Complex Variables 4/12/2018

3.5 Riemann Mapping Thm & Schwarz-Christoffel Transforms

Remarks

- Line = circle through ∞ in $\mathbb{CP}^1$
- $\exists!$ circle through 3 distinct points in $\mathbb{CP}^1$.
  line when ...
- open ball: points inside circle.
- what if circle = line?
- oriented circle in $\mathbb{CP}^1$. inside/outside.
- frac. lin. trans. circle $\longrightarrow$ circle.
  ball $\longrightarrow$ ball.

Riemann Mapping Thm

$D \subseteq \mathbb{C}$ simply-connected open set, with at least 2 points on boundary. Let $p \in D$.

$\exists!$ bijective analytic $\phi : D \longrightarrow \Delta = B(0,1)$

such that $\phi(p) = 0$ and $\phi'(p) \in \mathbb{R}_+$.

uniqueness: $\phi, \psi : D \xrightarrow{\cong} \Delta$, $\phi(p) = \psi(p) = 0$, $\phi'(p), \psi'(p) \in \mathbb{R}$

$g = \phi \circ \psi^{-1} : \Delta \xrightarrow{\cong} \Delta$, $g(0) = 0$.

$g(z) = \lambda z$, $|\lambda| = 1$. $g'(0) > 0 \Rightarrow \lambda = 1$.

$g(z) = id.$

Q: "2 pts on boundary"?
$\Rightarrow$ $D \subseteq \mathbb{CP}^1$ open simply-conn, $\mathbb{CP}^1 \setminus D$ contains 2 points.

Consequence: $D_1, D_2 \subsetneq \mathbb{C}$ open simply-conn.
$p_j \in D_j$.

$\exists!$ bijective analytic $\phi : D_1 \to D_2$

$\phi(p_1) = p_2$

$\phi'(p_1) > 0$.

Def D_1 and D_2 are conformally equivalent if

- which are conformally equivalent? $\{z \mid -\pi < \mathrm{Im}(z) < \pi\}$

Δ, $\mathbb{C}$, $\mathbb{CP}^1$, $\mathbb{C} - \mathbb{R}_{\leq 0}$, $\mathbb{C} \setminus \{0\}$, $\Delta - \{0\}$.

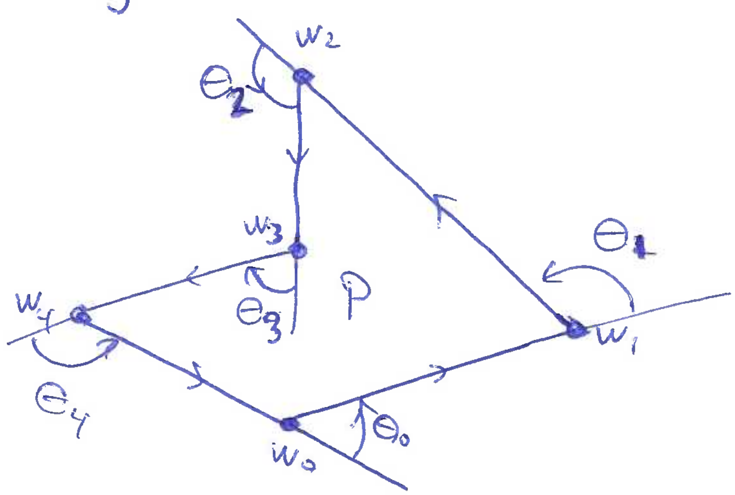
Schwarz-Christoffel Transforms

$P \subseteq \mathbb{C}$ polygon. (open). $U = \{z \mid \mathrm{Im}(z) > 0\}$ upper $\frac{1}{2}$ plane.

$\exists$ bijective analytic $f : U \xrightarrow{\cong} P$.

Trick: map $\mathbb{R} \longrightarrow \partial P$.

- by defining $f'(z)$.



Vertices: $w_0, w_1, \ldots, w_N \in \mathbb{C}$

turn-angles: $\theta_0, \theta_1, \ldots, \theta_N \in (-\pi, \pi)$

③

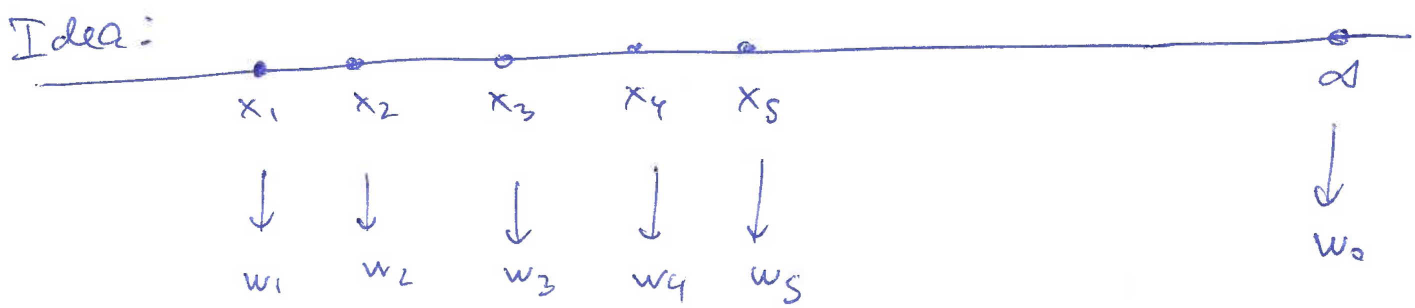
Let $x_0 < x_1 < x_2 < \cdots < x_N$ real numbers.

