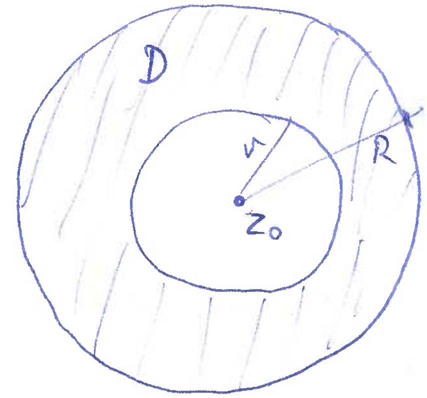
Laurent series

Annulus $0 \le r < |z-z_0| < R$:

$D = \{z \in \mathbb{C} \mid r < |z-z_0| < R\}$



Assume $f: D \longrightarrow \mathbb{C}$ analytic.

Goal: $f(z) = f_1(z) + f_2(z)$ where

$f_1 : B(z_0, R) \longrightarrow \mathbb{C}$ analytic

$f_2 : \{z \in \mathbb{C} \mid r < |z-z_0|\} \longrightarrow \mathbb{C}$ analytic.

And f_2 analytic at ∞ as well!

That is, $f_2(\frac{1}{z})$ has removable singularity at 0.

Def $a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z-z_0)^{n+1}} dz$, where $r < s < R$.

Note: a_n is well def, independent of choice of s.

Thm For $z \in D = \{z \in \mathbb{C} \mid r < |z-z_0| < R\}$ we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

This means: $f_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $B(z_0, R)$

$f_2(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0)^n$ converges on $\{z \in \mathbb{C} \mid |z-z_0| > r\}$

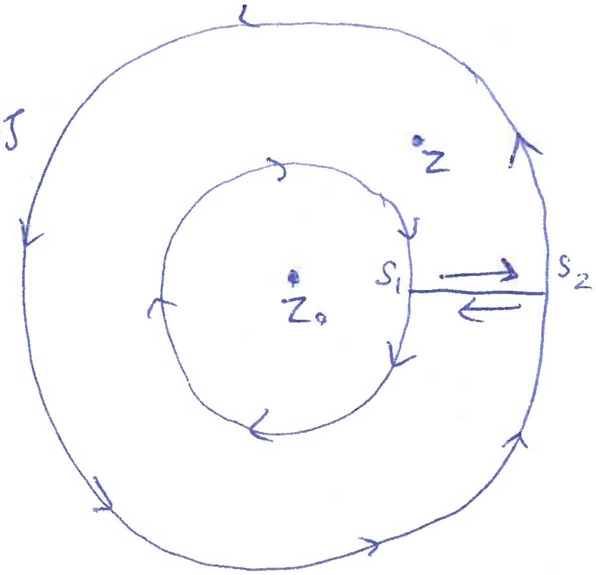
And $f(z) = f_1(z) + f_2(z)$ for $z \in D$.

Proof Let $z \in D$.

Choose s_1, s_2 : $0 \le r < s_1 < |z-z_0| < s_2 < R$

Cauchy's formula:

$$f(z) = \frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=s_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=s_1}\frac{f(\zeta)}{\zeta-z}\, d\zeta$$



First integral:

$$|\zeta - z_0| = s_2 > |z-z_0|$$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-z_0)-(z-z_0)}$$

$$= \frac{1}{\zeta-z_0}\,\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \frac{1}{\zeta-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \leftarrow \boxed{\left|\frac{z-z_0}{\zeta-z_0}\right| < 1}$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=s_2}\frac{f(\zeta)}{\zeta-z}\, d\zeta = \frac{1}{2\pi i}\int \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\, d\zeta = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Second integral:

$$|\zeta-z_0| = s_1 < |z-z_0|$$

$$\frac{1}{\zeta-z} = \frac{-1}{-(\zeta-z_0)+(z-z_0)} = \frac{1}{z-z_0}\,\frac{-1}{1-\frac{\zeta-z_0}{z-z_0}} = \frac{-1}{z-z_0}\sum_{k=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^k$$

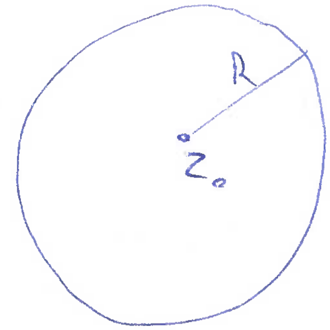
$$-\frac{1}{2\pi i}\int\limits_{|\zeta-z_0|=s_1}\frac{f(\zeta)}{\zeta-z}\, dz = \frac{1}{2\pi i}\int\sum_{k=0}^{\infty} f(\zeta)\cdot(\zeta-z_0)^k\cdot(z-z_0)^{-k-1}\, d\zeta$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta\right)(z-z_0)^n = \sum_{-\infty}^{-1} a_n (z-z_0)^n$$

$\boxed{n=-k-1 \Rightarrow -(n+1)=k}$ □

Special case

Assume f analytic on $B^*(z_0, R)$,
pole at z_0 of order m.



$$f(z) = \frac{H(z)}{(z-z_0)^m}, \qquad H : B(z_0,R) \longrightarrow \mathbb{C} \text{ analytic}$$

$$H(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad \text{valid on } B(z_0,R).$$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^{n-m} = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$$

$$a_n = c_{n+m}.$$

Example

$$f(z) = \frac{\exp(z)}{z^3}, \qquad z_0 = 0.$$

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$f(z) = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}, \qquad 0 < |z| < \infty.$$

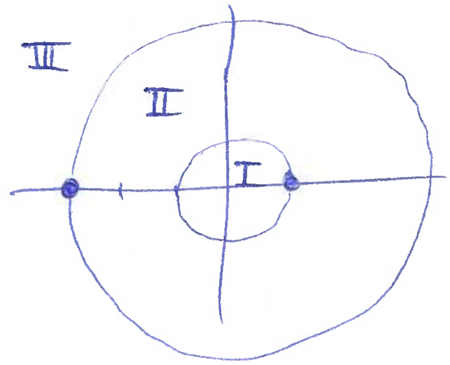
Example

$f(z) = \exp(1/z)$. essential sing at $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

