Lie algebra of GL(V)

E vector space $/k$, $\dim_k(E) < \infty$.

As variety: $k[E] = \text{Sym}^\circ(E^*)$

$E^* \subseteq k[E]$ linear fcns.

$p \in E$: $T_p E = \text{Der}_k(k[E], k(p)) = \text{Hom}_k(E^*, k) = E$

$X \in T_p E$, $f \in E^* \subseteq k[E]$: $X(f) = (f, X)$

Assume $E = \text{End}_k(V)$, $\dim(V) < \infty$.

Perfect pairing: $E \times E \longrightarrow k$, $(A, B) \mapsto \text{tr}(AB)$

For $A \in E$, def. $f_A \in E^*$ by $(f_A, B) = \text{tr}(AB)$.

$GL(V) \subseteq \text{End}(V)$ LAG.

$\mathfrak{gl}(V) = \text{End}(V)$ Lie algebra: $[X, Y] = XY - YX$.

$T_e GL(V) = T_e E = \mathfrak{gl}(V)$

Prop $\mathfrak{gl}(V) \xrightarrow{\cong} L(GL(V))$, $X \mapsto \bar{X}$ iso. of Lie algebras.

Proof

$X, A \in E$, $B, C \in GL(V)$.

$\lambda(B^{-1}) \cdot f_A = f_{AB}$: $(\lambda(B^{-1}) \cdot f_A)(C) = f_A(BC) = \text{tr}(ABC) = f_{AB}(C)$

$\bar{X} f_A = f_{XA}$: $\bar{X} f_A(B) = X(\lambda(B^{-1}) \cdot f_A) = X(f_{AB}) = \text{tr}(XAB) = f_{XA}(B)$

$[\bar{X}, \bar{Y}] \cdot f_A = (\bar{X}\bar{Y} - \bar{Y}\bar{X}) \cdot f_A = f_{XYA} - f_{YXA} = \overline{[X, Y]} \cdot f_A$.

$\square$

Lie algebra of LAG

Note: $\phi : V \longrightarrow W$ k-linear map, V, W finite dim.

Then $d\phi_v = \phi$ for all $v \in V$.

$$
\begin{array}{ccccccc}
T_v V & = & \mathrm{Der}_k(k[V], k(v)) & = & \mathrm{Hom}_k(V^*, k) & = & V \\
\downarrow d\phi_v & & \downarrow d\phi_v & & \downarrow \phi^{**} & & \downarrow \phi \\
T_{\phi(v)} W & = & \mathrm{Der}_k(k[W], k(\phi(v))) & = & \mathrm{Hom}_k(W^*, k) & = & W
\end{array}
$$

G LAG, $\nu : G \longrightarrow GL(V)$ rat. rep.

$d\nu : L(G) \longrightarrow \mathfrak{gl}(V)$ Lie algebra homomorphism.

Lemma $\phi : \mathrm{End}(V) \longrightarrow k$ k-linear map, $X \in T_e G$.

Then $\phi(d\nu(X)) = X(\nu^*(\phi))$.

Proof

$\phi \circ \nu : G \longrightarrow \mathbb{A}^1$ morphism, $T_{\phi(e)} \mathbb{A}^1 = k$.

$\phi(d\nu(X)) = d\phi_e(d\nu(X)) = d(\phi \circ \nu)_e(X) = X(\phi \circ \nu)$.

$\square$

Assume V has basis $\{v_1, \ldots, v_n\}$.

$GL(V) = GL_n$, $\mathfrak{gl}(V) = \mathfrak{gl}_n = \mathrm{Mat}(n \times n, k)$.

$A \in \mathrm{End}(V) : \quad A = (a_{ij}), \quad A.v_j = \sum_i a_{ij} v_i$

$k[\mathrm{End}(V)] = k[T_{ij}] : \quad T_{ij}(A) = a_{ij}$.

$\nu : G \longrightarrow GL_n, \quad \nu(g) = (\nu_{ij}(g)). \quad \nu_{ij} = \nu^*(T_{ij}) \in k[G].$

$X \in T_e G. \quad d\nu(X) = (b_{ij}) \in \mathfrak{gl}_n.$

$b_{ij} = T_{ij}(d\nu(X)) = X(\nu^*(T_{ij})) = X(\nu_{ij}).$

Prop G LAG, $V \subseteq k[G]$, $\dim(V) < \infty$, $\rho(x).V = V$ $\forall x \in G$.

$\qquad \rho: G \longrightarrow GL(V)$ rat. rep., $d\rho: T_e G \longrightarrow End(V)$.

$\qquad$ Then $\bar{X}f = d\rho(X).f$ $\forall X \in T_e G$, $f \in V$.

Proof

Fix $g \in G$, $f \in V$. Def. $\phi: End(V) \longrightarrow k$, $\phi(Y) = (Y.f)(g)$.

$\lambda(g^{-1}).f = \rho^* \phi \in k[G]$:

$\qquad (\lambda(g^{-1}).f)(x) = f(gx) = (\rho(x).f)(g) = \phi(\rho(x))$.

$(\bar{X}f)(g) = X(\lambda(g^{-1}).f) = X(\rho^* \phi) = \phi(d\rho(X)) = (d\rho(X).f)(g)$.

