GOODIES 2

A morphism of varieties \( \varphi : X \to Y \) is dominant if \( \varphi(X) = Y \).

**Problem 1.** The commutative algebra result lying over states that, if \( R \subset S \) is an integral extension of commutative rings and \( P \subset R \) is a prime ideal, then there is some prime \( Q \subset S \) such that \( Q \cap R = P \).

(a) Use lying over to show that if \( \varphi : X \to Y \) is a dominant morphism of irreducible varieties, then \( \varphi(X) \) contains a dense open subset of \( Y \).

(b) If \( \varphi : X \to Y \) is any morphism of varieties, then its image \( \varphi(X) \) is constructible, i.e. a finite union of locally closed subsets of \( Y \).

**Problem 2.** Let \( m_0, m_1, \ldots, m_N \in k[x_0, \ldots, x_n] \) be all the monomials of degree \( d \). The Veronese embedding is the map \( v_d : \mathbb{P}^n \to \mathbb{P}^N \) defined by

\[
v_d(x_0 : \cdots : x_n) = (m_0(x_0, \ldots, x_n) : \cdots : m_N(x_0, \ldots, x_n)).
\]

(a) Show that \( v_d \) is an isomorphism of \( \mathbb{P}^n \) with a closed subvariety in \( \mathbb{P}^N \).

(b) Let \( S \subset \mathbb{P}^n \) be a hypersurface of degree \( d \), i.e. \( S = V_d(f) \) where \( f \in k[x_0, \ldots, x_n] \) is an irreducible form of degree \( d \). Show that \( S = v_d^{-1}(H) \) for a unique hyperplane \( H \subset \mathbb{P}^N \).

**Problem 3.** Let \( L_1, L_2, \) and \( L_3 \) be lines in \( \mathbb{P}^3 \) such that none of them meet.

(a) There exists a unique quadric surface \( S \subset \mathbb{P}^3 \) containing \( L_1, L_2, \) and \( L_3 \). [Hint: Start by applying an automorphism of \( \mathbb{P}^3 \) to make the lines nice.]

(b) \( S \) is the disjoint union of all lines \( L \subset \mathbb{P}^3 \) meeting \( L_1, L_2, \) and \( L_3 \).

(c) Let \( L_4 \subset \mathbb{P}^3 \) be a fourth line which does not meet \( L_1, L_2, \) or \( L_3 \). Then the number of lines meeting \( L_1, L_2, L_3, \) and \( L_4 \) is equal to the number of points in \( L_4 \cap S \), which is one, two, or infinitely many.

**Problem 4.** An algebraic group is a pre-variety \( G \) together with morphisms \( m : G \times G \to G \) and \( i : G \to G \), and an identity element \( e \in G \), such that \( G \) is a group in the usual sense when \( m \) is used to define multiplication and \( i \) maps any element to its inverse element.

(a) Show that \( GL_n(k) \) is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that \( \mathbb{P}^1 \) is not an algebraic group, i.e. it is not possible to find morphisms \( m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) and \( i : \mathbb{P}^1 \to \mathbb{P}^1 \) satisfying the group axioms.

(d) Challenge: How about \( \mathbb{P}^n \) for \( n \geq 2 \)?

**Problem 5.** A morphism \( f : X \to Y \) of pre-varieties is called affine if, for every open affine subset \( V \subset Y \), the inverse image \( f^{-1}(V) \) is also affine. The morphism \( f \) is called finite if it is affine and \( k[f^{-1}(V)] \) is a finitely generated \( k[V] \)-module for every open affine \( V \subset Y \).

(a) Let \( Y = \bigcup V_i \) be an open affine covering of \( Y \) such that \( f^{-1}(V_i) \) is affine \( \forall i \). Show that \( f \) is affine. If \( k[f^{-1}(V_i)] \) is a finitely generated \( k[V_i] \)-module for all \( i \) then \( f \) is finite.

(b) If \( f \) is affine and \( Y \) is separated, then \( X \) is separated.
Problem 6. Let $X$ be a variety and $V \subset X$ any subset. Then $V$ inherits a structure of space with functions from $X$. Assume that $V$ is a variety with this structure. Show that $V$ is locally closed in $X$.

Problem 7. Set $E = V(y^2 - x^3 - 3) \subset \mathbb{C}^2$, $P = (1, 2) \in E$, and $U = E \setminus \{P\}$.
(a) Show that $U$ is an open affine subvariety of $E$.
(b) Challenge: $U$ is not of the form $D(f)$ for any regular function $f \in \mathcal{O}_E(E)$.

Problem 8. Set $E = k^{n+1}$ and recall that $\mathbb{P}^n = \{\ell \subset E \mid \ell$ is a line through the origin of $E\}$. Define $S = \{(\ell, v) \in \mathbb{P}^n \times E \mid v \in \ell\}$, and let $\rho : S \to \mathbb{P}^n$ be the projection.
(a) $S$ is a subbundle of rank 1 of the trivial vector bundle $\mathbb{P}^n \times E$.
Define an $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{L}$ by $\Gamma(U, \mathcal{L}) = \{\text{morphisms } s : U \to L \mid \rho s = 1_U\}$.
(b) $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.
Let $\pi : E \setminus \{0\} \to \mathbb{P}^n$ be the projection. For $d \in \mathbb{Z}$ we define an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n}(d)$ by $\Gamma(U, \mathcal{O}(d)) = \{s \in \mathcal{O}_E(\pi^{-1}(U)) \mid s(\lambda v) = \lambda^d s(v) \forall \lambda \in k, v \in E\}$.
(c) The sheaf $\mathcal{O}(d)$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.
(d) Find an integer $d \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(d)$ as an $\mathcal{O}_{\mathbb{P}^n}$-module.