Example $f(z) = \dfrac{3z+5}{(z-1)(z+3)}$ analytic on $\mathbb{C} - \{1, -3\}$



Partial fractions:

$$f(z) = \frac{2}{z-1} + \frac{1}{z+3}$$

I: Expansion on $|z| < 1$:

$$f(z) = -2\,\frac{1}{1-z} + \frac{1}{3}\,\frac{1}{1+\frac{z}{3}}$$

$$= -2\sum_{n=0}^{\infty} z^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(-2 - (-3)^{-n-1}\right) z^n$$

II: Expansion on $1 < |z| < 3$:

$$\frac{2}{z-1} = \frac{2}{z}\,\frac{1}{1-z^{-1}} = \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$f(z) = \sum_{n=-\infty}^{-1} \frac{1}{2} z^n + \sum_{n=0}^{\infty}\left(-3^{-n-1}\right) z^n$$

III: Exp on $|z| > 3$:

$$\frac{1}{z+3} = \frac{1}{z}\,\frac{1}{1+\frac{3}{z}} = \frac{1}{z}\sum_{n\geq 0}^{\infty}\left(-\frac{3}{z}\right)^n$$

$$= \sum_{n=-\infty}^{-1} (-3)^{-1-n} z^n$$

$$f(z) = \sum_{-\infty}^{-1}\left(\frac{1}{2} - (-3)^{-n}\right) z^n$$

$$\cancel{f(z) = \sum_{n=-\infty}^{-1} \left(-3^{-n-1} - (3)\right)}$$

Principal part of analytic fcn at pole

f analytic on $D \setminus \{z_0\}$, pole at z_0, order m.

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$$

Principal part: $\displaystyle\sum_{n=-m}^{-1} a_n (z-z_0)^n$

$$= P\left(\frac{1}{z-z_0}\right), \quad P \text{ poly of deg } m$$

with const term zero.

<u>Thm</u> $D \subseteq \mathbb{C}$ open, $z_1, z_2, \ldots, z_n \in D$.

f analytic on $D - \{z_1, z_2, \ldots, z_n\}$,

with poles at $z_1, \ldots, z_n$.

Then $f(z) = P_1\left(\frac{1}{z-z_1}\right) + \cdots + P_n\left(\frac{1}{z-z_n}\right) + g(z)$

where g analytic on D, P_i poly.

<u>Cor</u> $f(z)$ rational function with only simple poles.

Then $\exists$ polynomial $P(z)$ such that

$$f(z) = P(z) + \sum_{z_j \text{ pole of } f} \operatorname{Res}(f; z_j) \frac{1}{z-z_j}$$

<u>Example</u>

$$f(z) = \frac{z^3 + 5z^2 + 6z - 4}{(z-1)(z+3)} = z+3 + \frac{3z+5}{(z-1)(z+3)}$$

$$= z+3 + \frac{2}{z-1} + \frac{1}{z+3}.$$

Principal part of analytic function at pole

$D \subseteq \mathbb{C}$ open, $z_0 \in D$, $f : D \setminus \{z_0\} \to \mathbb{C}$ analytic ~~pole at z_0 of order m~~

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad , \quad z \in B^*(z_0, R) \subseteq D.$$

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad \text{for } s>0 \text{ small.}$$

$$a_n = \operatorname{Res}\left(\frac{f(z)}{(z-z_0)^{n+1}} ; z_0 \right)$$

Assume $f(z)$ has pole of order m at z_0.

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n \quad , \quad z \in B^*(z_0,R). \qquad a_n = 0 \text{ for } n < -m.$$

Principal part of $f(z)$ at z_0:

$$P\left(\frac{1}{z-z_0}\right) = \sum_{n=-m}^{-1} a_n (z-z_0)^n = \sum_{k=1}^{m} \frac{a_{-k}}{(z-z_0)^k}$$

$$a_{-k} = \operatorname{Res}\left(\frac{f(z)}{(z-z_0)^{1-k}} ; z_0 \right)$$

Note: $g(z) = f(z) - P\left(\frac{1}{z-z_0}\right)$

analytic on D (removable sing at z_0).

Several poles

$D \subseteq \mathbb{C}$ open, $z_1, \ldots, z_n \in D$, $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ analytic

f has pole at z_j of order m_j

Principal part at z_j: $$P_j\left(\frac{1}{1-z_j}\right) = \sum_{k=1}^{m_j} \operatorname{Res}\left(\frac{f(z)}{(z-z_j)^{1-k}} ; z_j \right) \cdot (z-z_j)^{-k}$$

$g_n(z) = f(z) - P_n\left(\frac{1}{z-z_n}\right)$ analytic on $D \setminus \{z_1, \dots, z_{n-1}\}$

poles at $z_1, \dots, z_{n-1}$ of orders $m_1, \dots, m_{n-1}$.

$g(z) = f(z) - P_n\left(\frac{1}{z-z_n}\right) - \dots - P_1\left(\frac{1}{z-z_1}\right)$ analytic on D.

$f(z) = g(z) + P_1\left(\frac{1}{z-z_1}\right) + \dots + P_n\left(\frac{1}{z-z_n}\right)$.

Facts about poles:
$q(z) = c_n z^n + \dots + c_1 z + c_0$
$q(\alpha) = 0 \Rightarrow q(z) = (z-\alpha)h(z)$
h poly of deg. $n-1$.
$q(z)$ has $\leq n$ roots
If α root then α zero of order $\leq n$.
$\Rightarrow \frac{1}{q(z)}$ has pole of order $\leq n$.

Special Case: Assume $f(z)$ rational function

$$f(z) = \frac{p(z)}{q(z)}, \quad p, q \text{ polynomials.}$$

Let $z_1, \dots, z_n$ be zeros of $q(z)$.

