Thm X scheme, $E \to X$ vector bundle of rank r. Then

$$A^*(X)[h] / \langle h^r + c_1(E) h^{r-1} + \dots + c_r(E) \rangle \xrightarrow{\cong} A^*(P(E))$$

Proof of injectivity

Set $h = c_1(\mathcal{O}_E(1)) \in A^1(P(E))$.

Segre class: $s_i(E) \cap \alpha = p_*(h^{r-1+i} \cap p^*(\alpha))$,

$\alpha \in A_*(X)$.

$p : P(E) \longrightarrow X$ proper + flat rel. dim. $r-1$.

p_* lifts to A^*:

$$\begin{tikzcd} A^*(P(E)) \arrow[r, "p_*"] \arrow[d, "\cdot [P(E)]"'] & A^*(X) \arrow[d, "\cdot [X]"] \\ A_*(P(E)) \arrow[r, "p_*"] & A_*(X) \end{tikzcd}$$

$c \in A^*(P(E))$. $\quad [p] \in A(P(E) \to X)$

$c \cdot [p] \in A(P(E) \to X)$

$p_*(c) := p_*(c \cdot [p]) \in A^*(X)$.

$\underline{\text{Note}}$: $p_*(h^i) = s_{i-r+1}(E) \in A^*(X)$.

$f : Y \longrightarrow X$, $\alpha \in A_*(Y)$.

$$\begin{tikzcd} P(f^*E) \arrow[r, "p'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ P(E) \arrow[r, "p"'] & X \end{tikzcd}$$

$p_*(h^i)(\alpha) =$

$p'_*\left(c_1(\mathcal{O}_{f^*E}(1))^i \cap p'^*(\alpha)\right) = s_{i-r+1}(f^*E) \cap \alpha$.

p_* $\quad$ h^i $\quad$ $[p]$

$A^*(X)[h]/\langle h^r + \cdots \rangle$ free $A^*(X)$-module with basis $\{1, h, \ldots, h^{r-1}\}$.

Show: $\{1, h, \ldots, h^{r-1}\}$ linearly independent in $A^*(P(E))$ over $A^*(X)$.

Let $c = p^*(c_0) + p^*(c_1)h + \cdots + p^*(c_i)h^i$,

$0 \leq i \leq r-1$, $c_i \neq 0 \in A^*(X)$.

Show: $c \neq 0 \in A^*(P(E))$.

$$p_*(h^{r-1-i} \cdot c) = s_{-i}(E) \cdot c_0 + \cdots + s_0(E) \cdot c_i$$
$$= c_i.$$

Chow ring of non-singular variety

X non-singular variety.

$$A^*(X) \xrightarrow{\cong} A_*(X)$$
$$c \longmapsto c \cdot [X]$$

Let $V, W \subseteq X$ closed subvarieties.

$c_V \in A^*(X)$: $\quad c_V \cdot [X] = [V] \in A_*(X)$

$c_W \in A^*(X)$: $\quad c_W \cdot [X] = [W]$.

Def $\quad V \cdot W = [V] \cdot [W]$

$$:= c_V \cdot c_W \cdot [X] \in A_*(X).$$
$$= c_V \cdot [W].$$

Recall: $\quad c_V = \gamma_f^! \circ pr_1^* : A_*(Y) \longrightarrow A(Y)$:

$f : Y \longrightarrow X$.

$$\begin{tikzcd}
f^{-1}(V) \arrow[r] \arrow[d] & Y \times V \arrow[r, "pr_1"'] \arrow[d] & Y \\
Y \arrow[r, hook, "\gamma_f"'] & Y \times X &
\end{tikzcd}$$

$Y = X$:
$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \times V \arrow[r, "pr_1"'] \arrow[d] & X \\ X \arrow[r, hook, "\delta"'] & X \times X & \end{tikzcd}$$

$$V \cdot W = c_V([W]) = \delta^!(pr_1^*([W]))$$
$$= \delta^!([V \times W]) \in A_*(V \cap W).$$

<u>Note</u>: $V \cdot W = W \cdot V$

$$\begin{tikzcd} & X \times X \arrow[d, "swap"] \\ X \arrow[ur, hook, "\delta"] \arrow[r, hook, "\delta"'] & X \times X \end{tikzcd}$$

<u>Example</u> Assume $V \cap W \hookrightarrow V \times W$ regular embedding of codim. n, $n = \dim(X)$.

