LAG 14 2026-03-05

G LAG, $H \subseteq G$ closed subgroup. $\mathfrak{g} = L(G)$, $\mathfrak{h} = L(H)$.

Lemma $\exists\ W \subseteq V \subseteq k[G]$:

(1) $\rho : G \longrightarrow GL(V)$ rational rep.

(2) $H = \{ g \in G \mid \rho(g).W = W \}$

(3) $\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\rho(X).W \subseteq W \}$

Proof

$I(H) = \langle f_1, \dots, f_r \rangle \subseteq k[G]$.

$\exists\ \mathrm{Span}_k \{ f_1, \dots, f_r \} \subseteq V \subset k[G]$:

$\rho : G \longrightarrow GL(V)$ rational rep.

$W = V \cap I(H)$.

$g \in G$: $\quad g \in H \Leftrightarrow \rho(g).I(H) = I(H) \Leftrightarrow \rho(g).W = W$.

$X \in \mathfrak{g}$: $\quad X \in \mathfrak{h} \Leftrightarrow \bar{X}.I(H) \subseteq I(H) \Leftrightarrow \bar{X}.W \subseteq W$.

$\square$

Thm $\exists$ rational rep. $\phi: G \longrightarrow GL(U)$, $0 \neq u \in U$:
$H = \{ g \in G \mid \phi(g).u \in ku \}$ and
$\mathfrak{h} = \{ X \in \mathfrak{g} \mid d\phi(X).u \in ku \}$.

Proof

Let $W \subseteq V \subseteq k[G]$ be as in Lemma, $d = \dim(W)$.

$U = \Lambda^d V$, $\quad 0 \neq u \in \Lambda^d W \subseteq U$.

$W = \{ v \in V \mid v \wedge u = 0 \in \Lambda^{d+1} V \}$ determined by u.

$\phi = \Lambda^d \rho : G \longrightarrow GL(U)$.

$x \in G$: $\quad x \in H \Leftrightarrow \rho(x).W = W \Leftrightarrow \phi(x).u \in ku$.

$u = w_1 \wedge \cdots \wedge w_d$, $\quad \{w_1, \ldots, w_d\}$ basis of W.

$$d\phi(X).u = \sum_{i=1}^{d} w_1 \wedge \cdots \wedge d\rho(X).w_i \wedge \cdots \wedge w_d$$

If $d\rho(X).W \nsubseteq W$: $\quad d\rho(X).w_j = w + v$, $\quad w \in W$, $v \notin W$.

$d\phi(X).u$ "contains" $w_1 \wedge \cdots \wedge w_{j-1} \wedge v \wedge w_{j+1} \wedge \cdots \wedge w_d$.

$X \in \mathfrak{h} \Leftrightarrow d\rho(X).W \subseteq W \Leftrightarrow d\phi(X).u \in ku$.

$\square$

Def $\phi: X \longrightarrow Y$ morphism of varieties.

ϕ is <u>separable</u> if $\forall$ conn. comp. $X' \subseteq X$:

X' is irred., $\overline{\phi(X')}$ is conn. comp. of Y,

$k(\overline{\phi(X')}) \subseteq k(X')$ separably generated.

<u>Cor</u> $\exists$ quasi-projective hom. G-variety X, $x \in X$:

(1) $H = G_x = \{g \in G \mid g.x = x\}$

(2) $\psi: G \longrightarrow X$, $g \mapsto g.x$ separable.

<u>Proof</u>

Let $\phi: G \to GL(U)$, $0 \neq u \in U$ be as in Theorem.

$\mathbb{P}(U) = \{[v] = kv \mid 0 \neq v \in U\}$

$G \circlearrowright \mathbb{P}(U)$, $g.[v] = [\phi(g).v]$

$x = [u]$, $X = G.x \subseteq \mathbb{P}(U)$. $H = G_x$ is clear.

$$\begin{array}{ccccccc} \psi: G & \xrightarrow{\phi} & GL(U) & \xrightarrow{A \mapsto A.u} & U \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}(U) \\ & & \cap\!| & & \cap\!| & & \\ & & End(U) & \xrightarrow{\text{linear}} & U & & \end{array}$$

$$\begin{array}{ccccccc} d\psi_e: T_eG & \xrightarrow{d\phi} & End(U) & \xrightarrow{A \mapsto A.u} & U & \longrightarrow & U/ku. \\ X & \longmapsto & d\phi(X) & & \longmapsto & & d\phi(X).u + ku \end{array}$$

$Ker(d\psi_e) = \mathfrak{h} \Rightarrow$

$\dim d\psi_e(\mathfrak{g}) = \dim G - \dim H = \dim X.$

$d\psi_e: T_eG \twoheadrightarrow T_xX$ surjective.