<u>Def</u> $g(z) = A(z-x_1)^{\alpha_1}(z-x_2)^{\alpha_2}\cdots(z-x_N)^{\alpha_N}$ $\quad A \in \mathbb{C}$.

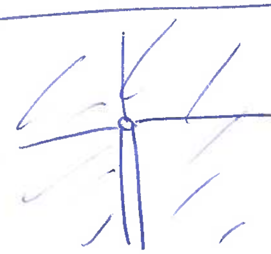
where $\alpha_j = -\dfrac{\theta_j}{\pi} \in \mathbb{R}$.

Choose A s.t. $\arg(A) =$ ~~direction~~ $\arg(w_1 - w_0)$.

Idea:



<u>Def</u> log on $D = \mathbb{C} - i\mathbb{R}_{\le 0}$



$f(z) = z^{\beta} = \exp(\beta \log(z))$

$f'(z) = \beta\, z^{\beta - 1}$.

$\cancel{g'(z) = A\sum_{j=1}^{N} \alpha_j (z-x_1)^{\alpha_1}\cdots(z-x_j)^{\alpha_j - 1}\cdots(z-x_N)^{\alpha_N}}$

$\cancel{x \in \mathbb{R}:\ x_j <}$

$x \in \mathbb{R}$: $\quad x_j < x$: $\quad (x-x_j)^{\alpha_j} \in \mathbb{R}^+ \quad > 0$

$x < x_j$: $\quad (x-x_j)^{\alpha_j} = \exp(\alpha_j \log(x-x_j))$

$$= \exp(\alpha_j \ln|x-x_j| + i\alpha_j \arg(x-x_j))$$

$$= |x-x_j|^{\alpha_j} \exp(i\pi\alpha_j)$$

$$\arg((x-x_j)^{\alpha_j}) = -\Theta_j$$

$\therefore$ If $x_j < x < x_{j+1}$ then

$$\arg(g(x)) = \cancel{\arg(A) + \Theta_1 + \Theta_2 + \cdots + \Theta_j}$$
$$\arg(A) - (\Theta_{j+1} + \cdots + \Theta_N)$$

Let $f(z) = \displaystyle\int_{x_1}^{z} g(z)\,dz + w_1$

Then f maps

$$x_1 \longmapsto w_1$$

$[x_1, x_2] \longmapsto$ line of slope $(w_2 - w_1)$

$[x_j, x_{j+1}] \longmapsto$ line of slope $(w_{j+1} - w_j)$.

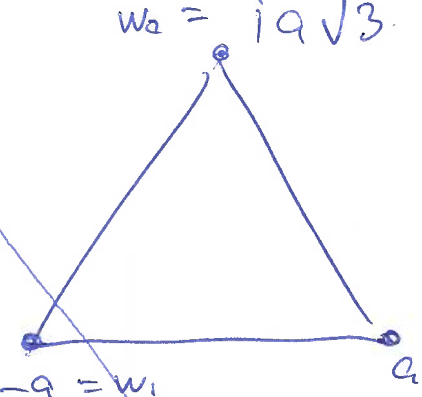
Schwarz – Christoffel Thm

One can choose $x_1 < x_2 < \cdots < x_N$ and $A \in \mathbb{C}$

such that $f(z) = w_1 + \int_{x_1}^{z} A\,(\zeta - x_1)^{\alpha_1} \cdots (\zeta - x_N)^{\alpha_N}\, d\zeta$

maps upper-half plane to polygon P.

Example



$\theta_j = \frac{2}{3}\pi$

$\alpha_j = -\frac{2}{3}$

$x_1 = -1, \quad x_2 = 1$

$$g(z) = A(z+1)^{-2/3}(z-1)^{-2/3}$$

$$g(z) = A(z^2-1)^{-2/3}$$

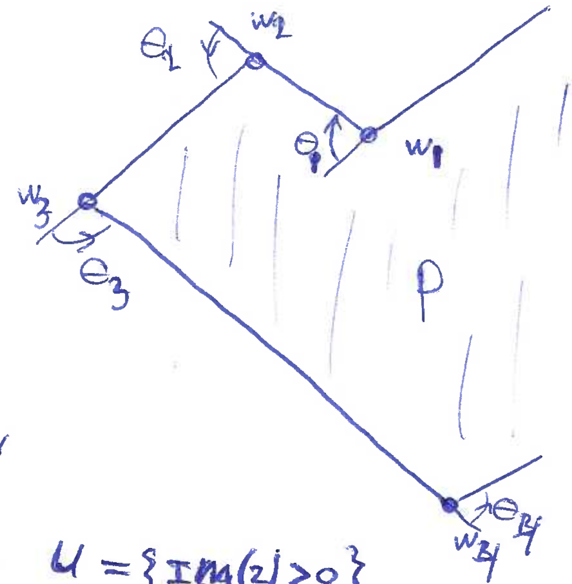
$$f(z) = A\int_{-1}^{z} (w^2-1)^{-2/3}\, dw \quad - a$$

$$\beta = \int_{-1}^{\infty} (t^2-1)^{-2/3}\, dt \in \mathbb{R}$$

$$i\sqrt{3}a = A\cdot\beta - a \quad \Rightarrow \quad A = \frac{(i\sqrt{3}+1)a}{\beta}$$