Then $f(z)$ analytic on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$

Poles at $z_1, \dots, z_n$.

$$f(z) = g(z) + P_1\left(\frac{1}{z-z_1}\right) + \dots + P_n\left(\frac{1}{z-z_n}\right)$$

$g(z)$ analytic on $\mathbb{C}$ (entire)

Note: $g(z)$ rational function without poles

$\Rightarrow g(z)$ polynomial!

uses that every non-const poly has root in $\mathbb{C}$

$$f(z) = g(z) + P_1\left(\frac{1}{z-z_1}\right) + \dots + P_n\left(\frac{1}{z-z_n}\right)$$

partial fractions expansion of $f(z)$!

Example Assume $f(z)$ rational function with simple poles at $z_1, \dots, z_n$.

Then $P_j\left(\frac{1}{z-z_j}\right) = \frac{A_j}{z-z_j}, \quad A_j = \operatorname{Res}(f(z); z_j)$

$$f(z) = g(z) + \frac{A_1}{z-z_1} + \dots + \frac{A_n}{z-z_n}, \quad g(z) \text{ polynomial.}$$

Example $f(z) = \frac{z^3+5z^2+6z-4}{(z-1)(z+3)} = z+3 + \frac{3z+5}{(z-1)(z+3)} = z+3 + \frac{2}{z-1} + \frac{1}{z+3}$.

Residue Theorem (section 2.6)

Problem: Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x-1}{(x^2+1)(x^2+9)}\,dx$

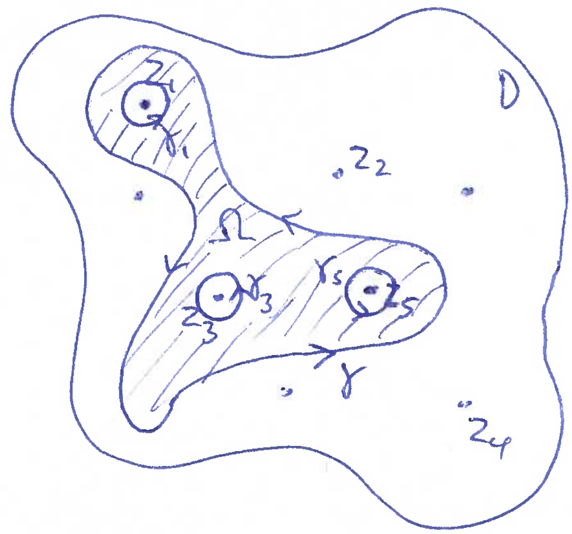
$D \subseteq \mathbb{C}$ simply connected, $z_1, \dots, z_N \in D$,
$f: D - \{z_1, \dots, z_N\} \longrightarrow \mathbb{C}$ analytic.

γ simple closed curve, positively oriented, $z_j \notin \gamma$ $\forall j$.

Then $$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \text{ inside } \gamma} \operatorname{Res}(f; z_j)$$

<u>Pf</u> Let γ_j be small circle around z_j.

Ω = points inside γ, outside γ_j $\forall j$



Green's Thm:

$$\int_{\Gamma = \partial\Omega} f(z)\,dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

$$= i \iint_\Omega 0\,dx\,dy = 0$$

$$0 = \int_{\partial\Omega} f(z)\,dz = \int_\gamma f(z)\,dz - \sum_{z_j \text{ inside } \gamma} \int_{\gamma_j} f(z)\,dz$$

$$\int_\gamma f(z)\,dz = \sum_{z_j \text{ inside } \gamma} \int_{\gamma_j} f(z)\,dz = \sum_{z_j \text{ inside } \gamma} 2\pi i \operatorname{Res}(f(z); z_j)$$

$\square$

Application

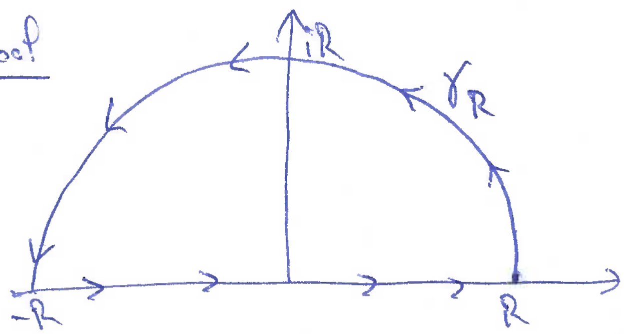
$$F(x) = \frac{P(x)}{Q(x)} \quad \text{real rational function.}$$

$P(x)$, $Q(x)$ polys with real coefs.

Assume $\deg(P) \leq \deg(Q) - 2$ and $Q(x) \neq 0 \ \forall x \in \mathbb{R}$.

Thm
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{\substack{z_j \in \text{upper} \\ \text{half plane}}} \operatorname{Res}\left(\frac{P(z)}{Q(z)}; z_j\right)$$

Proof



$$\int_{\gamma_R} \frac{P(z)}{Q(z)}\,dz = \text{RHS} \quad \text{for } R \text{ large}$$

$$\Rightarrow \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx = \text{RHS} - \int_{\substack{|z|=R \\ \operatorname{Im}(z) \geq 0}} \frac{P(z)}{Q(z)}\,dz$$

$Q(z) = b_n z^n + \dots + b_1 z + b_0$

$P(z) = a_m z^m + \dots + a_1 z + a_0.$ $\qquad m \leq n-2$

$|Q(z)| \geq \frac{1}{2}|b_n| \cdot |z|^n$ for $|z|$ large.