$\therefore$ $\psi: G \longrightarrow X$ separable.

$\square$

<u>Lemma</u> $h : X \longrightarrow Y$ surjective open map of top. spaces.

$Y' \subseteq Y$ subset. $h^{-1}(Y') \subseteq X$ closed $\Rightarrow$ $Y' \subseteq Y$ closed.

<u>Proof</u>: $Y - Y' = h(X - h^{-1}(Y'))$ is open. $\square$

<u>Lemma</u> $F \subseteq E$ separably gen. extension.

$a \in E$ alg. over F $\Rightarrow$ a separable over F.

<u>Proof</u>

Choose tr. basis $\{b_1, \dots, b_n\}$ of E/F s.t.

E/E' separable, $E' = F(b_1, \dots, b_n)$.

$p(T) \in E'[T]$ min. poly of a / E'.

Then $p(T)$ has distinct roots.

$q(T) \in F[T]$ min. poly of a / F.

$p(T) \mid q(T)$ in $E'[T]$ $\Rightarrow$ $p(T) \in \overline{F}[T] \cap E'[T] = F[T]$.

$\therefore$ $q(T) = p(T)$ has distinct roots.

$\square$

Prop X hom. G-variety, $x \in X$.
Assume $\psi: G \longrightarrow X$, $\psi(g) = g.x$ separable.
$U \subseteq X$ open, $f: U \longrightarrow k$ any function.
Then $f \in \mathcal{O}_X(U) \Leftrightarrow f\psi \in \mathcal{O}_G(\psi^{-1}(U))$.

Proof of $\Leftarrow$:
WLOG G connected.

$\Gamma = \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{A}^1$ subset.

$\psi: G \longrightarrow X$ equivariant of hom. G-varieties
$\Rightarrow \psi \times 1: G \times \mathbb{A}^1 \longrightarrow U \times \mathbb{A}^1$ is open.

$f\psi$ regular fcn $\Rightarrow$
$(\psi \times 1)^{-1}(\Gamma) = \{(g, f\psi(g)) \mid g \in \psi^{-1}(U)\} \subseteq \psi^{-1}(U) \times \mathbb{A}^1$ closed
$\Rightarrow \Gamma \subseteq U \times \mathbb{A}^1$ closed. (Lemma)
$\therefore \Gamma$ is a variety.

$$G \xrightarrow{(\psi, f\psi)} \Gamma \xrightarrow{pr_1} U$$

$$k(G) \supseteq k(\Gamma) \supseteq k(X)$$

$k(G)/k(X)$ separably gen. $\Rightarrow k(\Gamma)/k(X)$ separable.
(Lemma)

$\Gamma \longrightarrow U$ bijective & separable $\Rightarrow$ birational.

U non-singular.
Zariski's Main Thm. $\Rightarrow pr_1: \Gamma \xrightarrow{\cong} U$ iso.
$\therefore f: U \xrightarrow{\cong} \Gamma \xrightarrow{pr_2} \mathbb{A}^1$ regular.
$\square$

Quotients

X SWF. $\sim$ equiv. rel. on X.

$\pi : X \longrightarrow X'$ morphism.

Def π <u>respects</u> $\sim$ if $x_1 \sim x_2 \Rightarrow \pi(x_1) = \pi(x_2)$.

π is a <u>universal morphism respecting</u> $\sim$ if

$\forall$ morphism of SWF $f : X \longrightarrow Y$ respecting $\sim$

$\exists!$ morphism $\tilde{f} : X' \longrightarrow Y$ s.t. $f = \tilde{f}\pi$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & X' \arrow[ur, "\exists!"', "\tilde{f}"] & \end{tikzcd}$$

Construction

$X' = X/\sim$ as set. $\pi : X \longrightarrow X/\sim$

$U \subseteq X'$ open $\Leftrightarrow$ $\pi^{-1}(U) \subseteq X$ open.

$f : U \longrightarrow k$ regular $\Leftrightarrow$ $f\pi : \pi^{-1}(U) \longrightarrow k$ regular.

Exer $\pi : X \longrightarrow X/\sim$ univ. morphism respecting $\sim$.

X SWF, $X \circlearrowleft G$ right action.

$X/G = X/\sim$, $x_1 \sim x_2 \Leftrightarrow x_1.G = x_2.G$.

Example $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$.

Example $\mathbb{A}^1/\mathbb{G}_m = \{0, *\}$ SWF.

$\{0\}$ closed, $\{*\}$ open. $\mathcal{O}(\mathbb{A}^1/\mathbb{G}_m) = k$.

Def G alg. group, X G-variety.

The quotient X/G is <u>separable</u> if

(1) X/G alg. variety

(2) $\pi : X \longrightarrow X/G$ is separable.