Lines on cubic surface

$E = \mathbb{C}^4$. $\mathbb{P}^3 = \mathbb{P}(E)$.

$X \subseteq \mathbb{P}(E)$ cubic surface.

$X = Z(f)$, $f \in \mathrm{Sym}^3(E^\vee)$.

$\mathcal{O}(-1) \subseteq E_{\mathbb{P}^3} = E \times \mathbb{P}^3$.

$\mathcal{O}(-3) \subseteq \mathrm{Sym}^3(E)_{\mathbb{P}^3}$

$$X = Z(\mathcal{O}(-3) \longrightarrow \mathrm{Sym}^3(E)_{\mathbb{P}^3} \xrightarrow{f} \mathbb{C}_{\mathbb{P}^3})$$

$M = \mathrm{Gr}(2, E) = \{\text{lines in } \mathbb{P}(E)\}$.

$$0 \to S \to E_M \to Q \to 0$$

Locus of lines contained in X:

$$\Omega = Z(\mathrm{Sym}^3 S \longrightarrow \mathrm{Sym}^3 E \xrightarrow{f} \mathbb{C}) \subseteq M$$

<u>Claim:</u> f general $\Rightarrow \#\Omega = 27$.

$Y = Gr(4, Sym^3 E)$

$i : M \xrightarrow{\subseteq} Y \ , \quad V \longmapsto Sym^3 V$

$Y_0 = \{ W \in Y \mid W \subseteq Ker(f) \} \cong Gr(4, 19).$

$\Omega = i^{-1}(Y_0)$

$$\begin{tikzcd} M \arrow[r, "i"] & Y \\ \Omega \arrow[u] \arrow[r] & Y_0 \arrow[u] \end{tikzcd}$$

$GL(20) \circlearrowright Y$ transitive.

$M \cap g.Y_0$ transversal for all
$g \in$ dense open $\subseteq GL(20)$

$\Rightarrow \Omega = Z(Sym^3 S \xrightarrow{f} \mathbb{C}) = Z(\mathbb{C} \xrightarrow{f} Sym^3 S^\vee)$
reduced and of codim. 4 in M
for general $f \in Sym(E^\vee)$.

$\Rightarrow \# \Omega = \int [\Omega] = \int_M c_4(Sym^3 S^\vee).$

Let $a, b \in A^1(M)$ be Chern roots of $S^\vee$.

Chern roots of $\mathrm{Sym}^3 S^\vee$:

$$3a, \quad 2a+b, \quad a+2b, \quad 3b.$$

$$\begin{aligned} c_4(\mathrm{Sym}^3 S^\vee) &= 3a\cdot(2a+b)\cdot(a+2b)\cdot 3b \\ &= 9ab\cdot(2a^2+5ab+2b^2) \\ &= 9\, c_2(S^\vee)\cdot(2c_1(S^\vee)^2 + c_2(S^\vee)) \end{aligned}$$

<u>Compute</u>:

Given $V' \in M = Gr(2,E)$:

$$D(V') = \{V \in M \mid V \cap V' \neq 0\}$$
$$= \Omega_1(V' \longrightarrow E/V)$$

$$[D(V')] = c_1(E/S) = -c_1(S) = c_1(S^\vee)$$

Given $H \subseteq E = \mathbb{C}^4$, $\dim H = 3$:

$$\mathbb{P}^*(H) = \{V \in M \mid V \subseteq H\}$$
$$= Z(S \longrightarrow E/H) = Z(\mathbb{C} \longrightarrow S^\vee)$$

$$[\mathbb{P}^*(H)] = c_2(S^\vee).$$

$$c_2(S^\vee)^2 = [\mathbb{P}^*(H)] \cdot [\mathbb{P}^*(H')]$$
$$= [\mathbb{P}^*(H) \cap \mathbb{P}^*(H')]$$
$$= [\{H \cap H'\}] = [\text{point}].$$

$$c_2(S^\vee) \cdot c_1(S^\vee)^2 = [\mathbb{P}^*(H) \cap D(V') \cap D(V'')]$$
$$= [\{H \cap V' \oplus H \cap V''\}]$$
$$= [\text{point}].$$

$$\therefore 9\, c_2(S^\vee) \cdot (2\, c_1(S^\vee)^2 + c_2(S^\vee))$$
$$= 27 \cdot [\text{point}] \in A^4(M).$$

Products of Schur functions

$$\Lambda = \mathbb{Z}[s_1, s_2, s_3, \ldots]$$

$\mathbb{Z}$-basis: $\{ s_\lambda \mid \lambda \text{ partition} \}$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$

$s_\lambda = \det\left(s_{\lambda_i + j - i} \right)_{\ell \times \ell}$

Initial term: $s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_\ell}$.