$\square$

Cor $\bar{X}: k[G] \longrightarrow k[G]$ is locally finite $\forall X \in T_e G$.

Exer: G LAG. $Ad: G \longrightarrow GL(T_e G)$, $dAd: T_e G \longrightarrow End(T_e G)$.

$\qquad dAd(X)(Y) = [X, Y]$ $\forall X, Y \in T_e G$.

Exer: $\nu: G \longrightarrow GL(V)$ rat. rep.

$\qquad \wedge^n \nu: G \longrightarrow GL(\wedge^n V)$, $d(\wedge^n \nu): T_e G \longrightarrow End(\wedge^n V)$.

$\qquad d(\wedge^n \nu)(X).(v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{n} v_1 \wedge \cdots \wedge d\nu(X).v_i \wedge \cdots \wedge v_n$.

Jordan decomp in $L(G)$

G LAG, $X \in T_e G$.

$\bar{X}: k[G] \longrightarrow k[G]$ locally finite.

$\bar{X} = \bar{X}_s + \bar{X}_n$, $\bar{X}_s$ semi-simple, $\bar{X}_n$ nilpotent, $\bar{X}_s \bar{X}_n = \bar{X}_n \bar{X}_s$.

Thm (1) $\bar{X}_s, \bar{X}_n \in L(G)$ and $[\bar{X}_s, \bar{X}_n] = 0$.

$\qquad$ (2) $\phi: G \longrightarrow G'$ hom. of LAGs $\Rightarrow$

$\qquad\qquad d\phi(X_s) = d\phi(X)_s$, $d\phi(X_n) = d\phi(X)_n$

$\qquad$ (3) $G = GL_n \Rightarrow X = X_s + X_n$ is usual Jordan decomp. in M_n.

Fibers of morphisms

$\phi: X \longrightarrow Y$ dominant, X, Y irred. affine.

$\phi^*: k[Y] \subseteq k[X], \quad k(Y) \subseteq k(X).$

$k[X] = k[Y][f_1, \ldots, f_m] = k[Y][T_1, \ldots, T_m]/I$

$X \cong Z(I) \subseteq Y \times \mathbb{A}^m$ closed subvariety.

$x \longmapsto (\phi(x), f_1(x), \ldots, f_m(x))$

WLOG: $\{f_1, \ldots, f_r\}$ transcendence basis of $k(X)/k(Y)$.

$\quad\quad r = \dim X - \dim Y$ relative dimension.

$k[Y] \subseteq k[Y][f_1, \ldots, f_r] \subseteq k[X]$

$$Y \longleftarrow Y \times \mathbb{A}^v \xleftarrow{\text{gen. finite}} X$$

$$\phi(x) \longleftarrow (\phi(x), f_1(x), \ldots, f_r(x)) \longleftarrow x$$

Fact: $\phi^{-1}(y) \neq \emptyset \Rightarrow \dim \phi^{-1}(y) \geq v.$

Def Assume $\phi: X \longrightarrow Y$ dominant.

$\quad\quad \phi$ is generically finite: $k(X)/k(Y)$ finite ext.

$\quad\quad \phi$ is finite: $k[X]$ f.g. $k[Y]$-module.

Finite $\Rightarrow$ generically finite.

Assume $\phi: X \longrightarrow Y$ gen. finite, $k[X] = k[Y][f]$.
$f \in k(X)$ algebraic over $k(Y)$.
$$f^d + a_{d-1} f^{d-1} + \cdots + a_1 f + a_0 = 0, \qquad a_i \in k(Y)$$
$d = [k(X) : k(Y)]$

Choose $0 \neq h \in k[Y]$ s.t. $a_i \in k[Y]_h$ $\forall i$.
$Y_h = \{ y \in Y \mid h(y) \neq 0 \} \subseteq Y$ open affine.
$k[Y_h] = k[Y]_h$

$$k[X_h] \cong k[Y_h][T] / \langle T^d + \cdots + a_1 T + a_0 \rangle$$
$\qquad\qquad$ free $k[Y_h]$ - module gen. by $\{1, T, \ldots, T^{d-1}\}$.

$$X \xrightarrow{\;\text{gen. finite}\;} Y$$
$$\cup | \qquad\qquad\qquad \cup | \text{ open}$$
$$X_h \xrightarrow{\;\text{finite}\;} Y_h$$

Note: $\phi : X_h \longrightarrow Y_h$ surjective with finite fibers:
$$X_h \cong \{ (y, t) \in Y_h \times \mathbb{A}^1 \mid t^d + \cdots + a_1(y) t + a_0(y) = 0 \}.$$
$$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d + \cdots + a_1(y) t + a_0(y) = 0 \}$$

Assume $f \in k(X)$ separable over $k(Y)$.
$T^d + \cdots + a_1 T + a_0$ has d distinct roots in $\overline{k(Y)}$.
$\Rightarrow \forall y \in$ dense open $\subseteq Y_h$:
$\qquad T^d + \cdots + a_1(y) T + a_0(y)$ has d distinct roots in k.

Assume $f \in k(X)$ purely inseparable over $k(Y)$:
$d = p^j, \qquad f^d = a \in k(Y)$.
$$\phi^{-1}(y) = \{ t \in \mathbb{A}^1 \mid t^d = a(y) \} = \{ \sqrt[d]{a(y)} \}.$$