Example

point $= \mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \dots \subseteq \mathbb{P}^n$.

$K(\mathbb{P}^n)$ has basis $\{ [\mathcal{O}_{\mathbb{P}^r}] : 0 \le r \le n \}$

Generate:

$$K(\mathbb{P}^{n-1}) \xrightarrow{i_*} K(\mathbb{P}^n) \xrightarrow{j^*} K(\mathbb{A}^n) = \mathbb{Z}$$

Linearly independent:

Set $h = [\mathcal{O}_{\mathbb{P}^{n-1}}] = 1 - [\mathcal{O}(-1)] \in K(\mathbb{P}^n)$.

$$0 \to \mathcal{O}(-1) \otimes \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^{r-1}} \to 0$$

$$h \cdot [\mathcal{O}_{\mathbb{P}^r}] = \begin{cases} [\mathcal{O}_{\mathbb{P}^{r-1}}] & \text{if } r \ge 1 \\ 0 & \text{if } r = 0 \end{cases}$$

Assume $a_0 [\mathcal{O}_{\mathbb{P}^0}] + \dots + a_r [\mathcal{O}_{\mathbb{P}^r}] = 0$

Multiply with h^r: $\quad a_r [\mathcal{O}_{\mathbb{P}^0}] = 0$

$\Rightarrow a_r = \chi(a_r [\mathcal{O}_{\mathbb{P}^0}]) = 0$

Note: $[\mathcal{O}_{\mathbb{P}^r}] = h^{n-r} \in K(\mathbb{P}^n)$.

Basis: $\{ h^i : 0 \le i \le n \}$.

Alternative basis: $\{ \mathcal{O}(d) \mid 0 \le d \le n \}$.

Example

$X = V(f_1, \dots, f_c) \subseteq \mathbb{P}^n$ complete int.

$\deg(f_i) = d_i$.

Arithmetic genus:

$$p_a(X) = (-1)^{\dim X} \left(\chi(\mathcal{O}_X) - 1 \right)$$

$$[\mathcal{O}_X] = \prod_{i=1}^{c} \left(1 - [\mathcal{O}(-d_i)]\right) \qquad \in K(\mathbb{P}^n).$$

$$= \sum_{S \subseteq \{1,2,\dots,c\}} (-1)^{|S|} \left[\mathcal{O}\left(-\sum_{i\in S} d_i\right)\right]$$

$$\chi(\mathcal{O}_{\mathbb{P}^n}) = 1$$

$$\chi(\mathcal{O}(-d)) = (-1)^n \cdot \binom{d-1}{n}.$$

$$p_a(X) = \sum_{\emptyset \neq S \subseteq \{1,2,\dots,c\}} (-1)^{c-|S|} \binom{\sum d_i - 1}{n}$$

K-theoretic intersection theory

X alg. scheme, $\Omega \subseteq X$ closed subscheme.

$[\mathcal{O}_\Omega] \in K^\circ(\Omega) \longrightarrow K_\circ(\Omega) \longrightarrow K_\circ(X)$.

Properties:

(1) $\Omega_1, \Omega_2 \subseteq X$ closed.

$\Omega_1 \cup \Omega_2 \subseteq X$ closed subscheme.

$I_{\Omega_1 \cup \Omega_2} = I_{\Omega_1} \cap I_{\Omega_2} \subseteq \mathcal{O}_X$.

$$[\mathcal{O}_{\Omega_1 \cup \Omega_2}] = [\mathcal{O}_{\Omega_1}] + [\mathcal{O}_{\Omega_2}] - [\mathcal{O}_{\Omega_1 \cap \Omega_2}]$$
$$\in K_\circ(\Omega_1 \cup \Omega_2) \to K_\circ(X).$$

where $\Omega_1 \cap \Omega_2 \subseteq X$ scheme-theoretic intersection.

$$0 \longrightarrow \mathcal{O}_{\Omega_1 \cup \Omega_2} \longrightarrow \mathcal{O}_{\Omega_1} \oplus \mathcal{O}_{\Omega_2} \longrightarrow \mathcal{O}_{\Omega_1 \cap \Omega_2} \longrightarrow 0$$
$$(f_1, f_2) \longmapsto f_1 - f_2 .$$

$$0 \to R/I_1 \cap I_2 \longrightarrow R/I_1 \oplus R/I_2 \longrightarrow R/I_1 + I_2 \to 0$$

(2) $U \subseteq \mathbb{P}^1$ open.

$Z \subseteq X \times U$ closed irred. subvar.

Assume $Z \to U$ dominant.

For $t \in U$, $Z_t \subseteq X$ fiber over t.

Then $[\mathcal{O}_{Z_t}] \in K_0(X)$ independent of t.

