# Chern classes

$E$ vector bundle on scheme $X$.

$s_i(E) \in \mathrm{End}(A_*(X))$

Formal power series:

$$s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \cdots$$

## Chern polynomial:

$$c_t(E) = s_t(E)^{-1} = 1 + c_1(E)t + c_2(E)t^2 + \cdots$$

$c_i(E) \in \mathrm{End}(A_*(X))$.

$$c_1(E) = -s_1(E)$$

$$c_2(E) = s_1(E)^2 - s_2(E)$$

$p > 0$: $$\sum_{i=0}^{p} s_i(E)\, c_{p-i}(E) = 0$$

$$c_p(E) = -\sum_{i=1}^{p} s_i(E)\, c_{p-i}(E).$$

Assume $E$ has filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E \qquad r = \operatorname{rank}(E).$$

$L_i = E_i / E_{i-1}$ line bundle, $1 \leq i \leq r$.

<u>Lemma</u> Let $s \in \Gamma(X, E)$ be global section, let $\alpha \in A_k(X)$. Then

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha \in \text{image of } A_{k-r}(Z(s)).$$

<u>Note</u>: $Z(s) = \emptyset \Rightarrow \prod_{i=1}^{r} c_1(L_i) \cap \alpha = 0$

<u>Proof</u> $\bar{s}$ = image of $s$ in $\Gamma(X, L_r)$.

$D = (L_r, Z(\bar{s}), \bar{s})$ pseudo-div. on $X$.

$D \cdot \alpha \in A_{k-1}(Z(\bar{s}))$.

$c_1(L_r) \cap \alpha = D \cdot \alpha \in$ image of $A_{k-1}(Z(\bar{s}))$.

$$\begin{tikzcd}
0 \arrow[r] & E_{r-1}|_{Z(\bar{s})} \arrow[r] & E|_{Z(\bar{s})} \arrow[r] & L_r|_{Z(\bar{s})} \arrow[r] & 0 \\
& & Z(\bar{s}) \arrow[ul, dashed, "t"] \arrow[u, "s"'] \arrow[ur, "\bar{s}=0"'] & &
\end{tikzcd}$$

$t \in \Gamma(Z(\bar{s}), E_{r-1})$

$Z(t) = Z(s) \subseteq Z(\bar{s}) \subseteq X.$

Induction:

$\prod_{i=1}^{r-1} c_1(L_i) \cap (D_r \cdot \alpha) \in$ image of $A_{k-r}(Z(t))$.

$\square$

**Prop** $c_i(E) = e_i(c_1(L_1), \ldots, c_1(L_r))$
elementary sym. poly.

**Proof**
Show: $c_t(E) = \prod_{i=1}^{r} (1 + c_1(L_i) t)$.

$p : \mathbb{P}(E) \longrightarrow X$.

$\mathcal{O}_E(-1) \subseteq p^* E$ tautological line bundle

$\mathcal{O}_{\mathbb{P}(E)} \underset{s}{\subseteq} p^* E \otimes \mathcal{O}_E(1)$

$s \in \Gamma(\mathbb{P}(E), p^* E \otimes \mathcal{O}_E(1))$, $Z(s) = \varnothing$.

On $\mathbb{P}(E)$: $0 \subseteq p^* E_1 \otimes \mathcal{O}(1) \subseteq \cdots \subseteq p^* E_r \otimes \mathcal{O}(1)$

Quotients: $p^* L_i \otimes \mathcal{O}_E(1)$.

Lemma $\Rightarrow$ $\prod_{i=1}^{r} c_1(p^* L_i \otimes \mathcal{O}_E(1)) = 0$
$\in \mathrm{End}(A_*(\mathbb{P}(E)))$

$\zeta = c_1(\mathcal{O}_E(1))$

$\sigma_i = e_i(c_1(L_1), \dots, c_1(L_r))$

$\tilde{\sigma}_i = e_i(c_1(p^*L_1), \dots, c_1(p^*L_r))$.

Lemma $\Rightarrow \prod_{i=1}^{r} (c_1(p^*L_i) + \zeta) = 0$

$$\zeta^r + \tilde{\sigma}_1 \zeta^{r-1} + \dots + \tilde{\sigma}_{r-1} \zeta + \tilde{\sigma}_r = 0$$

$\alpha \in A_*(X)$, $j \geq 1$:

$$p_*(\zeta^{r+j-1} \cap p^*(\alpha)) + p_*(\tilde{\sigma}_1 \zeta^{r+j-2} \cap p^*(\alpha)) + \dots$$
$$\dots + p_*(\tilde{\sigma}_{r-1} \zeta^j \cap p^*(\alpha)) + p_*(\tilde{\sigma}_r \zeta^{j-1} \cap p^*(\alpha)) = 0.$$

Projection formula $\Rightarrow$

$$s_j(E) \cap \alpha + \sigma_1 s_{j-1}(E) \cap \alpha + \dots$$
$$\dots + \sigma_{r-1} s_{j-r+1}(E) \cap \alpha + \sigma_r s_{j-r}(E) \cap \alpha = 0.$$

$\therefore \sum_{p=0}^{j} s_{j-p}(E)\, \sigma_p = 0$, for $j \geq 1$.

$$s_t(E)(1 + \sigma_1 t + \dots + \sigma_r t^r) = 1$$

$$c_i(E) = \sigma_i = e_i(c_1(L_1), \dots, c_1(L_r)).$$

**Cor** $X$ scheme, $E$ vector bundle.

$$c_i(E) = 0 \quad \text{for } i > \operatorname{rank}(E).$$

## Whitney sum formula

$$0 \to E' \to E \to E'' \to 0$$

$$c_t(E) = c_t(E') \cdot c_t(E'').$$

$$c_p(E) = \sum_{i=0}^{p} c_i(E')\, c_{p-i}(E'').$$

## Splitting principle

$X$ scheme, $E \to X$ vector bundle.