$|P(z)| \leq 2|a_m| |z|^{n-2}$ for $|z|$ large,

$\therefore \left|\frac{P(z)}{Q(z)}\right| \leq C \cdot |z|^{m-n}$ for $|z|$ large. $\qquad C = 4\frac{|a_m|}{|b_n|}$

$$\left|\int_{\substack{|z|=R \\ \operatorname{Im}(z) \geq 0}} \frac{P(z)}{Q(z)}\,dz\right| \leq R\pi \cdot C \cdot R^{m-n} \longrightarrow 0 \quad \text{as } R \to \infty.$$

$\square$

Problem

5

$$\int_{-\infty}^{\infty} \frac{x^2-x-1}{(x^2+1)(x^2+9)}\,dx$$

Poles: $\pm i$, $\pm 3i$

$P(z) = z^2 - z - 1$

$Q(z) = (z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i)$

$$\operatorname{Res}\left(\frac{P(z)}{Q(z)}; i\right) = \left.\frac{z^2-z-1}{(z+i)(z-3i)(z+3i)}\right|_{z=i} = \frac{i^2-i-1}{(2i)\cdot(-2i)\cdot(4i)}$$

$$= \frac{-i-2}{16i} = i\,\frac{i+2}{16} = \frac{-1+2i}{16}$$

$$\operatorname{Res}\left(\frac{P(z)}{Q(z)}; 3i\right) = \left.\frac{z^2-z-1}{(z^2+1)(z+3i)}\right|_{z=3i} = \frac{-9-3i-1}{(-8)\cdot(6i)}$$

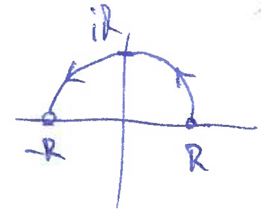
$$= \frac{-10-3i}{-48i} = i\,\frac{-10-3i}{48} = \frac{3-10i}{48}$$

$$\int_{-\infty}^{\infty} \frac{x^2-x-1}{(x^2+1)(x^2+9)}\,dx = 2\pi i\left(\frac{-1+2i}{16} + \frac{3-10i}{48}\right)$$

$$= \frac{\pi}{6}$$

Jordan's lemma

$$\left| \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} e^{iz}\,dz \right| < \pi$$



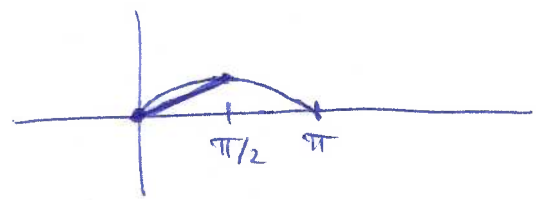
Pf

$\gamma(t) = R e^{it}, \quad 0 \le t \le \pi.$ $\qquad i\gamma(t) = iR\cos(t) - R\sin(t)$

$$\left| e^{i\gamma(t)} \right| = e^{-R\sin(t)} \le e^{-\frac{2Rt}{\pi}}$$

for $0 \le t \le \frac{\pi}{2}$.

Note: $\sin(t) \ge \frac{2t}{\pi}, \quad 0 \le t \le \frac{\pi}{2}$



$$\left| \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} e^{iz}\,dz \right| = \left| \int_0^\pi e^{i\gamma(t)} \gamma'(t)\,dt \right|$$

$$\le \int_0^\pi \left| e^{i\gamma(t)} \right| \cdot |\gamma'(t)|\,dt$$

$$\le \int_0^\pi e^{-R\sin(t)} \cdot R\,dt$$

$$= 2R \int_0^{\pi/2} e^{-R\sin(t)}\,dt$$

$$\le 2R \int_0^{\pi/2} e^{-\frac{2Rt}{\pi}}\,dt$$

$$= \left[-\pi e^{-\frac{2Rt}{\pi}} \right]_0^{\pi/2} = \pi\left(1 - e^{-R}\right) < \pi$$

$\square$

Complex Analysis 3/20/2018

2.6

Residue Thm $D \subseteq \mathbb{C}$ simply connected,
$f: D - \{z_1, \dots, z_N\} \longrightarrow \mathbb{C}$ analytic,
$\gamma \subseteq D$ pos. orient. simple closed curve, $z_j \notin \gamma$. $\forall j$.

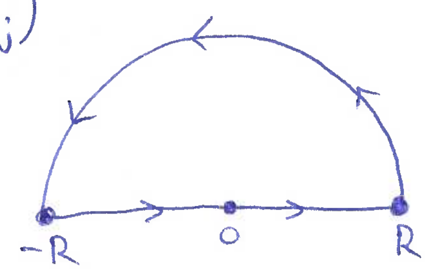
Then
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \text{ inside } \gamma} \operatorname{Res}(f; z_j)$$

Let f be analytic on $\mathbb{C}$ except finitely many isol. sings $\notin \mathbb{R}$.

Application

Assume $\int_{-\infty}^{\infty} f(x)\,dx$ exists (converges) AND $\int_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} f(z)\,dz \longrightarrow 0$ as $R \to \infty$.

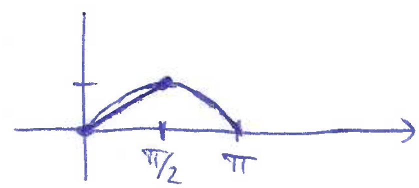
Then
$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{z_j \in \text{upper half plane}} \operatorname{Res}(f(z); z_j)$$



Jordan's Lemma

$$R\int_0^\pi e^{-R\sin(t)}\,dt < \pi$$

Proof Note: $\sin(t) \geq \frac{2t}{\pi}$ for $0 \leq t \leq \frac{\pi}{2}$



$$R\int_0^\pi e^{-R\sin(t)}\,dt = 2R\int_0^{\pi/2} e^{-R\sin(t)}\,dt$$
$$\leq 2R\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}}\,dt = \left[-\pi e^{-\frac{2Rt}{\pi}}\right]_0^{\pi/2} = \pi(1-e^{-R})$$

□

Cute Consequence:

$$\left|\int_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} e^{iz}\,dz\right| = \left|\int_0^\pi e^{i\gamma(t)}\gamma'(t)\,dt\right| \leq \int_0^\pi |e^{i\gamma(t)}|\cdot|\gamma'(t)|\,dt$$

$\boxed{\gamma(t) = Re^{it}}$

$$= \int_0^\pi e^{-R\sin(t)}\cdot R\,dt < \pi.$$

<u>Example</u> $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+\alpha^2}\,dx$, $\alpha > 0$ constant.