$\overline{Z} \subseteq X \times \mathbb{P}^1$ closure, reduced structure.

Projections:
$$\begin{tikzcd} \overline{Z} \arrow[r, "p"'] \arrow[d, "q"] & X \\ \mathbb{P}^1 & \end{tikzcd}$$

$\overline{Z}$ irred. variety, $K[t]$ PID $\Rightarrow$ q flat.

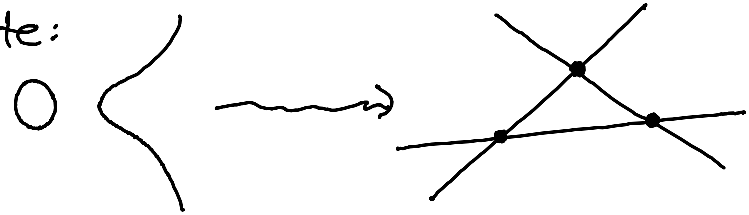
$\Rightarrow$ $[\mathcal{O}_{q^{-1}(t)}] = q^*[\mathcal{O}_{\{t\}}] \in K_0(\overline{Z})$.

p proper, $p: q^{-1}(t) \xrightarrow{\cong} Z_t$ iso.

$[\mathcal{O}_{Z_t}] = p_* q^*([\mathcal{O}_{\text{point}}])$.

<u>Example</u> $C \subseteq \mathbb{P}^2$ elliptic curve.

Degenerate:



$$[C] = 3\,[\text{line}] \in A^*(\mathbb{P}^2).$$

$$[\mathcal{O}_C] = 3\,[\mathcal{O}_{\mathbb{P}^1}] - 3\,[\mathcal{O}_{\mathbb{P}^0}] \in K(\mathbb{P}^2).$$

<u>Example</u> $\Omega \subseteq \mathbb{P}^n$ hypersurface of deg. d.

$$[\mathcal{O}_\Omega] = [\mathcal{O}_{H_1 \cup \dots \cup H_d}]$$

$$= \sum_{c \geq 1} (-1)^{c-1} \binom{d}{c} [\mathcal{O}_{\mathbb{P}^{n-c}}]$$

<u>Properties</u>

(3) $f : X \longrightarrow Y$ flat,

$\Omega \subseteq Y$ closed subscheme.

$$f^*[\mathcal{O}_\Omega] = [\mathcal{O}_{f^{-1}(\Omega)}] \in K_0(X).$$

Koszul complex

R Noetherian ring.

M f.g. R-module.

$s: M \longrightarrow R$ R-homomorphism.

$$d_p : \Lambda^p M \longrightarrow \Lambda^{p-1} M$$

$$m_1 \wedge \cdots \wedge m_p \longmapsto \sum_{i=1}^{p} (-1)^{i+1} s(m_i)\, m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge m_p$$

Complex $\Lambda^\bullet s = \Lambda^\bullet M$:

$$0 = \Lambda^N M \longrightarrow \Lambda^{N-1} M \longrightarrow \cdots \Lambda^2 M \longrightarrow M \xrightarrow{s} R \longrightarrow 0$$

$I(s) = s(M) \subseteq R$ ideal.

$$\Lambda^\bullet s \longrightarrow R/I(s) \longrightarrow 0 : \quad \cdots \longrightarrow M \xrightarrow{s} R \longrightarrow R/I(s) \longrightarrow 0$$

Example

Assume $I(s) = R$. Then $M = M' \oplus R$.

$\Lambda^p M = \Lambda^p M' \oplus \Lambda^{p-1} M'$.

$\mathrm{Im}(d_{p+1}) = \ker(d_p) = \Lambda^p M'$.

$\therefore \Lambda^\bullet(s)$ exact.

Regular section

Assume $M = R^n$,

$s = (s_1, \dots, s_n) : M \longrightarrow R$.

<u>Def</u> s <u>regular section</u>

iff $\Lambda^\bullet M \longrightarrow R/\mathcal{I}(s) \to 0$ exact.

<u>Fact</u> R local ring, $s_1, \dots, s_n \in \mathfrak{m}_R$.

$(s_1, \dots, s_n)$ regular sequence

$\Leftrightarrow$ regular section.

<u>Equivalent</u>: $s_1, \dots, s_n \in R$ any elts.

$s = (s_1, \dots, s_n) \in R^n$ regular section

$\Leftrightarrow \langle s_1, \dots, s_n \rangle = R$ or $(s_1, \dots, s_n)$ reg. seq.

Regular section of vector bundle

X alg. scheme.
E vector bundle of rank r.

$s \in \Gamma(X, E)$ section.