Complex Variables 4/17/2018

Schwarz – Christoffel Transform



$x_1 < x_2 < \cdots < x_N \quad \in \mathbb{R}$

$\theta_1, \theta_2, \ldots, \theta_N \in (-\pi, \pi)$

$\alpha_j = -\dfrac{\theta_j}{\pi}$

$$g(z) = A(z-x_1)^{\alpha_1}(z-x_2)^{\alpha_2}\cdots(z-\alpha_N)^{\alpha_N}$$

analytic on $U = \{\mathrm{Im}(z) > 0\}$

If $x_j < z < x_{j+1}$ then

$$\arg(g(z)) = \arg(A) - (\theta_{j+1} + \theta_{j+2} + \cdots + \theta_N)$$

$\boxed{\text{Choose } A \in \mathbb{C} \text{ such that } \arg(g(z)) = \text{right thing for } z \in \mathbb{R},\ z \ll 0.}$

<u>Def</u> $\quad f(z) = \displaystyle\int_{x_1}^{z} g(\zeta)\, d\zeta + w_1$

Image $f(\mathbb{R})$ = polygon with line segments going in correct directions.

<u>Thm</u> $\exists\ x_1 < x_2 < \cdots < x_N \in \mathbb{R}$ and $A \in \mathbb{C}$ such that

$f(x_j) = w_j$. $\qquad$ (So $f(\mathbb{R}) = \partial P$.)

<u>Note:</u> Can choose 3 of the x_j's freely.

– Replace $f(z)$ with $f(\phi(z))$,

ϕ: fractional linear transform.

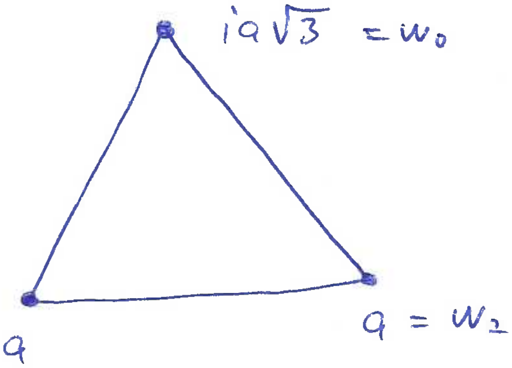
$$\boxed{\begin{aligned} x_1' &\longmapsto x_1 \\ x_2' &\longmapsto x_2 \\ x_3' &\longmapsto x_3 \end{aligned}}$$

Example

$$\theta_j = \frac{2\pi}{3}$$

$$\alpha_j = -\frac{2}{3}$$

$x_1 = -1, \quad x_2 = 1$



$$g(z) = A(z+1)^{-2/3}(z-1)^{-2/3} = A(z^2-1)^{-2/3}$$

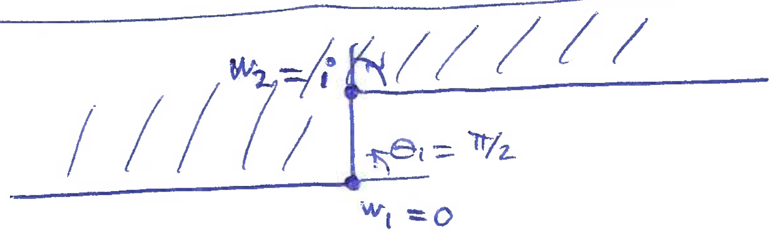
$$f(z) = A\int_{-1}^{z} (w^2-1)^{-2/3}\, dw \quad - a$$

Set $\beta = \int_{-1}^{1} (t^2-1)^{-2/3}\, dt$

$$a = f(1) = A\beta - a \quad \Rightarrow \quad A = \frac{2a}{\beta}$$

$$\therefore \; f(z) = -a + \frac{2a}{\beta}\int_{-1}^{z} (w^2-1)^{-2/3}\, dw$$

Example



$\theta_1 = \frac{\pi}{2} \quad \theta_2 = -\frac{\pi}{2}$

$x_1 = -1, \quad x_2 = 1$

$\alpha_1 = -\frac{1}{2} \quad \alpha_2 = \frac{1}{2}$

$$g(z) = A(z+1)^{-1/2}(z-1)^{1/2}$$

$$g(z) = A\left(\frac{z-1}{z+1}\right)^{1/2}$$

$$f(z) = \int g(z)\,dz = A\left[(z^2-1)^{1/2} - \mathrm{Log}\left(z + (z^2-1)^{1/2}\right)\right] + B.$$

$$f(1) = 0 + B = B$$

$$f(-1) = -A\log(-1) + B = -i\pi A + B$$

$B = i$

$i\pi A = i \Rightarrow A = \frac{1}{\pi}.$

Harmonic functions

$D \subseteq \mathbb{C}$ open, $u : D \longrightarrow \mathbb{C}$ function. (C^∞)

Laplacian: $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Harmonic: $\Delta u = 0$ on D.