<u>Use</u> $f(z) = \dfrac{e^{iz}}{z^2+\alpha^2}$. <u>Note:</u> $f(x) = \dfrac{\cos(x)}{x^2+\alpha^2} + i\,\dfrac{\sin(x)}{x^2+\alpha^2}$

(why not $f(z) = \dfrac{\cos(z)}{z^2+\alpha^2}$?)

Singularities : $z = \pm i\alpha$.

$$\operatorname{Res}(f\,;\, i\alpha) = \frac{e^{i(i\alpha)}}{(i\alpha + i\alpha)} = \frac{e^{-\alpha}}{2i\alpha}$$

<u>Check:</u> $\gamma(t) = Re^{it}$

$$\left|\int_{\substack{|z|=R\\ \operatorname{Im}(z)>0}} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)^2+\alpha^2}\,\gamma'(t)\,dt\right|$$

$$\le \int_0^{\pi} \frac{e^{-R\sin(t)}}{R^2-\alpha^2}\cdot R\,dt \le \frac{\pi}{R^2-\alpha} \longrightarrow 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+\alpha^2}\,dx = 2\pi i\,\frac{e^{-\alpha}}{2i\alpha} = \frac{\pi e^{-\alpha}}{\alpha}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+\alpha^2}\,dx = \frac{\pi e^{-\alpha}}{\alpha} \quad \text{AND} \quad \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+\alpha^2}\,dx = 0.$$

Example $\int_{-\infty}^{\infty} \frac{x^3 \sin(x)}{x^4+4}\,dx$

$$f(z) = e^{iz}\frac{z^3}{z^4+4} = e^{iz}\frac{z^3}{(z^2-2i)(z^2+2i)}$$

Poles: $z^2 = \pm 2i$. $\quad z = (1+i)\,e^{k\frac{\pi}{2}i}$, $k \in \mathbb{Z}$

$$z = \pm 1 \pm i$$

$$\operatorname{Res}(f;\ 1+i) = e^{i(1+i)}\frac{(1+i)^3}{2(1+i)\left((1+i)^2+2i\right)} = \frac{e^{-1+i}}{4}$$

$$\operatorname{Res}(f;\ -1+i) = e^{i(-1+i)}\frac{(-1+i)^3}{\left((-1+i)^2-2i\right)\cdot 2(-1+i)} = \frac{e^{-1-i}}{4}$$

Check:

$$\left|\int_{\substack{|z|=R \\ \operatorname{Im}(z)\geq 0}} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{e^{i\gamma(t)}\gamma(t)^3}{\gamma(t)^4+4}\gamma'(t)\,dt\right|$$

$$\leq \int_0^{\pi} \frac{e^{-R\sin(t)}\cdot R^3}{R^4-4}\cdot R\,dt < \frac{\pi R^3}{R^4-4} \to 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{1}{4}e^{-1+i} + \frac{1}{4}e^{-1-i}\right)$$

$$= \frac{2\pi i}{4e}\left(e^{i}+e^{-i}\right) = \frac{\pi i\cos(1)}{e}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^3\sin(x)}{x^4+4}\,dx = \frac{\pi\cos(1)}{e} \quad \text{AND} \quad \int_{-\infty}^{\infty}\frac{x^3\cos(x)}{x^4+4}\,dx = 0$$

Example $\displaystyle\int_0^{2\pi} \frac{1}{2+(\cos t)^2}\,dt$

Idea: $\gamma(t) = e^{it}$ $\qquad z = e^{it}$

$\gamma'(t) = i\gamma(t)$ $\qquad dz = iz\,dt$

$$\cos(t) = \frac{1}{2}\left(\gamma(t) + \frac{1}{\gamma(t)}\right)$$

$$\frac{1}{2+\cos(t)^2}\,dt = \frac{1}{2+\frac{1}{4}\left(z+\frac{1}{z}\right)^2}\,\frac{1}{iz}\,dz$$

Try: $\displaystyle\int_{|z|=1} \frac{1}{iz\left(2+\frac{1}{4}(z^2+2+z^{-2})\right)}\,dz =$

$$\int_0^{2\pi} \frac{1}{\cancel{i\gamma(t)}\left(2+\frac{1}{4}(\gamma(t)^2+2+\gamma(t)^{-2})\right)}\,\cancel{\gamma'(t)}\,dt =$$

$$\int_0^{2\pi} \frac{1}{2+\left[\frac{1}{2}(\gamma(t)+\gamma(t)^{-1})\right]^2}\,dt =$$

$$\int_0^{2\pi} \frac{1}{2+\cos(t)^2}\,dt.$$

Compute: $\displaystyle\int_{|z|=1} \frac{1}{iz\left(2+\frac{1}{4}\left(z^2+2+\frac{1}{z^2}\right)\right)}\,dz$

$$\frac{1}{i}\int_{|z|=1} \frac{z}{2z^2+\frac{1}{4}z^4+\frac{1}{2}z^2+\frac{1}{4}}$$

$$= 2\pi \cdot \frac{1}{2\pi i} \int_{|z|=1} \frac{4z}{z^4 + 10z^2 + 1} \, dz$$

$$R(z) = \frac{4z}{z^4 + 10z^2 + 1}$$

Poles: $z^2 = -5 \pm 2\sqrt{6}$

Poles in unit circle: $z^2 = -5 + 2\sqrt{6} < 0.$

$$z_1 = i\sqrt{-5 + 2\sqrt{6}}$$

$$z_2 = -i\sqrt{-5 + 2\sqrt{6}}$$

$$\text{Res}(R; z_1) = \frac{1}{2\sqrt{6}}$$

$$\text{Res}(R; z_2) = \frac{1}{2\sqrt{6}}$$

$$\therefore \int_0^{2\pi} \frac{1}{2 + (\cos t)^2} \, dt = 2\pi \int_{|z|=1} \frac{1}{2\pi i} \frac{4z}{z^4 + 10z^2 + 1} \, dz$$

$$= 2\pi \sum_{\substack{z_j \text{ inside} \\ \text{unit} \\ \text{circle}}} \text{Res}(R; z_j) = 2\pi \left(2 \cdot \frac{1}{2\sqrt{6}} \right)$$

$$= \frac{2\pi}{\sqrt{6}}.$$

Example

One can show: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \pi.$

Note: $\dfrac{\sin(z)}{z}$ analytic on $\mathbb{C}$.

