Push-Forward

$f: X \longrightarrow Y$ proper morphism.

$V \subseteq X$ closed subvariety.

$W = f(V) \subseteq Y$ closed subvariety.

$f: V \longrightarrow W$ dominant (surjective)

$\Rightarrow f^*: \; R(W) \xrightarrow{\subseteq} R(V)$ field ext.

$\text{tr.deg.}_K R(V) = \dim(V)$.

$\dim(W) = \dim(V) \Rightarrow R(V)/R(W)$ finite ext.

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if } \dim(V) = \dim(W). \\ 0 & \text{if } \dim(W) < \dim(V) \end{cases}$$

<u>Def</u> $f_*: Z_k(X) \longrightarrow Z_k(Y)$

$$f_*[V] = \deg(V/W)\cdot[W], \quad W = f(V).$$

<u>Prop</u> $f: X \to Y$ proper surjective of varieties. $r \in R(X)^*$. Then

$$f_*(\mathrm{div}(r)) = \begin{cases} 0 & \text{if } \dim Y < \dim X \\ \mathrm{div}(N(r)) & \text{if } \dim Y = \dim X. \end{cases}$$

$$N(r) \in R(Y)^* = \det\left(R(X) \xrightarrow{r\cdot} R(X)\right).$$

<u>Cor</u> $f: X \to Y$ any proper morphism.

$$f_*(\mathrm{Rat}_k(X)) \subseteq \mathrm{Rat}_k(Y).$$

$\Rightarrow f_*: A_k(X) \longrightarrow A_k(Y)$ well def.

<u>Example:</u>

1) X proper over $\mathrm{Spec}(K) = \{pt\}$ $\Rightarrow$

$$\exists\ A_0(X) \longrightarrow A_0(pt) = \mathbb{Z}$$

$$\alpha \longmapsto \int_X \alpha$$

$\alpha = \sum_P n_P [P] \in A_0(X)$. $\quad \int \alpha = \sum n_P [R(P):K]$

points well-defined / rat. equiv.

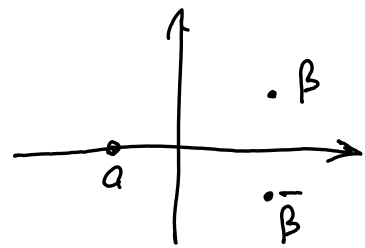
1a) $\mathbb{P}^1_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[x,y]) \longrightarrow \mathrm{Spec}\,\mathbb{R} = p$

$\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}\,\mathbb{R}[t] \subseteq \mathbb{P}^1_{\mathbb{R}}.$

$Q = \langle t-a \rangle \in \mathbb{A}^1_{\mathbb{R}}. \quad a \in \mathbb{R}.$

$R(Q) = \mathbb{R}[t]/_{Q} \xrightarrow{\cong} \mathbb{R}$

$\deg(Q/p) = 1.$



$\int_{\mathbb{P}^1_{\mathbb{R}}} [Q] = 1.$

$Q' = \langle t^2 + bt + c \rangle = \langle (t-\beta)(t-\bar{\beta}) \rangle$

$R(Q') = \mathbb{R}[t]/_{Q'} \cong \mathbb{C}.$

$\deg(Q'/p) = 2.$

$\int_{\mathbb{P}^1_{\mathbb{R}}} [Q'] = 2.$

2) $X \xrightarrow{f} Y \xrightarrow{g} Z$ both proper. $\Rightarrow$

$g_* f_* = (gf)_* : \quad Z_k(X) \to Z_k(Y) \to Z_k(Z)$

$A_k(X) \to A_k(Y) \to A_k(Z).$

<u>Exercise</u> Bezout's Thm for $\mathbb{P}^2$, $K = \bar{K}$.

$F, G \subseteq \mathbb{P}^2$ curves, no common comps.

$P \in F \cap G$.

$I(P, F \cdot G) = \dim_F \mathcal{O}_{P,\mathbb{P}^2} / \langle f, g \rangle$

where $I(F) = \langle f \rangle \subseteq \mathcal{O}_{P,\mathbb{P}^2}$

$I(G) = \langle g \rangle \subseteq \mathcal{O}_{P,\mathbb{P}^2}$

Then $\sum_{P \in F \cap G} I(P, F \cdot G) = \deg(F) \cdot \deg(G)$.