Note: $u = u_1 + i u_2$, $u_1, u_2 : D \longrightarrow \mathbb{R}$.

Then $\Delta u = \Delta u_1 + i \Delta u_2$

u Harmonic $\iff$ u_1 and u_2 real harmonic fcns.

Thm
If $f : D \longrightarrow \mathbb{C}$ analytic, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Harmonic on D.

Pf
$f = u + iv$. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = - \dfrac{\partial v}{\partial x}$

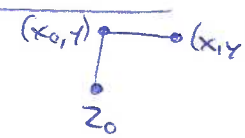
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

$\Delta v = 0$ $\quad \boxed{v = \mathrm{real}(-if)}$

$\square$

Thm If $u : D \longrightarrow \mathbb{R}$ is real harmonic fcn, then u LOCALLY has real harmonic conjugate, i.e. $v : B \longrightarrow \mathbb{R}$ such that $f = u + iv$ is analytic on B, where $B \subseteq D$ ball.

Pf $z_0 = (x_0, y_0) \in D$. $B = B(z_0, r) \subseteq D$.



$$v(x,y) = - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y)\, dt + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, s)\, ds$$

Claim: $f(x,y) = u(x,y) + i\, v(x,y)$ is analytic on B.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = -\int_{x_0}^{x} \frac{\partial^2 u}{\partial y}(t,y)\,dt + \frac{\partial u}{\partial x}(x_0,y)$$

$$\boxed{\cancel{= -\left[\frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x_0,y)\right] + \frac{\partial u}{\partial x}(x_0,y)}}$$

$$= \int_{x_0}^{x} \frac{\partial^2 u}{\partial x}(t,y)\,dt + \frac{\partial u}{\partial x}(x_0,y)$$

$$= \left[\frac{\partial u}{\partial x}(x,y) - \frac{\partial u}{\partial x}(x_0,y)\right] + \frac{\partial u}{\partial x}(x_0,y)$$

$$= \frac{\partial u}{\partial x}(x,y).$$

$\square$

<u>Thm</u> $u : D \longrightarrow \mathbb{R}$ is harmonic $\Leftrightarrow$ its restriction to every open ball contained in D is the real part of analytic function.

Example $u = x^2 - y^2$

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \Rightarrow v(x,y) = 2xy + p(x)$

$2y + p'(x) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \Rightarrow p'(x) = 0 \Rightarrow p(x) = \text{const}$

$v = 2xy + \text{const.}$

$f(z) = z^2 + \text{const.}$

<u>Examples</u>

(1) $u = 2xy \longrightarrow f(z) = -i z^2 = -i(x+iy)^2$

(2) $u = e^x \cos(y) \longrightarrow f(z) = \exp(z)$

(3) $u = (e^x + e^{-x})\cos(y) \longrightarrow f(z) = e^z + e^{-z}$

<u>Remark</u> Harmonic conj. of u is unique up to const.

$f_1 = u + iv_1$, $f_2 = u + iv_2$ both analytic.

CR $\Rightarrow \frac{\partial v_1}{\partial x} = \frac{\partial v_2}{\partial x}, \quad \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y} \Rightarrow v_2 = v_1 + \text{const.}$

<u>Example</u> $D = \mathbb{C} \setminus \{0\}$

$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$ harmonic on D.

But NOT the real part of analytic $f: D \to \mathbb{C}$.

<u>Note:</u> $u(z) = \frac{1}{2}\ln(|z|^2) = \ln|z| = \mathrm{Re}(\log(z))$.

<u>Thm</u> $u: D \to \mathbb{R}$ harmonic, non-constant.
Then u has no local max and no local min.

<u>Thm</u> Let $u: D \to \mathbb{C}$ be harmonic.
Let $\phi: \Omega \to \mathbb{C}$ be analytic.
Then $u \circ \phi$ is harmonic on $\phi^{-1}(D) \subseteq \Omega$.

<u>Pf</u> WLOG $u: D \to \mathbb{R}$ real fcn.
WLOG $u = \mathrm{Re}(f)$, $f: D \to \mathbb{C}$ analytic.
Then $f \circ \phi: \phi^{-1}(D) \to \mathbb{C}$ analytic
$\Rightarrow \mathrm{Re}(f \circ \phi) = u \circ \phi$ is harmonic.
$\square$

Mean Value Thm

$f : D \to \mathbb{C}$ analytic, $B(z_0, R) \subseteq D$, $0 < r < R$.

Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$

Proof

$$f(z_0) = \operatorname{Res}\left(\frac{f(z)}{z - z_0}; z_0\right)$$

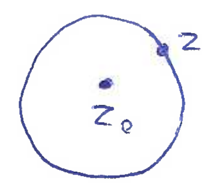
$$\gamma_r(t) = z_0 + re^{it}$$

$$\gamma_r'(t) = ire^{it}$$

$$= \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}}\, ire^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$



$f(z_0) =$ average of $f(z)$

$\square$

Thm $u : D \to \mathbb{C}$ harmonic, $B(z_0, R) \subseteq D$, $0 < r < R$.