Use $f(z) = \dfrac{e^{iz}}{z}$

Check:

$$\Bigg| \int_{\substack{|z|=R \\ \mathrm{Im}>0}} f(z)\,dz \Bigg| = \Bigg| \int_0^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)}\,\gamma'(t)\,dt \Bigg| \qquad \gamma(t) = Re^{it}$$

$$\leq \int_0^{\pi} \frac{e^{-R\sin(t)}}{R}\cdot R\,dt$$

$$< \frac{\pi}{R} \longrightarrow 0$$

$$\therefore\ \cancel{i\int_{-\infty}^{\infty}\frac{\sin(x)}{x}\,dx}\quad \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{z_j \in \text{upper half plane}} \mathrm{Res}(f; z_j) = 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = 0$$

WHAT IS WRONG?

3.1 Zeros of analytic functions

Assume f analytic on $B^*(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z-z_0| < r\}$, without essential singularity at z_0.

$f(z) = (z-z_0)^m g(z)$, $m \in \mathbb{Z}$, g analytic on $B(z_0, r)$, $g(z_0) \neq 0$.

$m < 0$: f has pole of order $-m$ at z_0

$m \geq 0$: removable sing. at z_0.

$m = 0$: $f(z_0) \neq 0$

$m > 0$: zero of order m.

<u>Lemma</u> $\mathrm{Res}\left(\frac{f'}{f}; z_0\right) = m$

<u>Proof</u> $f'(z) = m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)$

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}, \quad \mathrm{Res}\left(\frac{f'}{f}; z_0\right) = \mathrm{Res}\left(\frac{m}{z-z_0}; z_0\right) = m.$$

□ EXAMPLE $f(z) = z^m \exp(\frac{1}{z})$

<u>Zeros of analytic function are isolated from each other.</u>

<u>THM</u> $D \subseteq \mathbb{C}$ open, (domain) $f : D \to \mathbb{C}$ analytic, NOT identically zero.

Assume $f(z_0) = 0$ for some $z_0 \in D$.

Then $\exists\, \varepsilon > 0$ such that $f(z) \neq 0$ $\forall z \in B^*(z_0, \varepsilon) \subseteq D$.

<u>Proof</u>

Otherwise $\exists$ sequence $\{z_k\} \subseteq D$ s.t. $z_k \to z_0$ as $k \to \infty$ and $f(z_k) = 0$ $\forall k$.

Write $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $z \in B(z_0, r) \subseteq D$.

$a_0 = f(z_0) = 0$.

Example

$f(z) = z^m \exp(\frac{1}{z})$ analytic on $\mathbb{C} - \{0\}$

essential sing. at $z_0 = 0$.

$$f'(z) = m z^{m-1} \exp(\tfrac{1}{z}) + z^m \exp(\tfrac{1}{z}) \frac{-1}{z^2}$$

$$\frac{f'(z)}{f(z)} = \frac{m}{z} - \frac{1}{z^2} \qquad \operatorname{Res}\left(\frac{f'}{f}; 0\right) = m. \quad \text{— can be anything!}$$

Assume $a_0 = a_1 = \cdots = a_{N-1} = 0$.

$$g(z) = \frac{f(z)}{(z-z_0)^N} = \sum_{n=N}^{\infty} a_n (z-z_0)^{n-N} \quad \text{analytic on } B(z_0, r)$$

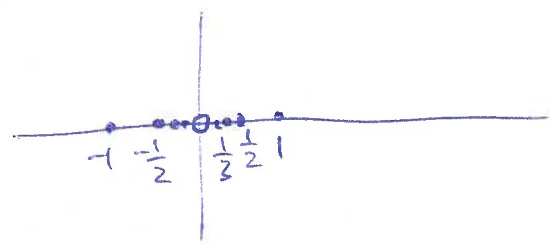
$$a_N = g(z_0) = \lim_{k\to\infty} g(z_k) = 0 \quad \text{since } f(z_k) = 0 \text{ and } g \text{ cont.}$$

$\therefore \; a_n = 0 \;\; \forall n, \quad f(z) = 0 \;\; \forall z$ ↯.

□

<u>Example</u> $f(z) = \sin\left(\frac{\pi}{z}\right)$ analytic on $D = \mathbb{C} \setminus \{0\}$.

$f\left(\frac{1}{n}\right) = \sin(n\pi) = 0$ and $\frac{1}{n} \longrightarrow z_0 = 0$. But $z_0 \notin D$.



Counting zeros and poles.

<u>Thm</u> $D \subseteq \mathbb{C}$ open,

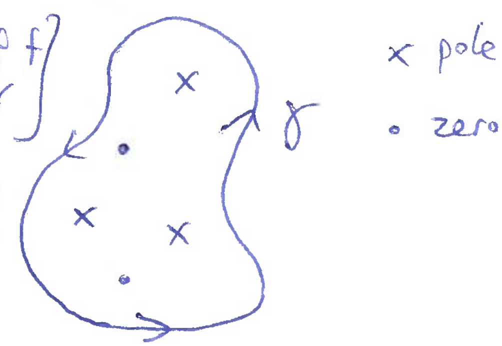
f analytic on D except finitely many poles.

$\gamma \subseteq D$ pos orient simple closed curve

inside$(\gamma) \subseteq D$, γ contains no zeros/poles of f.

Then $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \#\begin{bmatrix}\text{zeros of } f \\ \text{inside } \gamma\end{bmatrix} - \#\begin{bmatrix}\text{poles of } f \\ \text{inside } \gamma\end{bmatrix}$

COUNTED WITH MULTIPLICITY!!



