Some algebra review problems that will be needed in our course

1. (a) Use completion of squares to solve $2x^2 - 4x - 5 = 0$.

   (b) Use completion of squares to find the center and radius of the circle $x^2 + y^2 - 4x + 6y - 3 = 0$.

2. For the true-or-false questions, encircle true or false; then give a reason if the assertion is true or a counterexample if the assertion is false.

   (a) True or False: If $A$ and $P$ are $2 \times 2$ matrices with $P$ invertible and $\lambda$ is an eigenvalue for $A$, then $\lambda$ is an eigenvalue for $P^{-1}AP$.

   (b) True or False: If $A$ and $P$ are $2 \times 2$ matrices with $P$ invertible and $v$ is an eigenvector for $A$, then $v$ is an eigenvector for $P^{-1}AP$.

   (c) True or False: If $A$ is an $n \times n$ matrix such that $Ax = b$ is consistent for every vector $b$ in $\mathbb{R}^n$, then $Ax = 0$ has only the zero solution $x = 0$.

   (d) True or False: If $A$ is a $3 \times 2$ matrix whose columns $u, v$ are mutually orthogonal, then $A^TA$ is a diagonal matrix.

   (e) True or False: If $A$ and $B$ are two $n \times n$ invertible matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$. 
3. Suppose $A$ is a $3 \times 3$ matrix and $u, v, w$ are \textit{nonzero vectors} in $\mathbb{R}^3$ such that

$$Au = 2u, \quad Av = -2v, \quad Aw = 0.$$ 

(a) Let $P = [u \mid v \mid w]$ (the $3 \times 3$ matrix with columns $u, v, w$). Find a $3 \times 3$ matrix $D$ so that $AP = PD$. Prove that your answer is correct by calculating $AP$ and $PD$ separately.

(b) Let $x = au + bv + cw$, where $a, b, \text{ and } c$ are scalars. Write the vectors $Ax$ and $A^2x$ as linear combinations of $u, v, \text{ and } w$.

(c) Suppose $a, b, \text{ and } c$ are scalars such that $au + bv + cw = 0$. Prove that $a = 0, b = 0, \text{ and } c = 0$. (Hint: Use (b) with $x = 0$.)

(d) Prove that the matrix $P$ in (a) is invertible and the matrix $A$ is diagonalizable.

(e) Use (a) to find $\det A$ and the characteristic polynomial of $A$ in factorized form.
4. Let $W$ be the subspace of $\mathbb{R}^3$ spanned by the vector $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Let $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the orthogonal projection $w$ of $v$ onto $W$.

(b) Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $W^\perp$. Write down the equation satisfied by $x_1, x_2, x_3$. Use this to find a basis for $W^\perp$. 
5. Let \( W \) be the subspace of \( \mathbb{R}^4 \) with basis vectors

\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 0 \end{bmatrix}.
\]

(a) Apply the Gram-Schmidt process to the vectors \( \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \) given below to obtain an orthogonal set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) with \( \mathbf{v}_1 = \mathbf{u}_1 \). Fill in the table on the left as you calculate. (Don’t normalize \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \).)

\[
\begin{array}{c|c|c|c|c}
\mathbf{v}_1 \cdot \mathbf{v}_1 & \\
\mathbf{u}_2 \cdot \mathbf{v}_1 & \\
\mathbf{v}_2 \cdot \mathbf{v}_2 & \\
\mathbf{u}_3 \cdot \mathbf{v}_1 & \\
\mathbf{u}_3 \cdot \mathbf{v}_2 & \\
\end{array}
\]

\[
\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \end{bmatrix}.
\]

\[
\mathbf{v}_2 =
\]

\[
\mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 =
\]

(Problem 5 continues on next page)
(Continuation of Problem 5)

(b) Normalize the orthogonal basis for the subspace \( W \) from (a) to obtain an orthonormal basis \( \{w_1, w_2, w_3\} \) for \( W \).

(c) If \( v \in \mathbb{R}^4 \) and \( \{w_1, w_2, w_3\} \) is any orthonormal basis for the subspace \( W \), then the orthogonal projection of \( v \) onto \( W \) is the vector \( w = c_1w_1 + c_2w_2 + c_3w_3 \), where

\[
c_1 = \text{__________} \quad c_2 = \text{__________} \quad c_3 = \text{__________}
\]

(give a formula for the coefficients in terms of dot products).

(d) Take \( v = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 0 \end{bmatrix} \) and use the formulas from part (c) and the orthonormal basis from part (b) to calculate the orthogonal projection of \( v \) onto \( W \). Check your answer by calculating the vector \( z = v - w \) and showing that \( z \) is perpendicular to \( u_1, u_2, \) and \( u_3 \).
6. Let
\[ A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}. \]
Find the eigenvalues of \( A \) and prove that \( A \) is diagonalizable, \( i.e., \) there is an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonalizable. Furthermore, \( P \) can be chosen to be an orthogonal matrix, \( i.e., \) a matrix whose columns are unit vectors and whose distinct columns are orthogonal to each other.

7. Let \( A \) be a \( 3 \times 3 \) symmetric matrix with real entries, and have eigenvalues \( \lambda_1 = 0, \lambda_2 = 6, \) and \( \lambda_3 = 3. \) Let \( u_1, u_2, \) and \( u_3 \) be corresponding eigenvectors (normalized to have length one).

(a) Since \( A \) is symmetric and \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are all different, it follows that
\[ u_1 \cdot u_2 = u_1 \cdot u_3 = u_2 \cdot u_3 = \]

(b) The \( 3 \times 3 \) matrix \( P = [u_1 \mid u_2 \mid u_3] \) satisfies
\[ P^T P = \begin{bmatrix} \text{(fill in the entries of this 3 \times 3 matrix)} \end{bmatrix} \]

(c) Let \( P \) be the matrix of normalized eigenvectors from (b). Then \( A = PDP^T, \) where
\[ D = \begin{bmatrix} \text{(fill in the entries of this 3 \times 3 matrix)} \end{bmatrix} \]

(d) The characteristic polynomial of \( A \) is \(

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