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$$

Pf WLOG $u : D \to \mathbb{R}$ real harmonic.

$u(z) = \operatorname{Re} f(z)$ for $z \in B(z_0, R)$, f analytic.

$$u(z_0) = \operatorname{Re} f(z_0) = \operatorname{Re}\left(\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + e^{it})\, dt\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(z_0 + e^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$$

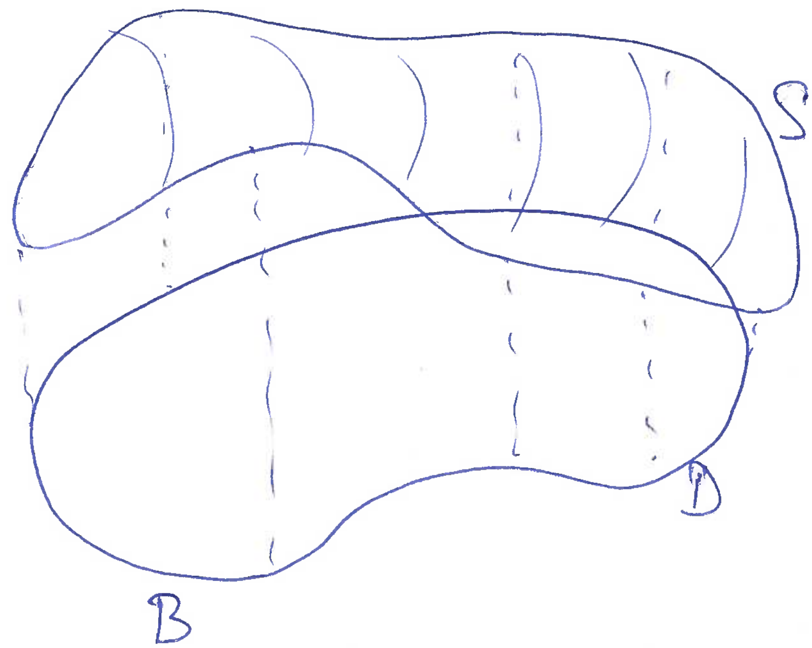
$\square$

$S = \{ (x, y, u(x,y)) \mid x+iy \in D \}$

Physics / Strain Energy

(7)

= surface of minimal strain energy
if $u: D \to \mathbb{R}$ harmonic.

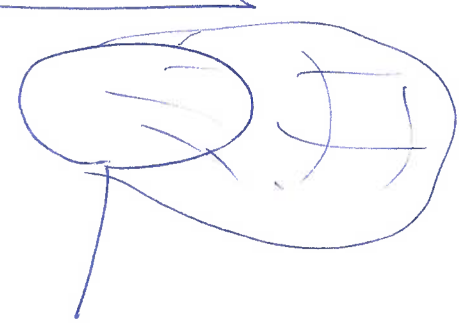


$B = \partial D$.

given real fcn:
$f : B \longrightarrow \mathbb{R}$

Frame: $\{ (x, y, f(x,y)) \mid (x,y) \in B \}$

— blow bubbles!!



Complex Variables 4/24/2018

<u>Gamma function</u> : $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, for $\mathrm{Re}(z) \geq 1$.

<u>Thm</u> $\Gamma(z+1) = z\Gamma(z)$ <u>Note:</u> $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$

<u>Cor</u> For $n \in \mathbb{Z}$, $n \geq 1$: $\Gamma(n) = (n-1)!$

<u>Def</u> For $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ define

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \quad \text{for any } n \in \mathbb{Z},\ n \geq 0,\ \mathrm{Re}(z+n) \geq 1$$

<u>Fact:</u> $\Gamma(z)$ is analytic on $D = \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with a simple pole at z_0 for each $z_0 \in \mathbb{Z}_{\leq 0}$. Satisfies $z\Gamma(z) = \Gamma(z+1)$ for $z \in D$.

<u>Riemann's Zeta function</u>

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, define

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \sum_{n=1}^\infty \exp(-s \cdot \ln(n))$$

Absolute convergence:

$$\sum_{n=1}^\infty |n^{-s}| = \sum_{n=1}^\infty n^{-p} < \infty \quad \text{where } p = \mathrm{Re}(+s) > +1.$$

$$\leq 1 + \int_1^\infty x^{-p} dx = 1 + \left[\frac{x^{1-p}}{1-p} \right]_1^\infty = \frac{-1}{1-p} +$$

<u>Examples</u>

$\zeta(2) = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. $\zeta(2k) = C_k \pi^{2k}$, $C_k \in \mathbb{Q}$ $(k \in \mathbb{N})$

m odd integer : $\zeta(m)$ hard to compute.

$$\sum_{n=1}^\infty n^{-p} \leq 1 + \int_1^\infty x^{-p} dx = 1 + \left[\frac{x^{1-p}}{1-p} \right]_1^\infty = 1 + \frac{1}{p-1} = \frac{p}{p-1}$$

Thm $\zeta(z)$ is analytic on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1\}$.