<u>Idea:</u> Choose line $L \subseteq \mathbb{P}^2$ such that $L \cap F \cap G = \emptyset$.

<u>Notation:</u> $F, G, L \subseteq K[x,y,z]$ homogeneous.

$G / L^{\deg(G)} \in K(F)^*$.

$[\operatorname{div}(G/L^{\deg G})] =$

$$\sum_{P \in F \cap G} I(P, F \cdot G) \cdot [P] - \deg(G) \sum_{P \in F \cap L} I(P, F \cdot L) \cdot [P]$$

$\int_F [\operatorname{div}(G/L^{\deg G})] = 0$.

WLOG: $G = L$.

WLOG: $F = L'$. <u>Clear!</u>

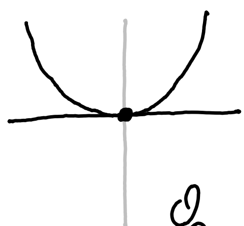
Cycle of scheme

X scheme, $X_1, \ldots, X_t$ irred. comps.

$\mathcal{O}_{X_i, X}$ Artinian.

Geometric mult of X_i in X: $\operatorname{length}(\mathcal{O}_{X_i, X})$

Fund. cycle: $[X] = \sum_{i=1}^{t} \operatorname{length}(\mathcal{O}_{X_i, X})\, [X_i] \in Z_*(X)$.

Example  $X = V(y, y - x^2) \subseteq \mathbb{A}^2$

$\mathcal{O}_{0,X} = \mathcal{O}(X) = k[x,y]/\langle x^2, y\rangle$.

$\operatorname{length}(\mathcal{O}_{0,X}) = 2$. $[X] = 2 \cdot [0]$

Warning: X non-singular variety,

$V, W \subseteq X$ intersect properly

$\not\Rightarrow [V] \cdot [W] = [V \cap W]$.

Flat pullback

$f : X \longrightarrow Y$ morphism.

Flat: $\forall\ U \subseteq Y$ open affine,

$\forall\ U' \subseteq f^{-1}(U) \subseteq X$ open affine,

$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(U')$

$\mathcal{O}_X(U')$ flat $\mathcal{O}_Y(U)$-module.

Relative dim. n:

$\forall\ V \subseteq Y$ subvariety

$\forall\ V' \subseteq f^{-1}(V)$ irred. comp.:

$\dim(V') = \dim(V) + n.$

Fact: $f : X \to Y$ flat, Y irred.,

X pure dim. $= \dim(Y) + n$

$\Rightarrow f$ has rel. dim n.

Fact:
- flat preserved by base-ext.
- flat of rel. dim n preserved by base ext.

Convention:

Any flat morphism assumed to have a relative dim.

Examples

1) open embedding $U \subseteq Y$.

2) Structure morphism $X \longrightarrow \mathrm{Spec}(k)$ if X pure dim. n

3) projection $X \times Y \longrightarrow Y$, X pure dim.

4) vector bundle $E \longrightarrow X$
(locally $\mathbb{A}^n \times X \to X$.)

5) X irred. variety, C non-sing curve,
$f : X \longrightarrow C$ dominant.
$U \subseteq C$ open affine. $U' \subseteq f^{-1}(U)$.
$\mathcal{O}_C(U)$ Dedekind domain.
$f^* : \mathcal{O}_C(U) \xrightarrow{\subseteq} \mathcal{O}_X(U')$ domain
$\Rightarrow$ torsion free $\mathcal{O}_C(U)$-module.

Def $f : X \longrightarrow Y$ flat of rel. dim n.

$V \subseteq Y$ subvariety.

$f^{-1}(V)$ scheme theoretic inv. image.

$f^*[V] = [f^{-1}(V)]$ fund. cycle.

$f^* : Z_k(Y) \longrightarrow Z_{k+n}(X)$.

Properties

1) $Z \subseteq Y$ any closed subscheme $\Rightarrow$

$f^*[Z] = [f^{-1}(Z)]$

2)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
fiber square ($X' = X \times_Y Y'$)
f flat, p proper.