$\exists$ flat morphism $f : X' \to X$ such that

(1) $f^* : A_*(X) \xrightarrow{\subseteq} A_*(X')$ injective.

(2) $f^*E$ has filtration

$$f^*E = E_r \supseteq E_{r-1} \supseteq \cdots \supseteq E_0 = 0,$$

with line bundle quotients $E_i/E_{i-1}$

**Proof**

On $P(E)$: $\quad 0 \to \mathcal{O}_E(-1) \to p^*E \to F \to 0$

Induction: $\exists\, h : X' \to P(E)$ s.t. $h^*F$ has filtration.

$\square$

**Cor** Chern class results hold for all vector bundles.

## Chern roots

If $\quad E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$, then

$$c_t(E) = \prod_{i=1}^{r} (1 + \alpha_i t) \quad , \quad \alpha_i = c_1(E_i/E_{i-1}).$$

Splitting principle $\Rightarrow$ OK to assume that $E$ has "Chern roots" $\alpha_1, \ldots, \alpha_r$.

**Example:** $E^\vee$ has Chern roots: $-\alpha_1, \dots, -\alpha_r$.

$$c_i(E^\vee) = e_i(-\alpha_1, \dots, -\alpha_r) = (-1)^i c_i(E).$$

$$\sum_{i=0}^{p} (-1)^i s_{p-i}(E^\vee)\, c_i(E) = 0 \quad , \quad p \geq 1.$$

$$s_p(E^\vee) = \sum_{i=1}^{p} (-1)^{i-1} s_{p-i}(E^\vee)\, c_i(E)$$

**Complete sym. poly:**

$$h_p(x_1, \dots, x_r) = \sum_{\Sigma a_i = p} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$$

sum of all monomials of degree $p$.

**Exercise:** $h_p = \sum_{i=1}^{p} (-1)^{i-1} h_{p-i}\, e_i \quad , \quad p \geq 1.$

**Hint:** $\prod_{i=1}^{r} (1 - x_i) \cdot \prod_{i=1}^{r} (1 - x_i)^{-1} = 1.$

$\therefore \quad s_p(E^\vee) = h_p(\alpha_1, \dots, \alpha_r)$

where $\alpha_1, \dots, \alpha_r$ Chern roots of $E$.

Easier than splitting principle:

$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ sum of line bundles.

$c_i(E) = e_i(c_1(L_1), \ldots, c_1(L_r))$.

<u>Example</u>

$E, F$ vector bundles of ranks $r, s$.

$c_i = c_i(E)$, $d_j = c_j(F)$.

$c_k(E \otimes F) = ? = \text{poly}(c_1, \ldots, c_r, d_1, \ldots, d_s)$.

$E = L_1 \oplus \cdots \oplus L_r$, $\quad F = M_1 \oplus \cdots \oplus M_s$.

$$E \otimes F = \bigoplus_{i,j} L_i \otimes M_j.$$

Chern roots: $c_1(L_i \otimes M_j) = \alpha_i + \beta_j$

$\alpha_i = c_1(L_i)$, $\beta_j = c_1(M_j)$.

$$c_t(E \otimes F) = \prod_{i,j} \left(1 + (\alpha_i + \beta_j) t\right)$$

$$c_1(E \otimes F) = \sum_{i,j} (\alpha_i + \beta_j) = s\, c_1(E) + r\, c_1(F).$$

$$c_{rs}(E \otimes F) = \prod_{i,j} (\alpha_i + \beta_j) = c_r(E)^s + \cdots + c_s(F)^r$$

Assume $F = L$ line bundle,

$\beta = c_1(L)$.

$$
\begin{aligned}
c_p(E\otimes L) &= e_p(\alpha_1+\beta, \alpha_2+\beta, \dots, \alpha_r+\beta) \\
&= \binom{r}{p}\beta^p + \binom{r-1}{p-1}(\alpha_1+\dots+\alpha_r)\beta^{p-1} + \cdots \\
&= \sum_{i=0}^{p} \binom{r-i}{p-i}\beta^{p-i} e_i(\alpha_1,\dots,\alpha_r) \\
&= \sum_{i=0}^{p} \binom{r-i}{p-i} c_i(E)\, c_1(L)^{p-i}
\end{aligned}
$$

$$
c_r(E\otimes L) = \sum_{i=0}^{r} c_i(E)\, c_1(L)^{r-i}.
$$

Example (Chern character)

$E$ vector bundle.

Chern roots $\alpha_1, \ldots, \alpha_r$.

$$ch(E) = \sum_{i=1}^{r} \exp(\alpha_i)$$

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$ch(E) = r + \left(\sum \alpha_i\right) + \frac{1}{2}\left(\sum \alpha_i^2\right) + \cdots$$

$$= r + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \cdots$$

Properties:

$$0 \to E' \to E \to E'' \to 0$$

$$\Rightarrow \quad ch(E) = ch(E') + ch(E'')$$

$$ch(E_1 \otimes E_2) = ch(E_1) \cdot ch(E_2)$$

$$\exp(\alpha_i + \beta_j) = \exp(\alpha_i)\exp(\beta_j).$$

$\therefore$ $ch : K(X) \longrightarrow A^*(X)$ ring hom.