Pf

$$\frac{d}{ds}\sum_{n=1}^{\infty} n^{-s} \overset{?}{=} \sum_{n=1}^{\infty} \frac{d}{ds}\left(n^{-s}\right) = \sum_{n=1}^{\infty}\left(-n^{-s}\cdot \ln(n)\right)$$

True if RHS series converges uniformly.

$p = \mathrm{Re}(s) > 1$. $\qquad \frac{p-1}{2} > 0$.

$\ln(n) < n^{\frac{p-1}{2}}$ for $n \gg 0$.

$n^{-p}\ln(n) < n^{-\frac{p+1}{2}}$ for $n \gg 0$.

$$\sum_{n=1}^{\infty}\left|-n^{-s}\ln(n)\right| = \sum_{n=1}^{\infty} n^{-p}\ln(n) < \text{const} + \sum_{n=1}^{\infty} n^{-\frac{p+1}{2}} < \infty$$

since $\frac{p+1}{2} > 1$.

uniform convergence follows from this.

$\square$

Cool Fact $\mathrm{Re}(s) > 1 \Rightarrow$

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

product over all primes!

Proof

$$\frac{1}{1-p^{-s}} = 1 + p^{-s} + (p^{-s})^2 + \cdots = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots$$
$$= 1^{-s} + p^{-s} + (p^2)^{-s} + \cdots$$

$$\prod_{\substack{p \leq N \\ \text{prime}}} \frac{1}{1-p^{-s}} = \sum_{\substack{n \geq 1: \\ \text{prod of primes} \leq N}} n^{-s}$$

$$\boxed{\begin{array}{l} n = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell} \\ n^{-s} = (p_1^{a_1})^{-s} \cdots (p_\ell^{a_\ell})^{-s} \end{array}}$$

Since $\sum n^{-s}$ conv. absolutely, same limit as $N \to \infty$ on both sides.

$\square$

Analytic on $\mathbb{C} \setminus \{1\}$:

~~Prop~~ $x \in \mathbb{R}$, $\boxed{\cancel{x \geq 0}}$

$[x]$ = integer part = $\max\{m \in \mathbb{Z} \mid m \leq x\}$. $\in \mathbb{Z}$

$\{x\}$ = frac part $= x - [x] \in [0,1)$.

Prop $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \frac{s}{s-1} - s \int_{x=1}^{\infty} \{x\}\, x^{-s-1}\, dx$$

Proof

$$\int_1^{\infty} \{x\}\, x^{-s-1}\, dx = \int_1^{\infty} x^{-s}\, dx - \int_1^{\infty} [x]\, x^{-s-1}\, dx$$

$$= \frac{1}{s-1} - \sum_{n=1}^{\infty} \int_{x=n}^{n+1} n\, x^{-s-1}\, dx$$

$$= \frac{1}{s-1} - \sum_{n=1}^{\infty} \left[\frac{n\, x^{-s}}{-s} \right]_n^{n+1}$$

$$\cancel{= \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\frac{n\, x^{-(n+1)}}{n+1} - \frac{n\, x^{-n}}{n} \right)}$$

$$= \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\frac{n\,(n+1)^{-s}}{s} - \frac{n \cdot n^{-s}}{s} \right)$$

$$= \frac{1}{s-1} + \frac{1}{s} \sum_{n=1}^{\infty} \left((n+1)^{1-s} - n^{1-s} - (n+1)^{-s} \right)$$

$$= \frac{1}{s-1} \cancel{+} - \frac{1}{s} \sum_{n=1}^{\infty} (n+1)^{-s} + \frac{1}{s} \sum_{n=1}^{\infty} \left((n+1)^{1-s} - n^{1-s} \right)$$

$$= \frac{1}{s-1} - \frac{1}{s}\left(\zeta(s) - 1\right) \cancel{+} - \frac{1}{s} = \frac{1}{s-1} - \frac{1}{s}\zeta(s)$$

$\square$

~~Prop~~

Def For $s \in \mathbb{C}$, $Re(s) > 0$, $s \neq 1$, set

$$\zeta(s) = \frac{s}{s-1} - s \int_{x=1}^{\infty} \{x\} x^{-s-1} dx$$

Thm $\zeta(s)$ is analytic on $\{s \in \mathbb{C} \mid Re(s) > 0, s \neq 1\}$

$\boxed{\text{SIMPLE POLE at } s=1}$

Point: $\int_{x=1}^{\infty} \{x\} x^{-s-1} dx$ abs. conv.

$$\left| \{x\} x^{-s-1} \right| \leq x^{-p-1}, \quad p = Re(s)$$

$-p-1 < -1$.

Riemann Hypothesis

~~Conjecture~~ Critical Strip: $\{s \in \mathbb{C} \mid 0 < Re(s) < 1\}$.

If $\zeta(s) = 0$ and s in crit. strip then $Re(s) = \frac{1}{2}$.

~~Thm~~ **Def** $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$, $Re(s) > 0$, $s \neq 1$.

