Schur functions of vector bundles

X scheme.

E vector bundle of rank r.

Chern roots: $x_1, \dots, x_r$.

Chern class: $c_i(E) = e_i(x_1, \dots, x_r) \in A^i(X)$.

Total Chern class:

$$c(E) = 1 + c_1(E) + \cdots + c_r(E) \in A^*(X)^\times \quad \text{(unit)}$$
$$= \prod_{i=1}^{r} (1 + x_i).$$

Whitney sum formula

$$0 \to E' \to E \to E'' \to 0$$
$$\Rightarrow \quad c(E) = c(E')\, c(E'')$$

$K(X)$ = Grothendieck group of alg. vector bundles on X.

Group hom: $c : K(X) \longrightarrow A^*(X)^\times$

$$c(E-F) = c(E)\, c(F)^{-1} = \frac{\prod (1+x_i)}{\prod (1+y_j)}$$

if F has Chern roots $y_1, \dots, y_s$.

$c_p(E-F) \in A^p(X)$ homogeneous comp. of deg. p in $c(E-F)$.

<u>Note</u>: $c_p(-F) = h_p(-y_1, \dots, -y_s) = s_p(F)$

Segre class

(Fulton's notation, small "s".)

<u>Def</u> $S_p(E-F) = S_p(x_1, \dots, x_r ; y_1, \dots, y_s)$

$= c_p(F^\vee - E^\vee)$

$=$ hom. comp. of deg. p in

$$\frac{\prod_j (1-y_j)}{\prod_i (1-x_i)}$$

<u>Note</u> $S_p(E) = c_p(-E^\vee) = (-1)^p s_p(E)$.

<u>Def</u> For $\lambda, \mu \in \mathbb{Z}^\ell$, $\mathcal{F} \in K(X)$, def.

$$S_{\lambda/\mu}(\mathcal{F}) = \det\left(S_{\lambda_i - i - \mu_j + j}(\mathcal{F})\right) \in A^{|\lambda/\mu|}(X).$$

<u>Note</u>: $\mathcal{F} \in K(X)$ defines ring hom.

$\Lambda \longrightarrow A^*(X)$; $S_p \mapsto S_p(\mathcal{F})$; $S_{\lambda/\mu} \mapsto S_{\lambda/\mu}(\mathcal{F})$.

Identities

E, F, G vector bundles on X.

$\mu \subseteq \lambda$ partitions.

- $S_{\lambda/\mu}(E-G) = \sum_{\nu : \mu \subset \nu \subset \lambda} S_{\nu/\mu}(E-F)\, S_{\lambda/\nu}(F-G)$.

- $S_{\lambda^T/\mu^T}(E-F) = S_{\lambda/\mu}(F^\vee - E^\vee)$

$S_p(E) = \det\left(c_{1+j-i}(E)\right)_{p\times p}$

$c_p(E) = \det\left(S_{1+j-i}(E)\right)_{p\times p}$

Set $n = \operatorname{rank}(E)$, $m = \operatorname{rank}(F)$.

- $\lambda_{n+1} \geq m+1 \Rightarrow S_\lambda(E-F) = 0$

- $S_{(m)^n}(E-F) = c_{mn}(E\otimes F^\vee)$ top Chern class.

- $\lambda \in \mathbb{N}^n$, $\mu \in \mathbb{Z}^\ell$:

$S_{(m)^n+\lambda,\mu}(E-F) =$

$$S_\mu(-F)\, S_{(m)^n}(E-F)\, S_\lambda(E)$$

Grassmann bundles

X alg. scheme, $E \to X$ vector bundle.

$n = \mathrm{rank}(E)$, $\quad 0 \le a \le n$.

$$Gr(a,E) = \left\{ (x,V) \;\middle|\; x \in X,\ \begin{array}{l} V \subseteq E(x) \text{ vector} \\ \text{subspace of} \\ \text{dim. } a \end{array} \right\}$$

Example: $Gr(1,E) = \mathbb{P}(E)$.

$\pi : Gr(a,E) \longrightarrow X$.

Note: If $U \subseteq X$ open, $E|_U \cong U \times K^n$,
then $\pi^{-1}(U) \cong U \times Gr(a,n)$.

$\dim Gr(a,E) = \dim(X) + a(n-a)$.

Glue: $Gr(a,E) \xrightarrow{\subseteq} \mathbb{P}(\Lambda^a E)$ closed emb.

$(x,V) \longmapsto (x, \Lambda^a V \subseteq \Lambda^a E(x))$

Tautological subbundle $A \subseteq \pi^* E$:

$\pi^* E = \{(x,V,u) : V \subseteq E(x),\ u \in E(x)\}$

$A = \{(x,V,u) : u \in V \subseteq E(x)\}$

$0 \to A \to \pi^* E \to Q \to 0$ on $Gr(a,E)$.

