarrow{\gamma} X \times Y \xrightarrow{\pi_Y} Y$, $\gamma(X) \subseteq X \times Y$ closed.

(4) X connected $\Rightarrow$ $\mathcal{O}_X(X) = k$.

<u>Pf</u>: $f : X \longrightarrow \mathbb{A}^1$, $f(X)$ closed, complete, connected.

(5) X affine $\Rightarrow$ X is finite.

Thm $\mathbb{P}^n$ is complete.

Thm X complete, C non-singular curve, $P \in C$.
Any morphism $\phi : C \setminus \{P\} \longrightarrow X$ extends to $\phi : C \longrightarrow X$.

Lemma $\phi: X \times Y \longrightarrow Z$ morphism, X, Y, Z irred., X complete. For $y \in Y$ set $\phi_y(x) = \phi(x,y)$.
Assume $\exists\, a \in Y$: $\phi_a : X \longrightarrow Z$ constant.
Then ϕ_y constant $\forall y \in Y$.

Proof

$\Gamma = \{(x, y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq X \times Y \times Z$ closed & irred. (graph)

$C = \{(y, \phi(x,y)) \mid x \in X, y \in Y\} \subseteq Y \times Z$ closed & irred. (image of Γ, X complete)

$\therefore$ C irred. variety.

$\pi_Y : C \longrightarrow Y$ surjective morphism.

ϕ_a constant $\Leftrightarrow$ $\pi_Y^{-1}(a) = $ point.

$\therefore$ $\dim(C) = \dim(Y)$.

$x \in X$: $C_x = \{(y, \phi(x,y)) \mid y \in Y\} \subseteq C$ closed & irred. (graph)

$C_x \cong Y \Rightarrow \dim(C_x) = \dim(C) \Rightarrow C_x = C$.

$\pi_Y : C \xrightarrow{\cong} Y$ isomorphism.

$\pi_Y^{-1}(y) = $ point $\Rightarrow$ ϕ_y constant $\forall y \in Y$.

$\square$

Cor G complete alg. group $\Rightarrow$ G is commutative.

Proof $\phi: G \times G \longrightarrow G$, $\phi(x,y) = xyx^{-1}$.

ϕ_e constant $\Rightarrow$ ϕ_y const. $\forall y \in G$.

$\square$

Exer $\mathbb{P}^n$ is not an alg. group for $n \geq 1$.

Lemma $\phi : X \longrightarrow Y$ bijective equivariant morphism of homogeneous G-varieties.

Then X complete $\Leftrightarrow$ Y complete.

Proof: $\phi \times 1_Z : X \times Z \longrightarrow Y \times Z$ homeomorphism $\forall Z$. $\square$

Def G LAG, $P \subseteq G$ closed subgroup.

P is parabolic if G/P is complete.

Def $P \subseteq G$ subgroup, Z set. $A \subseteq G \times Z$ is P-stable if $(g, z) \in A$, $p \in P$ $\Rightarrow$ $(gp, z) \in A$.

Lemma $P \subseteq G$ is parabolic $\Leftrightarrow$

$\forall$ var. Z $\forall$ $A \subseteq G \times Z$ closed P-stable: $\pi_Z(A) \subseteq Z$ closed.

Proof

$A \subseteq G/P \times Z$ closed

$\updownarrow$

$\pi^{-1}(A) \subseteq G \times Z$ closed, P-stable.

$$\begin{tikzcd} G \times Z \arrow[r] \arrow[d, "\pi"'] & Z \\ G/P \times Z \arrow[ur] & \end{tikzcd}$$

Same image in Z.

$\square$

Lemma G LAG, $Q \subseteq P \subseteq G$ closed subgroups.
$Q \subseteq G$ parabolic $\Leftrightarrow$ $Q \subseteq P$ and $P \subseteq G$ parabolic.

Proof

$\Rightarrow$: $P/Q \subseteq G/Q$ closed and $G/Q \twoheadrightarrow G/P$.

$\Leftarrow$: $P \times G \times Z \underset{\pi}{\overset{\alpha}{\rightrightarrows}} G \times Z \xrightarrow{\pi_Z} Z$

$$\alpha(p, g, z) = (gp, z), \quad \pi(p, g, z) = (g, z).$$

Let $A \subseteq G \times Z$ be closed, Q-stable.

$\alpha^{-1}(A) \subseteq P \times G \times Z$ closed, Q-stable.