Then $f^* p_* = p'_* f'^* : Z_k(Y') \to Z_{k+n}(X)$

Thm $f^*\left(\mathrm{Rat}_k(Y)\right) \subseteq \mathrm{Rat}_{k+n}(X)$.

$f^* : A_k(Y) \longrightarrow A_{k+n}(X)$ well def.

Exact seq: $Y \overset{i}{\subseteq} X$ closed subscheme.

$U = X \setminus Y \underset{j}{\subseteq} X$ open.

$$\begin{tikzcd} Z_k Y \arrow[r, "i_*"] \arrow[d] & Z_k X \arrow[r, "j^*"] \arrow[d] & Z_k U \arrow[r] \arrow[d] & 0 \\ A_k Y \arrow[r, "i_*"] & A_k X \arrow[r, "j^*"] & A_k U \arrow[r] & 0 \end{tikzcd}$$

exact.

exact.

Pf: Assume $\alpha \in Z_k X$, $j^*\alpha = 0 \in A_k U$:

$\exists$ $W_i \subseteq X$, $r_i \in R(W_i)$ s.t.

$$j^*\left(f - \sum_i \operatorname{div}(r_i)\right) = 0 \in Z_k(U).$$

$$\Rightarrow f - \sum_i \operatorname{div}(r_i) \in \operatorname{Image}(i_*).$$

□

Affine bundle

$p: E \longrightarrow X$ morphism of schemes.

Def p affine bundle of rank n if $\exists\ X = \cup U_\alpha$ open cover s.t.

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "p"] \arrow[d, phantom, "\cong" rotate=90] & U_\alpha \\ U_\alpha \times \mathbb{A}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Note Affine bundle $\not\Rightarrow$ vector bundle.

Note: Affine bundle $\Rightarrow$ flat.

Prop $p: E \longrightarrow X$ affine bundle of rank n

$\Rightarrow p^*: A_k X \longrightarrow\!\!\!\!\rightarrow A_{k+n} E$ surjective $\forall k$.

Examples

1) $A_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & \text{if } k=n \\ 0 & \text{if } k\neq 0. \end{cases}$

$p : \mathbb{A}^n \longrightarrow \{pt\}.$

2) $L^k \subseteq \mathbb{P}^n$ linear subspace of dim k,

$0 \le k \le n.$

$A_k(\mathbb{P}^n)$ gen. by $[L^k]$.

$\mathbb{P}^n - L^{n-1} \cong \mathbb{A}^n.$

$$A_k(L^{n-1}) \twoheadrightarrow A_k(\mathbb{P}^n) \longrightarrow A_k(\mathbb{A}^n) = 0$$

gen. by $[L^k]$ by induction. for $0 \le k \le n-1$.

3) $A_k(\mathbb{P}^n) = \mathbb{Z}[L^k] \cong \mathbb{Z}.$

$k=n$: clear.

$k=n-1$: $Cl(\mathbb{P}^n) = \mathbb{Z}$.

$R(\mathbb{P}^n) = \{ F/G \mid F, G \text{ homog. of same degree} \}.$

$k \leq n-2$:

Assume $d[L^k] = 0 \in A_k(\mathbb{P}^n)$.

$d[L^k] = \sum [\mathrm{div}(r_i)]$, $r_i \in R(V_i)$,

$V_i \subseteq \mathbb{P}^n$ subvar, $\dim V_i = k+1$.

$Z = \bigcup V_i \subseteq \mathbb{P}^n$ closed subsch, $\dim Z = k+1$

$d[L^k] = 0 \in A_k(Z)$

Choose $M \subseteq \mathbb{P}^n$ linear subspace, $\dim M = n-k-2$, such that $M \cap Z = \emptyset$. (Bertini).

$p : \mathbb{P}^n - M \longrightarrow \mathbb{P}^{k+1}$ projection from M.

$p : Z \longrightarrow \mathbb{P}^{k+1}$ proper.

$p_*(d[L^k]) = 0 \in A_k(\mathbb{P}^{k+1}) = \mathbb{Z}$.

But $p : L^k \xrightarrow{\cong} p(L^k) \subseteq \mathbb{P}^{k+1}$ hyperplane.

$p_*[L^k] = [p(L^k)] \neq 0$ $\lightning$.