Littlewood-Richardson coefs:

Define $c^\nu_{\lambda\mu} \in \mathbb{Z}$ by

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu .$$

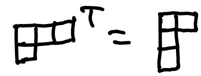
Note: Uniquely determined by products of single Schur polynomials:

$$s_\lambda(x) \cdot s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x)$$

$\forall\, x = (x_1, \ldots, x_N), \ \forall\, N \geq 0.$

Involution of Λ

$$S_{\nu^T/\lambda^T}(x;y) = (-1)^{|\nu/\lambda|} S_{\nu/\lambda}(y;x)$$



Ring involution:

$$\Lambda \longrightarrow \Lambda$$

$$S_{\nu/\lambda} \longmapsto S_{\nu^T/\lambda^T}$$

$$S_p \longmapsto S_{(1)^p}$$

$$S_\lambda \cdot S_\mu = \sum_\nu C^\nu_{\lambda\mu} S_\nu$$

$$\Rightarrow S_{\lambda^T} \cdot S_{\mu^T} = \sum_\nu C^\nu_{\lambda\mu} S_{\nu^T}$$

$$\therefore C^{\nu^T}_{\lambda^T\mu^T} = C^\nu_{\lambda\mu}$$

Semistandard Young tableaux

ν/λ skew diagram.

$$SSYT(\nu/\lambda) = \left\{ \begin{array}{l} \text{labelings } \nu/\lambda \to \mathbb{N} \text{ s.t.} \\ \text{rows weakly incr.} \\ \text{cols strictly incr.} \end{array} \right\}$$

$$\begin{array}{cc} & \longrightarrow \ \leq \\ \wedge \ \downarrow & \begin{array}{|c|c|c|c|c|c|} \hline &&&&1&2\\ \hline &&&2&2&3\\ \hline &1&1&3&5&\\ \hline 2&3&4&4&& \\ \hline \end{array} \end{array} = T$$

<u>Row word</u>: $(2,3,4,4,1,1,3,5,2,2,3,1,2)$

<u>content</u> of $T = \mu = (\mu_1, \mu_2, \dots)$

$\mu_i = \#$ boxes containing i

$= (3, 4, 3, 2, 1)$

$$x^T = \prod_{i \geq 1} x_i^{\# \text{boxes} \ni i} = x_1^3 x_2^4 x_3^3 x_4^2 x_5$$

<u>Thm</u> $$s_{\nu/\lambda}(x) = \sum_{T \in SSYT(\nu/\lambda)} x^T$$

Jeu de taquin slides

$$T = \begin{array}{llll} \square & 1 & 2 & 4 \\ 1 & 3 & 5 & \\ 2 & 4 & 4 & \\ 3 & & & \end{array} \longrightarrow \begin{array}{llll} 1 & 1 & 2 & 4 \\ \square & 3 & 5 & \\ 2 & 4 & 4 & \\ 3 & & & \end{array} \longrightarrow \begin{array}{llll} 1 & 1 & 2 & 4 \\ 3 & \square & 5 & \\ 2 & 4 & 4 & \\ 3 & & & \end{array} \longrightarrow \begin{array}{llll} 1 & 1 & 2 & 4 \\ 3 & 4 & 5 & \\ 2 & 4 & \square & \\ 3 & & & \end{array} = T'$$

Note: $x^T = x^{T'}$.

Reverse jdt slides: opposite direction.

Exercise $T \in SSYT(\nu/\lambda)$, λ/λ' horiz strip. Let T' be the result of applying jdt slides starting from boxes in λ/λ', in right-to-left order.

Then $T' \in SSYT(\nu'/\lambda')$, where $\nu' \subseteq \nu$, ν/ν' horiz strip, $|\nu/\nu'| = |\lambda/\lambda'|$.