Then
$$\begin{tikzcd} V \cap W \arrow[r, hook, "\delta'"'] \arrow[d] & V \times W \arrow[d] \\ X \arrow[r, hook, "\delta"'] & X \times X \end{tikzcd}$$

$$V \cdot W = \delta^!([V \times W])$$
$$= \delta'^*([V \times W])$$
$$= [V \cap W]$$

Example

$V \overset{i}{\hookrightarrow} X$ regular embedding of codim. d,
X non-singular.

Then $V \cdot V = c_d(N_V X) \cap [V]$.

$V \cdot V = i^*([V])$.

$$\begin{tikzcd}
0 = C_V V \subseteq N \arrow[r] \arrow[d] & V \arrow[l, bend right, "s"'] \arrow[r, "="] \arrow[d] & V \arrow[d] \\
N_V X \arrow[r] & V \arrow[r, hook, "i"'] & X
\end{tikzcd}$$

$$i^!([V]) = s^*([C_V V]) = s^*(s_*([V]))$$
$$= c_d(N_V X) \cap [V].$$

Example X non-singular, $\dim(X) = n$.

$\delta : X \hookrightarrow X \times X$.

$N_X(X \times X) = T_X$:

$$0 \to T_X \to T_X \oplus T_X \to N_X(X \times X) \to 0$$

$$X \cdot_\delta X = c_n(T_X) \cap [X].$$

Intersection Multiplicities

X non-singular variety, $\dim(X) = n$

$Y_1, Y_2 \subseteq X$ closed subschemes,
Y_i of pure dim. m_i.

$Z \subseteq Y_1 \cap Y_2$ irred. component.

$\dim(Z) \geq m_1 + m_2 - n$.

$\operatorname{codim}(Z, X) \leq \operatorname{codim}(Y_1, X) + \operatorname{codim}(Y_2, X)$.

Def Z is a proper component if

$$\dim(Z) = \dim(Y_1) + \dim(Y_2) - \dim(X).$$

$Y_1 \cdot Y_2 \in A_{m_1+m_2-n}(Y_1 \cap Y_2)$.

Z proper component $\Rightarrow$

$i(Z, Y_1 \cdot Y_2 ; X) :=$ coef. of $[Z]$ in $Y_1 \cdot Y_2$

well defined.

Note: Coef. of $[Z]$ in $[Y_1 \cap Y_2]$

$$= \operatorname{length}(\mathcal{O}_{Z, Y_1 \cap Y_2}).$$

Prop X non-sing variety,

$Y_1, Y_2 \subseteq X$ closed subschemes,

$Z \subseteq Y_1 \cap Y_2$ proper component.

(a) $1 \le i(Z, Y_1 \cdot Y_2 ; X) \le \text{length}(\mathcal{O}_{Z, Y_1 \cap Y_2})$

(b) $\mathcal{O}_{Z, Y_1 \cap Y_2}$ Cohen-Macaulay $\Rightarrow$

$i(Z, Y_1 \cdot Y_2 ; X) = \text{length}(\mathcal{O}_{Z, Y_1 \cap Y_2})$

(c) Assume Y_1 and Y_2 are varieties:

$$i(Z, Y_1 \cdot Y_2 ; X) = 1$$
$$\Updownarrow$$
$$I(Z) = I(Y_1) + I(Y_2) \subseteq \mathcal{O}_{Z,X}$$
$$\Downarrow$$
$\mathcal{O}_{Z,Y_1}$ and $\mathcal{O}_{Z,Y_2}$ regular local rings.

Example

X non-singular,

$Y_1, Y_2 \subseteq X$ closed CM subschemes.

Assume $Y_1 \cap Y_2$ has pure dim. $= \dim(Y_1) + \dim(Y_2) - \dim(X)$.

(b) $\Rightarrow$ $Y_1 \cdot Y_2 = [Y_1 \cap Y_2] \in A_*(X)$.

Example

X non-singular variety,

$Y_1, Y_2 \subseteq X$ closed subvarieties.

Assume $Y_1 \cap Y_2$ is (generically) reduced of pure dim. $= \dim(Y_1) + \dim(Y_2) - \dim(X)$.

Then $$Y_1 \cdot Y_2 = [Y_1 \cap Y_2] = \sum_{\substack{Z \subseteq Y_1 \cap Y_2 \\ \text{component.}}} [Z]$$

Proof

We always have

$$I(Z) = \sqrt{I(Y_1) + I(Y_2)} \subseteq \mathcal{O}_{Z,X}.$$

$Y_1 \cap Y_2$ reduced $\Rightarrow$

$I(Y_1) + I(Y_2)$ radical.