<u>Proof</u>

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz \underset{\text{Residue Thm}}{=} \sum_{\substack{z_j \text{ pole of} \\ f'/f \text{ inside } \gamma}} \text{Res}\left(\frac{f'}{f}; z_j\right)$$

□

Argument Principle

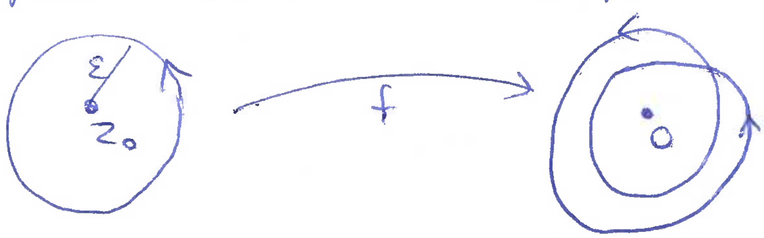
Note: $\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)}$

$$\Rightarrow \int_\gamma \frac{f'(z)}{f(z)}dz = \log(f(\text{end point})) - \log(f(\text{start point}))$$
$$= 0 \qquad ???$$

Problem

$\log(z)$ can only be defined on $\mathbb{C}$ − ray from 0.

If f has zero or pole at z_0, then $\gamma(t) = f(z_0 + \varepsilon e^{it})$ loops around z_0.

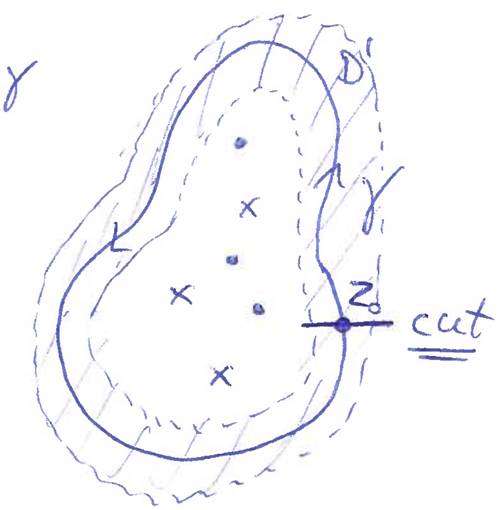


goes m times around, $f(z) = (z-z_0)^m g(z)$, $g(z_0) \neq 0$

$\therefore$ $\log(f(z))$ not defined!

Fix: $f(z)$ has no zeros/poles on γ.

Let $D' \subseteq D$ be "thin" open nbhd. of γ s.t. $f(z)$ has no zeros/poles on D'.



Let $D'' = D -$ "cut".

Now: D'' is simply-connected!

$\exists$ analytic function $h : D'' \to \mathbb{C}$ such that $\exp(h(z)) = f(z)$, $z \in D''$.

Recall: $h(z) = \int_{\text{fixed pt}}^{z} \frac{f'(z)}{f(z)}dz$

So we have: $h'(z) = \frac{f'(z)}{f(z)}$

$\{z_0\} = \gamma \cap \text{cut}.$

Note: $h(z) = \log(f(z))$
$= \ln|f(z)| + i\arg(f(z))$

$$\int_{\substack{\gamma \\ z_0 \to z_0}} \frac{f'(z)}{f(z)}\,dz = \int_{\substack{\gamma \\ z_0 \to z_0}} h'(z)\,dz = \underset{\substack{\uparrow \\ \text{end point}}}{h(z_0)} \cancel{\text{end}} - \underset{\substack{\uparrow \\ \text{start point}}}{h(z_0)}$$

$$= i\arg(f(\underset{\substack{\uparrow \\ \text{end}}}{z_0})) - i\arg(\underset{\substack{\uparrow \\ \text{start.}}}{f(z_0)})$$

Thm

$$\frac{1}{2\pi}\begin{bmatrix}\text{change in } \arg(f(z)) \\ \text{as } z \text{ traverses } \gamma\end{bmatrix} = \#\begin{bmatrix}\text{zeros of } f \\ \text{inside } \gamma\end{bmatrix} - \#\begin{bmatrix}\text{poles of } f \\ \text{inside } \gamma\end{bmatrix}$$

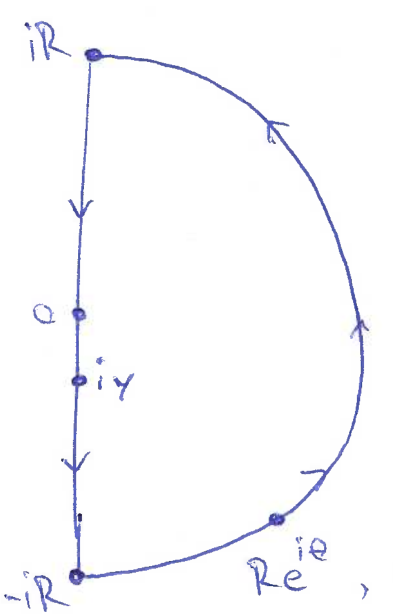
WITH MULT!

Example Let $\lambda \in \mathbb{R}$, $\lambda > 1$.

Then $\exists!\, z \in \mathbb{C}$ s.t. $\mathrm{Re}(z) > 0$ and $z + e^{-z} = \lambda$.

Pf. $f(z) = z + e^{-z} - \lambda$ has exactly one zero with $\mathrm{Re}(z) > 0$.

Count zeros: $R > 0$ large

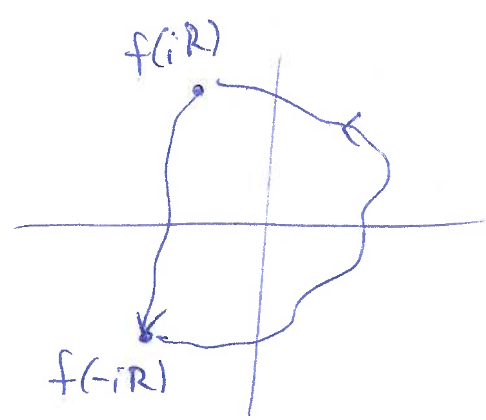


$f(iy) = (\cos(y) - \lambda) + i(y - \sin(y))$

$\mathrm{Re}(f(iy)) < 0$ in left half plane.

