Enumerative Geometry Jan 20, 2022

M Th 12:10 – 1:30.

Online: first 3 classes.

Hill 425: starting Monday Jan 31.

Course website: math.rutgers.edu/rasbach

Goal of Enumerative Geometry:

Count the geometric figures of specified type that satisfy list of conditions.

- easier to count than to list.

Example: $\{x \in \mathbb{C} \mid x^4 + 2x^3 + 3x^2 + 4x + 5 = 0\}$.

- multiple solutions?

Intersection theory

X non-singular proj. variety $/\mathbb{C}$, $\dim X = n$.

$\Omega \subseteq X$ non-sing. closed subvariety, $\dim = m$.

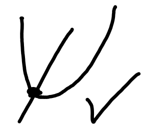
Cohom. class: $[\Omega] \in H^{2(n-m)}(X)$.

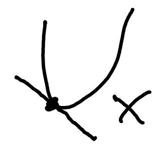
$$\left([\Omega] \in H_{2m}(\Omega) \longrightarrow H_{2m}(X) \cong H^{2(n-m)}(X). \right)$$

Fundamental Fact:

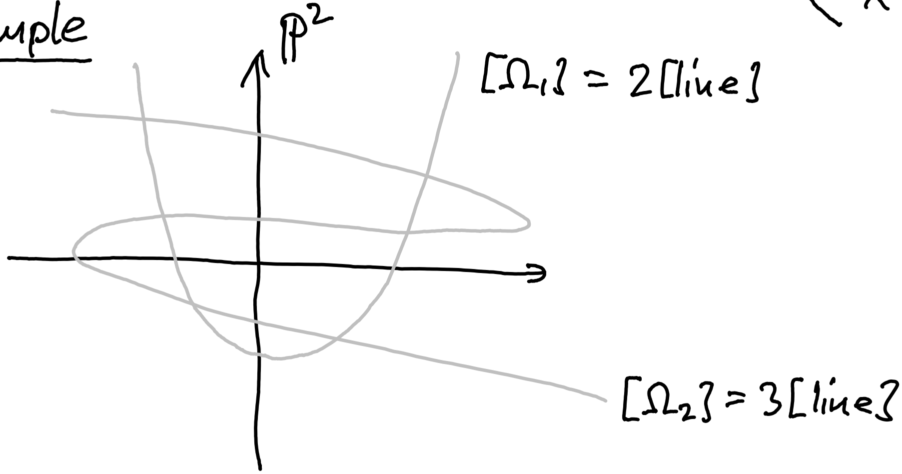
Let $\Omega_1, \Omega_2 \subseteq X$ be non-singular closed subvarieties that meet transversally.

Then $[\Omega_1 \cap \Omega_2] = [\Omega_1] \cdot [\Omega_2]$ holds in $H^*(X)$.





Example



$$[\Omega_1 \cap \Omega_2] = 2[\text{line}] \cdot 3[\text{line}] = 6[\text{point}]$$

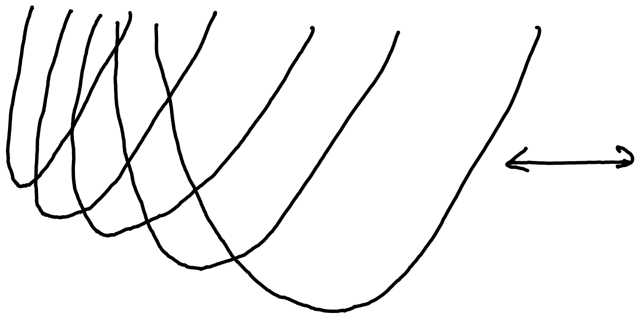
$\therefore$ We can count # points in intersection of subvarieties (subj. to conds.)

Count geometric figures sat. conditions

X = moduli space of all geom. figures of given type.

- One point for each figure.
- two points near each other $\Leftrightarrow$ figures similar.

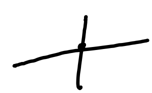
E.g. all curves in $\mathbb{P}^r$ of degree 2.



curve in X

Ideal situation:

- X non-singular projective variety.
  - sometimes "projective" requires adding extra "degenerate" figures.



- List of conditions $\Leftrightarrow$ closed subvarieties $\Omega_1, \dots, \Omega_k \subseteq X$ that meet transversally.

$$\# \text{ solutions } = \# \, \Omega_1 \cap \cdots \cap \Omega_k$$

$$[\Omega_1 \cap \cdots \cap \Omega_k] = [\Omega_1] \cdot \cdots \cdot [\Omega_k] \in H^*(X).$$

Sub-Problems

1) Construct (compact) moduli space X.

