

Thm  $G$  LAG,  $H \subseteq G$  closed subgroup.

- (1)  $G/H$  is quasi-projective.
- (2)  $G \rightarrow G/H$  is separable.
- (3)  $\dim(G/H) = \dim(G) - \dim(H)$ .

Proof

Let  $(X, x)$  be as in Corollary:

- $X$  quasi-projective homogeneous  $G$ -variety.
- $G_x = H$
- $\psi: G \rightarrow X, g \mapsto g \cdot x$  is separable.

$H = G_x \Rightarrow X = G/H$  as set.

$\psi$  open:  $U \subseteq X$  open  $\Leftrightarrow \psi^{-1}(U) \subseteq G$  open.

$f: U \rightarrow k$  regular  $\Leftrightarrow f\psi: \psi^{-1}(U) \rightarrow k$  regular.

$\therefore X = G/H$  as SWF.

□

Notation:

$G/H = \{g \cdot H \mid g \in G\}, \pi: G \rightarrow G/H, \pi(g) = g \cdot H.$

## Fiber bundles

$G$  LAG,  $H \subseteq G$  closed subgroup,  $H \curvearrowright X$ .

$G \times X \curvearrowright H$ ,  $(g, x) \cdot h = (gh, h^{-1} \cdot x)$ .

$G \times^H X = (G \times X) / H$  SWF

$$= \{ [g, x] \mid g \in G, x \in X \} / \{ [gh, x] = [g, h \cdot x] \forall h \in H \}$$

$p: G \times^H X \longrightarrow G/H$  morphism  
 $[g, x] \longmapsto g \cdot H$

$$\begin{array}{ccc} G \times X & \longrightarrow & G \\ \downarrow & & \downarrow \pi \\ G \times^H X & \xrightarrow[\rho]{\exists!} & G/H \end{array}$$

Note:  $\gamma = g \cdot H \in G/H$

$\Rightarrow X \longrightarrow p^{-1}(\gamma)$ ,  $x \longmapsto [g, x]$  bijective morphism.

Local section of  $\pi$ :

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H \\ & \searrow \sigma & \downarrow \iota \\ & & U \end{array} \quad \begin{array}{l} \pi \sigma = 1_U \\ U \neq \emptyset \text{ open} \end{array}$$

Note:  $\exists$  local section  $\Rightarrow G/H$  "covered" by sections.

$$\sigma^g: g \cdot U \longrightarrow G, \quad \sigma^g(\gamma) = g \sigma(g^{-1} \cdot \gamma).$$

$$\pi \sigma^g = 1_{g \cdot U}$$

Example  $G/H$  projective  $\Rightarrow \exists$  local sections.

Prop Assume  $\pi: G \rightarrow G/H$  has local sections.

Then  $G \times^H X$  is a variety and  $p: G \times^H X \rightarrow G/H$  is locally trivial with fiber  $X$ .

$$\begin{array}{ccc} G \times^H X & \xrightarrow{p} & G/H \\ \cup & & \cup \\ p^{-1}(U) \cong U \times X & \xrightarrow{p|_U} & U \text{ open} \end{array}$$

Proof

Assume  $\sigma: U \rightarrow G$  local section of  $\pi$ .

$$\begin{array}{ccc} U \times X & \xrightarrow{\cong} & p^{-1}(U) \\ (y, x) & \longmapsto & [\sigma(y), x] \\ (\underbrace{(\pi(g), \sigma(\pi(g))^{-1}g \cdot x)}_H) & \longleftarrow & [g, x] \end{array}$$

□

Cor  $\pi$  has local sections  $\Leftrightarrow \pi$  locally trivial (Fiber  $H$ )

Proof:  $G = G \times^H H \xrightarrow{\pi = p} G/H$ . □

Example  $H \rightarrow GL(V)$  rational rep.  $\Rightarrow$

$G \times^H V \rightarrow G/H$  vector bundle.

Thm  $\phi: X \rightarrow Y$  morphism of irred. vars.

$\phi$  is separable (and dominant)  $\Leftrightarrow$

$\exists$  dense open  $U \subseteq X: \forall x \in U: d\phi_x: T_x X \rightarrow T_{\phi(x)} Y$  surj.

Cor  $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ ,  $X, Y, Z$  irred.

$\phi$  and  $\psi$  separable  $\Rightarrow \psi \circ \phi$  separable  $\Rightarrow \psi$  separable.

Cor  $\phi: X \rightarrow X', \psi: Y \rightarrow Y', \phi \times \psi: X \times Y \rightarrow X' \times Y'$

$\phi$  and  $\psi$  separable  $\Leftrightarrow \phi \times \psi$  separable.

### Quotients of products

$X, Y$  irred. varieties,  $\sim_X, \sim_Y$  equiv rels.

$\sim = (\sim_X, \sim_Y)$  on  $X \times Y$ .