And T is the result of applying reverse jdt slides to T', starting from boxes in ν/ν', in right-to-left order.

$$\begin{array}{ccccc} & & \square & \square & 3 \\ \square & 1 & 1 & 2 & 4 \\ 2 & 2 & 4 & & \end{array} \longrightarrow \begin{array}{ccccc} & & \square & 2 & 3 \\ \square & 1 & 1 & 4 & \square \\ 2 & 2 & 4 & & \end{array} \longrightarrow \begin{array}{ccccc} & & 1 & 2 & 3 \\ \square & 1 & 4 & 4 & \square \\ 2 & 2 & \square & & \end{array} \longrightarrow \begin{array}{ccccc} & & 1 & 2 & 3 \\ 1 & 2 & 4 & 4 & \square \\ 2 & \square & \square & & \end{array}$$

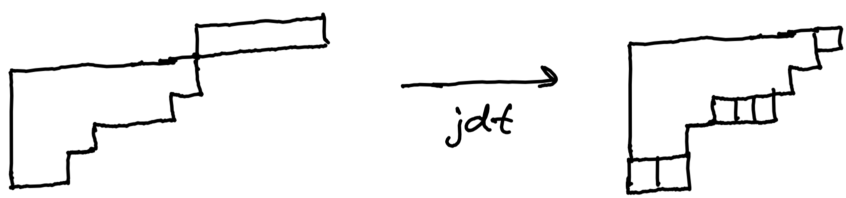
Pieri Formula

$$S_\lambda \cdot S_p = \sum_{\nu} S_\nu$$

sum over $\nu \supseteq \lambda$ s.t. ν/λ horiz. strip, $|\nu/\lambda| = p$.

Proof

$$SSYT(\lambda) \times SSYT((p)) \longleftrightarrow \coprod_{\nu} SSYT(\nu)$$



$\square$

Dual Pieri formula:

$$S_\lambda \cdot S_{(1)^p} = \sum_{\nu} S_\nu$$

sum over $\nu \supseteq \lambda$ s.t. ν/λ vertical strip, $|\nu/\lambda| = p$.

Example

$X = Gr(m,n)$, $\quad 0 \to M \to K_X^n \to Q \to 0.$

Schubert classes:

$S_\lambda(M^\vee) \in A^*(X)$, $\quad \lambda \subseteq$ 

Note:

$\ell(\lambda) = \#\text{ rows in } YD(\lambda) = \max\{i : \lambda_i > 0\}.$

$\ell(\lambda) > m \Rightarrow S_\lambda(M^\vee) = 0$

$\lambda_1 > n-m \Rightarrow \ell(\lambda^T) > n-m \Rightarrow$

$$\begin{aligned} S_\lambda(M^\vee) &= S_\lambda(M^\vee - K_X^n) \\ &= S_{\lambda^T}(K_X^n - M) \\ &= S_{\lambda^T}(Q) = 0 \end{aligned}$$

Fact: $\{S_\lambda(M^\vee) \mid \lambda \subseteq (n-m)^m\}$ basis of $A^*(X)$.

Point class:

Let $V \in X$. $\quad \{V\} = Z(V_X \to K_X^n \to Q)$

$$\begin{aligned} [\text{point}] &= S_{(m)^{n-m}}(Q) \\ &= S_{(n-m)^m}(M^\vee) \in A^*(X). \end{aligned}$$

Example

$X = Gr(2,4)$. $\quad 0 \to M \to K^4 \to Q \to 0$

$$S_{(1,1)} \cdot S_{(1,1)} = S_{(1,1,1,1)} + S_{(2,1,1)} + S_{(2,2)}$$

$$c_2(M^\vee)^2 = S_{(1,1)}(M^\vee)^2 = S_{(2,2)}(M^\vee) = [\text{point}].$$

$$S_{(1,1)} \cdot S_{(1)}^2 = \left(S_{(1,1,1)} + S_{(2,1)}\right) \cdot S_{(1)}$$

$$= S_{(1,1,1,1)} + 2\, S_{(2,1,1)} + S_{(2,2)} + S_{(3,1)}$$

$$c_2(M^\vee) \cdot c_1(M^\vee)^2 = S_{(2,2)}(M^\vee) = [\text{point}].$$