2) Understand $H^*(X)$.
   E.g. generators + relations. Chow.

3) Understand $[\Omega_i] \in H^*(X)$.
   E.g. polynomial expression in generators.
   Degeneracy locus / Giambelli formulas.

4) Ensure sufficient transversality.
   E.g. use group action to move subvarieties to "general position".

Note: (2) + (3) involve combinatorics!

Example

Given general lines $L_1, L_2, L_3, L_4 \subseteq \mathbb{P}^3$,
how many lines $M \subseteq \mathbb{P}^3$ meet $L_1, \dots, L_4$?

"general" = ?

$\mathbb{P}^3 = \{$ 1-dim. vector subspaces $p \subseteq \mathbb{C}^4\}$.

Line in $\mathbb{P}^2 \Leftrightarrow$ 2-dim. subspace $M \subseteq \mathbb{C}^4$

$$M \Leftrightarrow \{p \in \mathbb{P}^3 : p \subseteq M \subseteq \mathbb{C}^4\} = \mathbb{P}^1$$

$X = \{$ lines in $\mathbb{P}^2 \}$

$= \{ M \subseteq \mathbb{C}^4 \mid \dim M = 2 \}$

$= Gr(2,4)$ Grassmann variety.

$M \in X$.

$M =$ row span $\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{bmatrix}$

Plucker coordinates: $p_{ij} = p_{ij}(M) = \begin{vmatrix} m_{1i} & m_{1j} \\ m_{2i} & m_{2j} \end{vmatrix}$.

$$p(M) = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \mathbb{P}^5$$

<u>well-defined</u> !

Plucker embedding: $X \subseteq \mathbb{P}^5$, $M \mapsto p(M)$.

<u>Note</u>: Lines in $\mathbb{P}^3$ / points in $X = Gr(2,4)$:

$$M = \text{span}\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \end{bmatrix}$$

$$N = \text{Span}\begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \end{bmatrix}$$

$$\det\begin{vmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & & & m_{24} \\ n_{11} & & & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \end{vmatrix} = \begin{aligned} & P_{12}(M)\cdot P_{34}(N) \\ & - P_{13}(M)\, P_{24}(N) \\ & + P_{14}(M)\cdot P_{23}(N) \\ & + P_{23}(M)\, P_{14}(N) \\ & - P_{24}(M)\, P_{13}(N) \\ & + P_{34}(M)\, P_{12}(N). \end{aligned}$$

$\therefore \quad P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$

$X = Gr(2,4) = Z(P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}) \subseteq \mathbb{P}^5.$

non-singular, projective, $\dim(X) = 4$.

Note: M meets $N \Leftrightarrow \det = 0$

$\Leftrightarrow p_{12}(M) \cdot p_{34}(N) - \cdots + p_{34}(M)\, p_{12}(N) = 0.$

$\therefore \{\text{lines } M \subseteq \mathbb{P}^3 \text{ meeting } N\} = X \cap H \subseteq \mathbb{P}^5$

$H = H(N) \subseteq \mathbb{P}^5$ hyperplane.

Solutions:

Lines meeting $L_1, L_2, L_3, L_4 \longleftrightarrow$

$$H(L_1) \cap H(L_2) \cap H(L_3) \cap H(L_4) \cap X \subseteq \mathbb{P}^5.$$

L_1, L_2, L_3, L_4 "general":

$H(L_1) \cap \cdots \cap H(L_4) \subseteq \mathbb{P}^5$ must be a line not contained in X.

Possible to choose "general" lines thanks to transitive group action $GL_4(\mathbb{C}) \circlearrowright X$ coming from $GL_4(\mathbb{C}) \circlearrowright \mathbb{C}^4$!

Later: Kleiman's transversality Thm.

Count: Line $\cap$ conic $= 2$ points (with multiplicity)

Use $H^*(\mathbb{P}^5) = \mathbb{Z}[h]/\langle h^6 \rangle$:

$$[H(L_1) \cap H(L_2) \cap H(L_3) \cap H(L_4) \cap X]$$

$$= [H]^4 \cdot 2[H] = 2\,[\text{point}].$$

Use $H^*(X) = H^*(Gr(2,4))$:

- Schubert Calculus.

or use Plücker embedding: $p : X \hookrightarrow \mathbb{P}^5$.

Calculation happens in $p^* H^*(\mathbb{P}^5) \subseteq H^*(X)$.

$\Omega_i = \{\text{lines meeting } L_i\} = p^{-1}(H(L_i))$.