Thm $0 < Re(s) < 1 \Rightarrow$

$$\xi(1-s) = \xi(s).$$

Cor $\zeta(s)$ extends to analytic function on $\mathbb{C} \setminus \{1\}$. pole at $z_0 = 1$.

Proof $\quad \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{-1+s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ 5

~~Def~~ For $\mathrm{Re}(s) < 1$ define

$$\zeta(s) = \pi^{\frac{2s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \, \Gamma\left(\frac{s}{2}\right)^{-1}$$

analytic everywhere except (?) $s = 0$.

$\zeta(1-s)$ has simple pole at $s=0$

$\Gamma\left(\frac{s}{2}\right)$ has simple pole at $s=0$

$\Gamma\left(\frac{s}{2}\right)$ has simple zero at $s=0$.

$\square$

Harmonic

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Harmonic $\Leftrightarrow$ $\Delta u = 0$

$u : D \to \mathbb{R}$ harmonic $\Leftrightarrow$ u locally real part of analytic fcn.

Harmonic conjugate: $v : D \to \mathbb{R}$ such that $f = u + iv$ analytic.

— unique up to constant.

$\to$ Page 5 from 4/17/2018.

Catchup + Review

Laurent series of $\cot(z)$ $= \dfrac{\cos(z)}{\sin(z)}$ on $\overset{*}{B}(0,\pi)$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots$$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

Note: $\cot(z) = -\cot(-z)$ — odd function.

$$\cot(z) = \sum_{k=0}^{\infty} a_k z^{2k-1} = a_0 z^{-1} + a_1 z + a_2 z^3 + \cdots$$

$$\sin(z)\cot(z) = \cos(z)$$

1: $a_0 = 1$

z^2: $a_1 - \frac{1}{6}a_0 = -\frac{1}{2} \Rightarrow a_1 = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$

z^4: $a_2 - \frac{1}{6}a_1 + \frac{1}{120}a_0 = \frac{1}{24} \Rightarrow a_2 = \frac{1}{24} + \frac{1}{6}a_1 - \frac{1}{120}a_0 = -\frac{1}{45}$

z^{2k}: $a_k - \frac{1}{6}a_{k-1} + \frac{1}{120}a_{k-2} - \cdots + \frac{(-1)^k}{(2k+1)!}a_0 = \frac{(-1)^k}{(2k)!}$

$$\boxed{a_k = \frac{(-1)^k}{(2k)!} - \sum_{j=1}^{k} \frac{(-1)^j}{(2j+1)!} a_{k-j}}$$

Example

$$a_3 = -\frac{1}{6!} + \frac{1}{3!}a_2 - \frac{1}{5!}a_1 + \frac{1}{7!}a_0 = -\frac{2}{945}$$

Note: $a_k \in \mathbb{Q} \quad \forall k.$

Claim: $a_k < 0$ for $k \geq 1$.

Relation to Riemann's Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1$$

Let $m \in \mathbb{N}$.

$$f_m(z) = \pi z^{-m} \cot(\pi z)$$

Pole of order $m+1$ at $z_0 = 0$

Simple pole at $z_0 = n$, $n \in \mathbb{Z}$, $n \neq 0$.

$$\mathrm{Res}(f_m(z); n) = \mathrm{Res}\left(\frac{\pi z^{-m}\cos(\pi z)}{\sin(\pi z)}; n\right) = \frac{\pi n^{-m}\cos(\pi n)}{\pi \sin'(\pi n)}$$

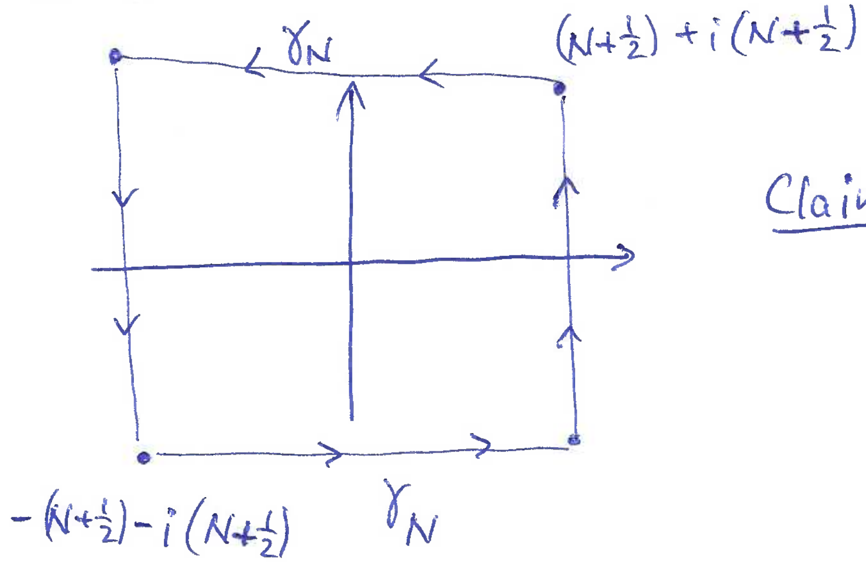
$$= \frac{1}{n^m}, \quad n \in \mathbb{Z},\ n \neq 0$$

$$\mathrm{Res}(f_m(z); 0) = \text{coef. of } z^{m-1} \text{ in } \pi\cot(\pi z) = \sum_{k=0}^{\infty} \pi a_k (\pi z)^{2k-1}$$

$$= \begin{cases} 0 & \text{if } m \text{ odd} \\ \pi^{2k} a_k & \text{if } m = 2k \text{ is even.} \end{cases}$$

$$\therefore \sum_{n \in \mathbb{Z}} \mathrm{Res}(f_m(z); n) = \begin{cases} -\zeta(m) + 0 + \zeta(m) = 0 & \text{if } m \text{ odd } \geq 3 \\ \pi^{2k} a_k + 2\zeta(2k) & \text{if } m = 2k \text{ even.} \end{cases}$$

