LAG 15 2026-03-10

Thm G LAG, $H \subseteq G$ closed subgroup.

(1) G/H is quasi-projective.

(2) $G \longrightarrow G/H$ is separable.

(3) $\dim(G/H) = \dim(G) - \dim(H)$.

Proof

Let (X, x) be as in Corollary:

- X quasi-projective homogeneous G-variety.
- $G_x = H$
- $\psi : G \longrightarrow X$, $g \longmapsto g.x$ is separable.

$H = G_x \Rightarrow X = G/H$ as set.

ψ open: $U \subseteq X$ open $\Leftrightarrow$ $\psi^{-1}(U) \subseteq G$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow$ $f\psi : \psi^{-1}(U) \longrightarrow k$ regular.

$\therefore$ $X = G/H$ as SWF.

$\square$

Notation:

$G/H = \{ g.H \mid g \in G \}$, $\pi : G \longrightarrow G/H$, $\pi(g) = g.H$.

Fiber bundles

G LAG, $H \subseteq G$ closed subgp, $H \circlearrowright X$.

$G \times X \circlearrowleft H$, $(g,x).h = (gh, h^{-1}.x)$.

$G \times^H X = (G \times X)/H$ SWF

$= \{[g,x] \mid g \in G, x \in X\} / \{[gh,x] = [g, h.x] \ \forall h \in H\}$

$p : G \times^H X \longrightarrow G/H$ morphism
$[g,x] \longmapsto g.H$

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\ G \times^H X \arrow[r, dashed, "\exists !", "p"'] & G/H \end{tikzcd}$$

<u>Note</u>: $y = g.H \in G/H$

$\Rightarrow X \longrightarrow p^{-1}(y)$, $x \longmapsto [g,x]$ bijective morphism.

<u>Local section of π</u>:

$$\begin{tikzcd} G \arrow[r, "\pi"] & G/H \\ & \mathcal{U} \arrow[u, phantom, "\cup\!\!\shortmid"] \arrow[ul, "\sigma"] \end{tikzcd}$$
$\mathcal{U} \neq \emptyset$ open

$\pi \sigma = 1_{\mathcal{U}}$.

<u>Note</u>: $\exists$ local section $\Rightarrow$ G/H "covered" by sections.

$\sigma^g : g.\mathcal{U} \longrightarrow G$, $\sigma^g(y) = g\,\sigma(g^{-1}.y)$.

$\pi \sigma^g = 1_{g.\mathcal{U}}$

<u>Example</u> G/H projective $\Rightarrow$ $\exists$ local sections.

Prop Assume $\pi : G \longrightarrow G/H$ has local sections. Then $G \times^H X$ is a variety and $p : G\times^H X \longrightarrow G/H$ is locally trivial with fiber X.

$$\begin{array}{ccc} G\times^H X & \xrightarrow{\quad p \quad} & G/H \\ \cup\!\!| & & \cup\!\!| \\ p^{-1}(U) \cong U\times X & \xrightarrow{\quad pr_1 \quad} & U \text{ open} \end{array}$$

Proof

Assume $\sigma : U \longrightarrow G$ local section of π.

$$\begin{array}{ccc} U\times X & \xrightarrow{\quad \cong \quad} & p^{-1}(U) \\ (y,x) & \longmapsto & [\sigma(y), x] \\ (\pi(g), \underbrace{\sigma(\pi(g))^{-1} g}_{\in H} . x) & \longleftarrow\!\shortmid & [g,x] \end{array}$$

$\square$

Cor π has local sections $\Leftrightarrow$ π locally trivial (Fiber H)

Proof: $G = G\times^H H \xrightarrow{\pi = p} G/H$. $\square$

Example $H \longrightarrow GL(V)$ rational rep. $\Rightarrow$

$G\times^H V \longrightarrow G/H$ vector bundle.

Thm $\phi: X \longrightarrow Y$ morphism of irred. vars.
ϕ is separable (and dominant) $\Leftrightarrow$
$\exists$ dense open $U \subseteq X$: $\forall x \in U$: $d\phi_x: T_x X \twoheadrightarrow T_{\phi(x)} Y$ surj.

Cor $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$, X, Y, Z irred.
ϕ and ψ separable $\Rightarrow$ $\psi\phi$ separable $\Rightarrow$ ψ separable.

Cor $\phi: X \longrightarrow X'$, $\psi: Y \longrightarrow Y'$, $\phi \times \psi: X \times Y \longrightarrow X' \times Y'$.
ϕ and ψ separable $\Leftrightarrow$ $\phi \times \psi$ separable.