$[\Omega_i] = p^*[H]$

$$[\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4] =$$

$$[\Omega_1] \cdot [\Omega_2] \cdot [\Omega_3] \cdot [\Omega_4] =$$

$$p^*([H]^4)$$

$$\Rightarrow \quad p_* [\Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4] =$$

$$p_* p^*(h^4) = h^4 \cdot [X] = 2\,[pt].$$

Intersection Theory

Conventions

k fixed field. $k = \bar{k}$ in lectures. $k = \mathbb{C}$.

Variety: reduced + irreducible

Subvariety: closed subvariety.

Scheme: alg. scheme = scheme of finite type / k.

order of vanishing

X variety.

$R(X) = K(X) =$ field of rational fcns on X.

Let $f \in R(X)^*$ non-zero rat. fcn.

$V \subseteq X$ subvariety of codim. 1. $\boxed{\text{prime divisor}}$

$\mathrm{ord}_V(f) =$ order of vanishing of f along V.

$\mathcal{O}_{V,X} =$ local ring of X along V

$= \{ h \in R(X)^* \mid f \text{ is defined at some point in } V \}$.

local ring of dim. 1.

$\mathfrak{m}_{V,X} \subseteq \mathcal{O}_{V,X}$ max. ideal.

Assume X normal.

Then $\mathcal{O}_{V,X}$ Noetherian local normal dim 1.

$\Rightarrow \mathcal{O}_{V,X}$ DVR.

$\mathfrak{m}_{V,X} = \langle t \rangle$.

$f = ut^n$, $n \in \mathbb{Z}$, $u \in \mathcal{O}_{V,X}$ unit.

$\mathrm{ord}_V(f) = n$.

Note: $f \in \mathcal{O}_{V,X} \Leftrightarrow \mathrm{ord}_V(f) \geq 0$

$f \in \mathfrak{m}_{V,X} \Leftrightarrow \mathrm{ord}_V(f) > 0$.

Note: $\mathrm{ord}_V : R(X)^* \longrightarrow \mathbb{Z}$ group hom.

$\mathrm{ord}_V(fg) = \mathrm{ord}_V(f) \cdot \mathrm{ord}_V(g)$.

X any variety.

Assume first $f \in \mathcal{O}_{V,X}$.

Then $\mathcal{O}_{V,X} / \langle f \rangle$ Artinian.

Def $\mathrm{ord}_V(f) = \mathrm{length}(\mathcal{O}_{V,X}/\langle f \rangle)$.

$\mathrm{ord}_V(f/g) = \mathrm{length}(\mathcal{O}_{V,X}/\langle f \rangle) - \mathrm{length}(\mathcal{O}_{V,X}/\langle g \rangle)$

for $f,g \in \mathcal{O}_{V,X}$ non-zero.

Prop: $\mathrm{ord}_V : R(X)^* \longrightarrow \mathbb{Z}$ group hom.

Principal Divisors

X variety.

Weil divisors:

$Div(X)$ = free abelian gp. gen. by prime divisors V. $[V] \in Div(X)$.

$f \in R(X)^*$ Principal divisor:

$$div(f) = \sum_V ord_V(f)\,[V] \in Div(X).$$

<u>Note</u>: finite sum.

<u>Note</u>: $div : R(X)^* \longrightarrow Div(X)$ group hom.

$$Cl(X) = Div(X) / \{div(f) \mid f \in R(X)^*\}$$

$Cl(X) = A_{n-1}(X) = (n-1)$-st Chow group.

$n = \dim(X)$.

Chow Groups

X alg. scheme $/K$.

$0 \le k \le \dim(X)$.

k-cycles:

$Z_k(X)$ = free abelian group gen. by subvarieties $V \subseteq X$ of dim k.

$$= \left\{ \sum_{\text{finite}} n_i [V_i] \;\middle|\; \begin{matrix} n_i \in \mathbb{Z}, \\ V_i \subseteq X \text{ subvar.} \\ \dim V_i = k \end{matrix} \right\}.$$

Rational equivalence:

Given $W \subseteq X$ subvar. of $\dim(W) = k+1$ and $f \in R(W)^*$:

$\operatorname{div}(f) \in \operatorname{Div}(W) \subseteq Z_k(X)$.

$\operatorname{Rat}_k(X) \subseteq Z_k(X)$ subgroup gen by $\operatorname{div}(f)$ for all $W \subseteq X$ of dim $k+1$, all $f \in R(W)^*$.

Def $A_k(X) = Z_k(X) / \operatorname{Rat}_k(X)$. (Chow group.)

Example

1) $Y \subseteq X$ closed subscheme.

$$A_k(Y) \longrightarrow A_k(X) \quad \text{group hom.}$$

2) X_1, X_2 closed subschemes.

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \longrightarrow A_k(X_1 \cup X_2) \to 0$$