$\Lambda^\bullet E^\vee \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0$:

$$0 \to \Lambda^r E^\vee \to \cdots \to \Lambda^2 E^\vee \to E^\vee \xrightarrow[s]{} \mathcal{O}_X \to \mathcal{O}_{Z(s)} \to 0$$

s regular section $\Longleftrightarrow$ exact.

$\Rightarrow$ $Z(s) \subseteq X$ regular embedding of codim. r,
with normal bundle
$N_{Z(s)} X = E|_{Z(s)}$.

$$\begin{tikzcd}
E|_{Z(s)} \arrow[r] \arrow[d] & Z(s) \arrow[r] \arrow[d] & X \arrow[d, "s"] \\
E \arrow[r] & X \arrow[r, "s_E"'] & E
\end{tikzcd}$$

Pullback along regular embedding

$f : X \hookrightarrow Y$ regular embedding of codim d.

$$f^! : K_0(Y) \longrightarrow K_0(X)$$

$$f^![\mathcal{F}] = \sum_{i \geq 0} (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})]$$

Well defined

$\mathrm{Tor}_i^Y(\mathcal{O}_X, \mathcal{F}) = 0$ for $i > d$. (Next page!)

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

LES $\Rightarrow$ $f^![\mathcal{F}] = f^![\mathcal{F}'] + f^![\mathcal{F}'']$

Compatible with pullback:

$$\begin{array}{ccc} K^0(Y) & \xrightarrow{f^*} & K^0(X) \\ \downarrow & & \downarrow \\ K_0(Y) & \xrightarrow{f^!} & K_0(X) \end{array}$$

Prop $f: X \hookrightarrow Y$ regular of codim d.
$V \subseteq Y$ closed subscheme.
$W = f^{-1}(V) \subseteq X$ scheme th. inv. image.
Assume V is CM of pure dim. k
and W has pure dim $k-d$.
Then W is CM and $f^![\mathcal{O}_V] = [\mathcal{O}_W] \in K_\circ(X)$.

Proof

Note: $\mathcal{O}_W = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_V = \mathrm{Tor}_0^Y(\mathcal{O}_X, \mathcal{O}_V)$.

Show: $\mathrm{Tor}_i^Y(\mathcal{O}_X, \mathcal{O}_V) = 0$ for $i > 0$.

Local question.
WLOG: Y affine, $I(X) = \langle f_1, \dots, f_d \rangle$,
$f_1, \dots, f_d \in \mathcal{O}_Y$ regular seq.

Koszul resolution $\Lambda^\bullet(f)$, $f = (f_1, \dots, f_d)$:

$$0 \to K_d \to K_{d-1} \to \cdots \to K_1 \to K_0 \to \mathcal{O}_X \to 0$$

Koszul complex of $f|_V$:

$$0 \to K_\bullet \otimes \mathcal{O}_V \to \mathcal{O}_W \to 0. \qquad (*)$$

V CM, $W = Z(f_1, \dots, f_d) \subseteq V$ codim. d
$\Rightarrow f|_V$ regular section $\Rightarrow (*)$ exact.

$\square$

Cor

X non-sing. variety.

$V_1, V_2 \subseteq X$ closed subschemes of pure dim.

Assume V_1 and V_2 are CM and

$$\operatorname{codim}(V_1 \cap V_2, X) = \operatorname{codim}(V_1, X) + \operatorname{codim}(V_2, X)$$

Then $V_1 \cap V_2$ is CM and

$$[\mathcal{O}_{V_1}] \cdot [\mathcal{O}_{V_2}] = [\mathcal{O}_{V_1 \cap V_2}] \in K(X).$$

Proof

$\Delta : X \hookrightarrow X \times X$ regular emb.

Prop $\Rightarrow$ $\Delta^* [\mathcal{O}_{V_1 \times V_2}] = [\mathcal{O}_{\Delta^{-1}(V_1 \times V_2)}] = [\mathcal{O}_{V_1 \cap V_2}]$.

$X \times V_2 \xrightarrow{i} X \times X \xrightarrow[p_2]{} X$

$$[\mathcal{O}_{V_1 \times V_2}] = i_* [\mathcal{O}_{V_1 \times V_2}] = i_*\big((p_2 i)^* [\mathcal{O}_{V_1}] \cdot [\mathcal{O}_{X \times V_2}] \big)$$

$$= p_2^* [\mathcal{O}_{V_1}] \cdot i_* [\mathcal{O}_{X \times V_2}] = [\mathcal{O}_{V_1 \times X}] \cdot [\mathcal{O}_{X \times V_2}] \in K(X \times X).$$

$\Rightarrow \Delta^* [\mathcal{O}_{V_1 \times V_2}] = [\mathcal{O}_{V_1}] \cdot [\mathcal{O}_{V_2}] \in K(X)$.

$\square$