Quotients of products

X, Y irred. varieties, $\sim_X$, $\sim_Y$ equiv rels.
$\sim = (\sim_X, \sim_Y)$ on $X \times Y$.
Bijective morphism of SWF:

$$(X \times Y)/\sim \longrightarrow (X/\sim_X) \times (Y/\sim_Y)$$

Prop Assume $X/\sim_X$, $Y/\sim_Y$, $(X \times Y)/\sim$ are <u>varieties</u>,
$X/\sim_X$ and $Y/\sim_Y$ are <u>normal</u>,
and $X \longrightarrow X/\sim_X$ and $Y \longrightarrow Y/\sim_Y$ <u>separable</u>.
Then $(X \times Y)/\sim \;\cong\; (X/\sim_X) \times (Y/\sim_Y)$.

Proof

$$\begin{tikzcd} X \times Y \arrow[d] \arrow[dr, "\text{separable}"] & \\ (X \times Y)/\sim \arrow[r, "\substack{\text{bijective} \\ \text{separable,} \\ \text{normal target.}}"'] & (X/\sim_X) \times (Y/\sim_Y) \end{tikzcd}$$

Zariski $\Rightarrow$ iso.

$\square$

<u>Prop</u> G LAG, $H \subseteq G$ closed normal subgroup.
Then G/H is a LAG.

<u>Proof</u>

$$\begin{tikzcd}
G \times G \arrow[r, "{(x,y) \mapsto xy^{-1}}"] \arrow[d] & G \arrow[d] \\
G \times G / H \times H \arrow[r, dashed, "\exists!"] \arrow[d, phantom, "\parallel"] & G/H \\
G/H \times G/H \arrow[ur] &
\end{tikzcd}$$

$(x.H, y.H) \mapsto xy^{-1}.H$

$\therefore$ G/H alg. group.

Choose $\phi : G \longrightarrow GL(V)$, $0 \neq v \in V$:

$H = \{ g \in G \mid \phi(g).v \in kv \}$

$\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\phi(X).v \in kv \}$

Given character $\chi : H \longrightarrow \mathbb{G}_m$:

$V_\chi = \{ u \in V \mid \phi(h).u = \chi(h)u \ \forall h \in H \}$

$g \in G \Rightarrow g.V_\chi = V_{g.\chi}$ where $(g.\chi)(h) = \chi(g^{-1}hg)$:

$g.u \in g.V_\chi$: $h.(g.u) = gg^{-1}hg.u = \chi(g^{-1}hg)\, g.u$

<u>Note</u>: $v \in V_\chi$ for some χ.

<u>Note</u>: $\sum V_\chi = \bigoplus V_\chi$.

WLOG: $V = \bigoplus_\chi V_\chi$.

<u>Def</u>: $W = \bigoplus_\chi \mathrm{End}_k(V_\chi) \subseteq \mathrm{End}_k(V)$.

<u>Note</u>: Given $\alpha : V \longrightarrow V$ linear:

$\alpha f = f\alpha \ \forall f \in W \Leftrightarrow$

$\alpha : V_\chi \longrightarrow V_\chi$ mult. by scalar $\forall \chi$

Def: $\psi : G \longrightarrow GL(W)$, $\psi(g).f = \phi(g) f \phi(g)^{-1}$, $f \in W$.

$Ker(\psi) = H$: $\psi(g) = 1_W \Leftrightarrow \phi(g) f = f \phi(g) \ \forall f \in W$

$\Rightarrow \phi(g).v \in kv \Rightarrow g \in H$

$$\begin{tikzcd}
G \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & GL(W) \\
& G/H \arrow[ur, dashed, "\exists! \ \lambda"'] &
\end{tikzcd}$$

λ injective hom. of alg. groups.

$\lambda(G/H) = \psi(G) \subseteq GL(W)$ closed subgroup.

$$\begin{tikzcd}
T_eG \arrow[rr, "d\psi"] \arrow[dr, "d\pi_e"'] & & T_e(\psi(G)) \\
& T_e(G/H) \arrow[ur, "d\lambda_e"'] &
\end{tikzcd}$$

Exer: $X \in \mathfrak{g}$, $f \in W \Rightarrow d\psi(X).f = d\phi(X) f - f\, d\phi(X)$

$Ker(d\psi) \subseteq T_eH$:

$X \in Ker(d\psi) \Leftrightarrow d\phi(X) f = f\, d\phi(X) \quad \forall f \in W$

$\Rightarrow d\phi(X).v \in kv \Rightarrow X \in T_eH$

$\dim d\psi(T_eG) = \dim G - \dim Ker(d\psi)$

$\geq \dim G - \dim H = \dim G/H = \dim \psi(G)$.

$\lambda : G/H \longrightarrow \psi(G)$ bijective separable group hom.

$\Rightarrow$ isomorphism.

$\square$