Universal Property

Z any scheme:

$$\mathrm{Mor}(Z, \mathrm{Gr}(a,E)) = \left\{ \begin{array}{l} (f, A') : \\ f : Z \to X \text{ morphism} \\ A' \subseteq f^*E \text{ subbundle} \\ \qquad\qquad \text{of rank } a \end{array} \right\}$$

Given $h : Z \to \mathrm{Gr}(a,E)$:

$$\begin{array}{ccccc} A' = h^*A \subseteq f^*E & & A \subseteq \pi^*E & & E \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow[h]{} & \mathrm{Gr}(a,E) & \xrightarrow[\pi]{} & X \end{array}$$

Flag bundles

$E \to X$ vector bdl. rank n,

$0 \le a_1 \le a_2 \le n$

$$Fl(a_1, a_2; E) = \left\{ (x, V_1, V_2) \;\middle|\; \begin{array}{l} x \in X,\ V_1 \subset V_2 \subset E(x), \\ \dim(V_i) = a_i \end{array} \right\}$$

Tautological flag: $A_1 \subset A_2 \subset \pi^*E$.

Abuse: $A_1 \subset A_2 \subset E$ on $Fl(a_1, a_2; E)$.

Construction:

$$Fl(a_1, a_2; E) = \mathrm{Gr}(a_1, A_2) \to \mathrm{Gr}(a_2, E) \to X$$

$$Fl(a_1, a_2; E) = \mathrm{Gr}(a_2 - a_1, A_1) \to \mathrm{Gr}(a_1, E) \to X$$

Pragacz's Gysin formula

X alg. scheme.

E, F vector bundles of ranks e, f.

$f = a + q$.

$\pi : Gr(a, E) \longrightarrow X$ Grassmann bundle

$0 \to A \to \pi^* E \to Q \to 0$ taut. exact. seq.

$\lambda \in \mathbb{Z}^q$, $\mu \in \mathbb{Z}^\ell$, $\lambda_i \geq e \ \forall i$. Then:

$$\pi_* \left(S_\lambda(Q - \pi^* E) \cdot S_\mu(A - \pi^* E) \right) = S_{\lambda - (a)^q, \mu}(F - E)$$
$$\in A^*(X).$$

<u>Example</u> $a = 0$, $Gr(a, F) = X$:

$$S_\lambda(F - E) \cdot S_\mu(-E) = S_{\lambda, \mu}(F - E)$$

(Equiv. to factorization formula.)

<u>Special case</u>: $a = f - 1$, $q = 1$, $E = 0$.

$$Gr(f-1, F) = \mathbb{P}^*(F) = \mathbb{P}(F^\vee) \xrightarrow{\pi} X.$$

$Q = \mathcal{O}_{F^\vee}(1)$. $\qquad (Q^\vee \subseteq \pi^* F^\vee)$

$S_p(Q) = c_1(\mathcal{O}_{F^\vee}(1))^p$. Def. of Segre classes $\Rightarrow$

$$\pi_*(S_p(Q)) = s_{p-f+1}(F^\vee) = S_{p-f+1}(F).$$

$$\begin{tikzcd}[row sep=small]
A \subseteq H \subseteq F & A \subseteq F \\
Fl(a, f-1; F) \arrow[r, "\tau'"'] \arrow[d, "\pi'"'] & Gr(a, F) \arrow[d, "\pi"] \\
\mathbb{P}^*(F) \arrow[r, "\tau"'] & X \\
H \subseteq F &
\end{tikzcd}$$

$\tilde{\lambda} = (\lambda_2, \dots, \lambda_q) \in \mathbb{Z}^{q-1}$

<u>Induction</u>:

$$\begin{tikzcd}[column sep=tiny]
S_{\lambda_1+q-1}(F/H - E) \cdot S_{\tilde{\lambda}}(H/A - E) \cdot S_\mu(A-E) \arrow[d, mapsto, "\pi'_*"'] \arrow[r, mapsto, "\tau'_*"] & S_\lambda(F/A - E) \cdot S_\mu(A-E) \arrow[d, mapsto, "\pi_*"', "(!)"] \\
S_{\lambda_1+q-1}(F/H - E) \cdot S_{\tilde{\lambda}-(a)^{q-1}, \mu}(H-E) \arrow[r, mapsto, "\tau_*"] & S_{\lambda-(a)^q, \mu}(F-E)
\end{tikzcd}$$

Enough to prove formula for $a = f-1$.

$\pi : \mathbb{P}^*(F) \longrightarrow X, \quad p \geq e = \mathrm{rank}(E).$

$0 \to A \to F \to Q \to 0$ on $\mathbb{P}^*(F)$.

Show:

$$\pi_*\big(S_p(Q-E)\,S_\mu(A-E)\big) = S_{p-f+1,\mu}(F-E)$$

Proof:

$$S_p(Q-E)\,S_\mu(A-E) = S_p(Q-E)\,S_\mu(F-E\oplus Q)$$

$$= \sum_{k\geq 0} S_k(Q)\,S_{p-k}(-E) \sum_{\ell\geq 0} S_{(1^\ell)}(-Q)\,S_{\mu/(1^\ell)}(F-E)$$

$$= \sum_{k,\ell\geq 0} (-1)^\ell S_{k+\ell}(Q)\,S_{p-k}(-E)\,S_{\mu/(1^\ell)}(F-E)$$

$$\xmapsto{\pi_*}$$

$$\sum_{k,\ell\geq 0} (-1)^\ell S_{k+\ell-f+1}(F)\,S_{p-k}(-E)\,S_{\mu/(1^\ell)}(F-E)$$

$$= \sum_{\ell\geq 0} S_{p-f+1+\ell}(F-E)\,S_{\mu/(1^\ell)}(F-E)$$

$$= S_{p-f+1,\mu}(F-E) \quad \leftarrow \text{expand determinant after first row.}$$

$\square$