Bijjective morphism of SWF:

$$(X \times Y) / \sim \longrightarrow (X / \sim_X) \times (Y / \sim_Y)$$

Prop Assume  $X / \sim_X, Y / \sim_Y, (X \times Y) / \sim$  are varieties,

$X / \sim_X$  and  $Y / \sim_Y$  are normal,

and  $X \rightarrow X / \sim_X$  and  $Y \rightarrow Y / \sim_Y$  separable.

Then  $(X \times Y) / \sim \cong (X / \sim_X) \times (Y / \sim_Y)$ .

Proof

$$\begin{array}{ccc} X \times Y & \xrightarrow{\text{separable}} & (X / \sim_X) \times (Y / \sim_Y) \\ \downarrow & & \\ (X \times Y) / \sim & \xrightarrow{\text{bijjective, separable, normal target. Zariski} \Rightarrow \text{iso.}} & (X / \sim_X) \times (Y / \sim_Y) \end{array}$$

□

Prop  $G$  LAG,  $H \subseteq G$  closed normal subgroup.

Then  $G/H$  is a LAG.

Proof

$$\begin{array}{ccc}
 G \times G & \xrightarrow{(x,y) \mapsto xy^{-1}} & G \\
 \downarrow & & \downarrow \\
 G \times G / H \times H & \xrightarrow{\exists!} & G/H \\
 \parallel & \nearrow & \nearrow \\
 G/H \times G/H & \xrightarrow{(x.H, y.H)} & XY^{-1}.H
 \end{array}$$

$\therefore G/H$  alg. group.

Choose  $\phi: G \rightarrow GL(V)$ ,  $0 \neq v \in V$ :

$$H = \{g \in G \mid \phi(g).v \in kv\}$$

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid d\phi(X).v \in kv\}$$

Given character  $\chi: H \rightarrow \mathbb{C}^*$ :

$$V_\chi = \{u \in V \mid \phi(h).u = \chi(h)u \quad \forall h \in H\}$$

$$g \in G \Rightarrow g.V_\chi = V_{g.\chi} \quad \text{where } (g.\chi)(h) = \chi(g^{-1}hg):$$

$$g.u \in g.V_\chi: h.(g.u) = gg^{-1}hg.u = \chi(g^{-1}hg)g.u$$

Note:  $v \in V_\chi$  for some  $\chi$ .

$$\text{Note: } \sum V_\chi = \bigoplus V_\chi.$$

$$\text{WLOG: } V = \bigoplus_{\chi} V_\chi.$$

$$\text{Def: } W = \bigoplus_{\chi} \text{End}_k(V_\chi) \subseteq \text{End}_k(V).$$

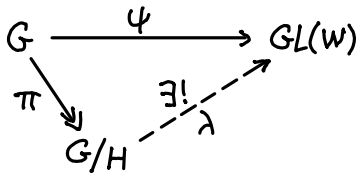
Note: Given  $\alpha: V \rightarrow V$  linear:

$$\alpha f = f \alpha \quad \forall f \in W \Leftrightarrow$$

$$\alpha: V_\chi \rightarrow V_\chi \quad \text{mult. by scalar } \forall \chi$$

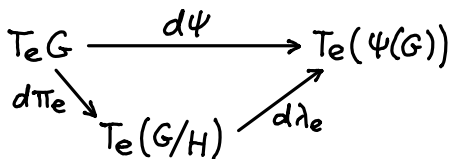
Def:  $\psi: G \rightarrow GL(W)$ ,  $\psi(g).f = \phi(g)f\phi(g)^{-1}$ ,  $f \in W$ .

$\text{Ker}(\psi) = H: \psi(g) = 1_W \Leftrightarrow \phi(g)f = f\phi(g) \forall f \in W$   
 $\Rightarrow \phi(g).v \in \mathfrak{k}_V \Rightarrow g \in H$   $\curvearrowright$



$\lambda$  injective hom. of  
 alg. groups.

$\lambda(G/H) = \psi(G) \subseteq GL(W)$  closed subgroup.



Exer:  $X \in \mathfrak{g}$ ,  $f \in W \Rightarrow d\psi(X).f = d\phi(X)f - f d\phi(X)$

$\text{Ker}(d\psi) \subseteq T_e H:$

$X \in \text{Ker}(d\psi) \Leftrightarrow d\phi(X)f = f d\phi(X) \forall f \in W$   
 $\Rightarrow d\phi(X).v \in \mathfrak{k}_V \Rightarrow X \in T_e H$

$\dim d\psi(T_e G) = \dim G - \dim \text{Ker}(d\psi)$

$\geq \dim G - \dim H = \dim G/H = \dim \psi(G).$

$\lambda: G/H \rightarrow \psi(G)$  bijective separable group hom.

$\Rightarrow$  isomorphism.

□