Generically reduced $\Rightarrow$

$$I(Z) = I(Y_1) + I(Y_2) \subseteq \mathcal{O}_{Z,X}$$

(c) $\Rightarrow$ $i(Z, Y_1 \cdot Y_2 ; X) = 1.$

$\square$

Example

X non-singular variety over $K = \overline{K}$.
$Y_1, Y_2 \subseteq X$ closed subvarieties.

$Z \subseteq Y_1 \cap Y_2$ irred. component.

Assume Y_1 and Y_2 <u>meet generically transversally</u> along Z:

$\exists P \in Z$: P non-singular closed point of both Y_1 and Y_2, and

$$T_{Y_1}(P) + T_{Y_2}(P) = T_X(P).$$

Then $Y_1 \cap Y_2 = Z$ in neighborhood of P, P is a non-singular point of Z, and $i(Z, Y_1 \cdot Y_2; X) = 1$.

Zariski tangent spaces

X alg. scheme over $K = \overline{K}$.

$P \in X$ closed point.

$\mathfrak{m}_P \subseteq \mathcal{O}_{P,X}$ max. ideal.

$$T_X(P) := \left(\mathfrak{m}_P / \mathfrak{m}_P^2\right)^\vee$$

Intuition:

$f \in \mathcal{O}_{P,X}$: $df = f - f(P) \in \mathfrak{m}_P / \mathfrak{m}_P^2$.

$\vec{v} \in T_X(P)$: $\vec{v}(df)$ = directional derivative.

Note: $\dim_K T_X(P)$ = minimal number of generators of $\mathfrak{m}_P$.

$$\dim \mathcal{O}_{P,X} \leq \dim_K T_X(P)$$

$$\dim \mathcal{O}_{P,X} = \dim_K T_X(P)$$

$\Updownarrow$ $P \in X$ non-singular point.

Tangent space of subscheme

Let $Y \subseteq X$ closed subscheme, $P \in Y$.

$I(Y) \subseteq \mathcal{O}_{P,X}$ ideal of Y.

$$T_Y(P) = \left(\mathfrak{m}_P / I(Y) + \mathfrak{m}_P^2\right)^\vee \subseteq T_X(P)$$

vector subspace.

Note: $I(Y) \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \longrightarrow \mathfrak{m}_P/I(Y)+\mathfrak{m}_P^2 \longrightarrow 0$

$$\Rightarrow T_Y(P) = \{\vec{v} \in T_X(P) \mid \vec{v}(f) = 0 \ \forall f \in I(Y)\}$$
$$= I(Y)^\perp \subseteq T_X(P).$$

Note Let $Y_1, Y_2 \subseteq X$ closed subschemes, $P \in Y_1 \cap Y_2$.

$$\begin{aligned} T_{Y_1 \cap Y_2}(P) &= I(Y_1 \cap Y_2)^\perp \\ &= (I(Y_1) + I(Y_2))^\perp \\ &= I(Y_1)^\perp \cap I(Y_2)^\perp \\ &= T_{Y_1}(P) \cap T_{Y_2}(P) \subseteq T_X(P). \end{aligned}$$

Generically transversal intersection

Let X non-sing. variety / $k = \bar{k}$.

$Y_1, Y_2 \subseteq X$ closed subvarieties.

$Z \subseteq Y_1 \cap Y_2$ irred. component.

$P \in Z$ closed point.

Assume P non-sing. in Y_1 and Y_2,

and $T_{Y_1}(P) + T_{Y_2}(P) = T_X(P)$.

$$0 \to T_{Y_1 \cap Y_2}(P) \longrightarrow T_{Y_1}(P) \oplus T_{Y_2}(P) \to T_X(P) \to 0$$

$$\begin{aligned} \dim T_Z(P) &\leq \dim T_{Y_1 \cap Y_2}(P) \\ &= \dim(Y_1) + \dim(Y_2) - \dim(X) \\ &\leq \dim \mathcal{O}_{P,Z} \leq \dim \mathcal{O}_{P, Y_1 \cap Y_2} \end{aligned}$$

$\therefore$ $\mathcal{O}_{P,Z} = \mathcal{O}_{P, Y_1 \cap Y_2}$ regular local ring of $\dim = \dim(Y_1) + \dim(Y_2) - \dim(X)$.

$\Rightarrow I(Z) = I(Y_1) + I(Y_2) \subseteq \mathcal{O}_{P,X}$

$\Rightarrow I(Z) = I(Y_1) + I(Y_2) \subseteq \mathcal{O}_{Z,X}$.