$Q \subseteq P$ parab. $\Rightarrow$ $\pi(\alpha^{-1}(A)) \subseteq G \times Z$ closed, P-stable.

$P \subseteq G$ parab. $\Rightarrow$ $\pi_Z(A) = \pi_Z(\pi(\alpha^{-1}(A))) \subseteq Z$ closed.

$\square$

Cor $P \subseteq G$ parabolic $\Leftrightarrow$ $P^\circ \subseteq G^\circ$ parabolic.

Proof

$$\begin{array}{ccc} P & \subset & G \\ \cup & & \cup \\ P^\circ & \subset & G^\circ \end{array}$$

Note:
$G^\circ \subseteq G$ parabolic.

$\square$

Thm G connected LAG. TFAE:

(a) G has no proper parabolic subgroups.

(b) $G \circlearrowright X$, X complete $\Rightarrow$ $X^G \neq \emptyset$.

(c) $G \subseteq GL_n \Rightarrow \exists\, x \in GL_n : xGx^{-1} \subseteq B_n$ (upper Δ)

(d) G is solvable.

Proof

(a) $\Rightarrow$ (b): Choose closed orbit $\Omega \subseteq X$.

$x \in \Omega$, $G_x \subseteq G$ isotropy group.

$G/G_x \longrightarrow \Omega$, $g.G_x \longmapsto g.x$

bijective morphism of hom. G-varieties.

X complete $\Rightarrow$ Ω complete $\Rightarrow$ G/G_x complete

$\Rightarrow$ $G_x \subseteq G$ parabolic $\Rightarrow$ $G_x = G$ $\Rightarrow$ $x \in X^G$.

(b) $\Rightarrow$ (c): $G \subseteq GL(V)$ closed subgroup.

$Fl(V) = \{ V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V) \mid \dim(V_i) = i \}$

Exer: $Fl(V)$ projective variety.

(b) $\Rightarrow \exists$ G-stable flag $V_\bullet \subseteq V$ $\Rightarrow$ (c).

(c) $\Rightarrow$ (d): B_n is solvable.

(d) $\Rightarrow$ (a): Choose minimal parabolic $P \subseteq G$.

$P \subsetneq G \Rightarrow G/P$ not affine $\Rightarrow P \subseteq G$ not normal $\Rightarrow (G,G) \not\subseteq P$.

$(G,G)/(G,G)\cap P \longrightarrow (G,G)P/P$

bijective equiv. morphism of homogeneous (G,G)-varieties.

$P \subsetneq (G,G)P$ parab. $\Rightarrow$ $(G,G) \cap P \subsetneq (G,G)$ parab.

Induction on $\dim(G)$ $\Rightarrow$ $\lightning$

$\square$

Lemma $H \subseteq G$ connected solvable, $P \subseteq G$ parabolic.

$\exists g \in G : gHg^{-1} \subseteq P$.

Proof

$H \circlearrowright G/P$. Let $g.P \in (G/P)^H$ be a fixed point.

$\forall h \in H : hg.P = g.P \Rightarrow g^{-1}Hg \subseteq P$.

□

Def G LAG. A Borel subgroup of G is a maximal closed connected solvable subgroup.

Thm G LAG, $B \subseteq G$ closed subgroup. TFAE:

(1) $B \subseteq G$ Borel

(2) $B \subseteq G$ min. parabolic.

(3) $B \subseteq G$ connected solvable parabolic.

Proof

(3) $\Rightarrow$ (1) + (2): Lemma.

(1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3):

Choose $B \subseteq G$ Borel, $P \subseteq G$ min. parabolic.

WLOG $B \subseteq P$.

$P = P^\circ$ is connected, contains no proper parabolic.

$\Rightarrow$ P closed connected solvable.

$\therefore$ $B = P$ satisfy (3).

□

Cor All Borel subgroups are conjugate.

Cor $\phi : G \twoheadrightarrow G'$ surjective homomorphism of LAGs.

$P \subseteq G$ Borel/parabolic $\Rightarrow$ $\phi(P) \subseteq G'$ Borel/parabolic.

Proof

$G/P \twoheadrightarrow G'/\phi(P)$. $P \subseteq G$ parab. $\Rightarrow$ $\phi(P) \subseteq G'$ parab.

P Borel $\Rightarrow$ $\phi(P)$ connected solvable parabolic.

□