Use Residue Thm



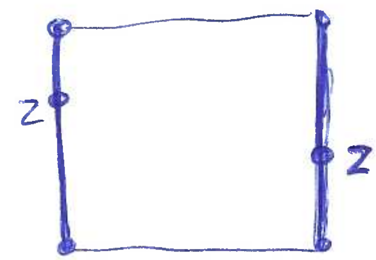
$N \in \mathbb{N}$ positive integer

Claim: $|\cot(\pi z)| < 2$ for all $z \in \gamma_N$.

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{\frac{1}{2}\left(e^{\pi i z} + e^{-\pi i z}\right)}{\frac{1}{2i}\left(e^{\pi i z} - e^{-\pi i z}\right)} = i\,\frac{1+e^{-2\pi i z}}{1-e^{-2\pi i z}} \qquad 3$$

$$|\cot(\pi z)| = \left|\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1}\right| = \left|\frac{1 + e^{-2\pi i z}}{1 - e^{-2\pi i z}}\right|$$

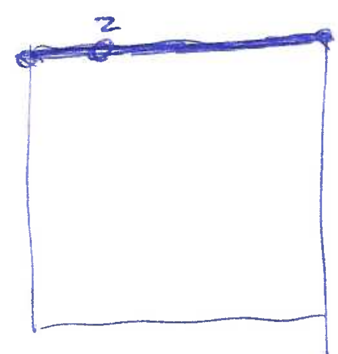
$$z = \pm\left(N+\tfrac{1}{2}\right) + iY$$



$$e^{-2\pi i z} = e^{i\pi}\, e^{2\pi Y} = -e^{2\pi Y}$$

$$|\cot(\pi z)| = \left|\frac{1 - e^{2\pi Y}}{1 + e^{2\pi Y}}\right| < 1$$

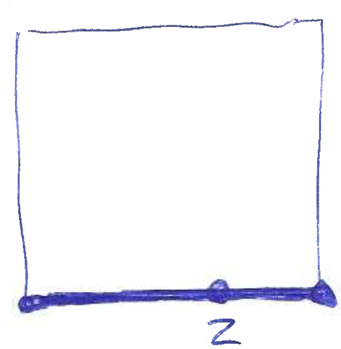
$$z = x + i\left(N+\tfrac{1}{2}\right)$$



$$\left|e^{2\pi i z}\right| = e^{-2\pi\left(N+\frac{1}{2}\right)} \leq e^{-3\pi}$$

$$|\cot(\pi z)| \leq \frac{1 + e^{-3\pi}}{1 - e^{-3\pi}} < 2$$

$$z = x - i\left(N+\tfrac{1}{2}\right)$$



$$\left|e^{-2\pi i z}\right| = e^{-2\pi\left(N+\frac{1}{2}\right)} \leq e^{-3\pi}$$

$$|\cot(\pi z)| \leq \frac{1 + e^{-3\pi}}{1 - e^{-3\pi}} < 2 .$$

Estimate: $|f_m(z)| \leq \pi \cdot N^{-m} \cdot 2$ for $z \in \gamma_N$.

$$\left| \int_{\gamma_N} f_m(z)\,dz \right| \leq \text{length}(\gamma_N) \cdot \frac{2\pi}{N^m} < 8N \cdot \frac{2\pi}{N^m} \longrightarrow 0 \quad \text{as } N \to \infty$$

$\boxed{\text{when } m \geq 2.}$

Residue Thm: $\displaystyle \frac{1}{2\pi i} \int_{\gamma_N} f_m(z)\,dz = \sum_{n=-N}^{N} \text{Res}(f_m; n)$

$$\boxed{\therefore \sum_{n=-\infty}^{\infty} \text{Res}(f_m; n) = 0 \qquad \text{when } m \geq 2}$$

$m = 2k$: $\quad \pi^{2k} a_k + 2\zeta(2k) = 0$
$k \geq 1$

$$\zeta(2k) = -\frac{\pi^{2k} a_k}{2}, \qquad k \in \mathbb{N},\ k \geq 1$$

$$a_k = -\frac{2\zeta(2k)}{\pi^{2k}} = -\frac{2}{\pi^{2k}} \sum_{n=1}^{\infty} n^{-2k} < 0$$

Cool formula: $\displaystyle \boxed{\cot(z) = z^{-1} - \sum_{k=1}^{\infty} \frac{2\zeta(2k)}{\pi^{2k}} z^{2k-1}}$