$f(iR) \in$ 2nd quadrant

$f(-iR) \in$ 3rd quadrant.



$z = Re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$\mathrm{Re}(z) = R\cos(\theta) > 0$.

$|e^{-z}| < 1$.

$$f(z) = \underbrace{z}_{\text{big}} + \underbrace{e^{-z} - \lambda}_{\text{not so big}} = Re^{i\theta}(1 + c(\theta)), \quad c(\theta) \text{ small.}$$

$\arg(f(z)) \approx \theta$ moves from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\therefore$ total increase in $\arg(f(z))$ is 2π.

Rouché's Thm

f and g analytic on open set $D \subseteq \mathbb{C}$.
$\gamma \subseteq D$ simple closed curve, inside$(\gamma) \subseteq D$.

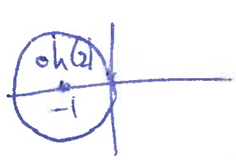
Assume $|f(z) + g(z)| < |f(z)| \quad \forall z \in \gamma$.

Then # zeros of f inside γ = # zeros of g inside γ. $\boxed{\text{WITH MULT}}$

Note: $f(z) \neq 0$ and $g(z) \neq 0$ for $z \in \gamma$.

WLOG: f and g have no common zeros inside γ.
(Replace with $\frac{f(z)}{z - z_0}$, $\frac{g(z)}{z - z_0}$.)

$h(z) = \frac{g(z)}{f(z)}$. $\quad |h(z) + 1| < 1$ for $z \in \gamma$.



total change of $\arg(h(z))$ as z traverses γ: 0

zeros of h = # poles of h.

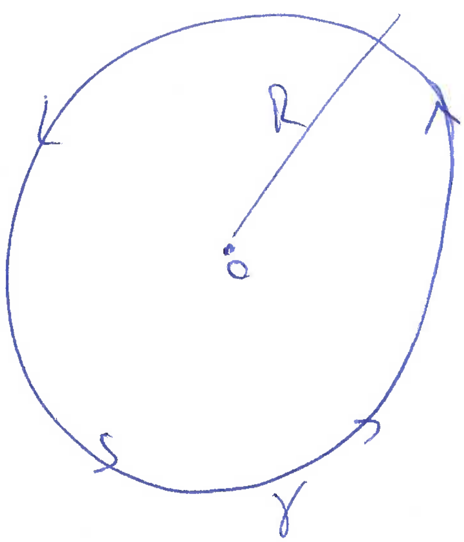
Fundamental Thm of Algebra

A polynomial of deg. n has exactly n zeros.

Proof

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Count zeros:



For $|z| = R$:

$$\left| \frac{f(z) - z^n}{z^n} \right| = \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| < 1.$$

$\therefore \quad |f(z) - z^n| < |z^n| \quad$ for $z \in \gamma$.

$\Rightarrow$ # zeros of f = # zeros of z^n

$= n.$

$\square$

3.2 Max Modulus & Mean Value.

$D \subseteq \mathbb{C}$ open, $f: D \to \mathbb{C}$ analytic, non-constant.

Consider equation $f(z) = w_0$, $w_0 \in \mathbb{C}$.

Solution $z_0 \in D$:

$f(z-z_0) - w_0$ has zero at z_0

$f(z-z_0) - w_0 = (z-z_0)^m g(z)$, g analytic on D, $m \geq 1$.

z_0 is solution to $f(z) = w_0$ of order m.

Simple solution: order $m = 1$.

Note z_0 simple solution $\Leftrightarrow$ $f'(z_0) \neq 0$ and $f(z_0) = w_0$

Thm $f: D \to \mathbb{C}$ non-constant analytic function.

Then $f(D) \subseteq \mathbb{C}$ is open.

Proof

Let $w_0 \in f(D)$.

Show: $B(w_0, \delta) \subseteq f(D)$ for some $\delta > 0$.

Choose $z_0 \in D$ such that $f(z_0) = w_0$.

$f(z) - w_0 = (z-z_0)^m g(z)$, $g(z)$ analytic on D, $g(z_0) \neq 0$.
$m \geq 1$

Choose $r > 0$ such that

$\overline{B(z_0, r)} \subseteq D$ and $g(z) \neq 0$ for all $z \in \overline{B(z_0, r)}$.

Set $\delta = \min_{|z-z_0|=r} |f(z) - w_0|$.

Recall: Any cont. fcn. on compact set attains its min / max.

Then $\delta > 0$.

Claim: $B(w_0, \delta) \subseteq f(D)$.

Let $w \in B(w_0, \delta)$.

For $|z - z_0| = r$ we have

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \delta \le |f(z) - w_0|$$

Rouché's Thm $\Rightarrow$ $f(z) - w$ has same # zeros as $f(z) - w_0$ in $B(z_0, r)$.

$\therefore$ $\exists z \in B(z_0, r)$: $f(z) = w$, so $w \in f(D)$.

□

<u>Thm</u> $f : D \longrightarrow \mathbb{C}$ analytic, $z_0 \in D$.
Assume $f(z) - f(z_0)$ has zero of order m at z_0.
Then f is m-to-1 near z_0.

<u>Means:</u> $\exists$ $0 < \varepsilon < r$ such that $B(z_0, r) \subseteq D$ and for each $z \in B^*(z_0, \varepsilon)$, the equation $f(\zeta) = f(z)$ has exactly m solutions (<u>simple</u>) ζ in $B(z_0, r)$.

<u>Proof</u> Repeat above proof.

Set $w_0 = f(z_0)$. $\qquad f(z) - w_0 = (z - z_0)^m g(z)$

Choose $r > 0$ s.t. $\qquad g(z) \neq 0$ for $z \in \overline{B(z_0, r)} \subseteq D$

<u>AND</u> $f'(z) \neq 0$ for $z \in B^*(z_0, r)$.

$$\delta = \min_{|z - z_0| = r} |f(z) - w_0| > 0$$

Choose $\varepsilon > 0$ s.t. $\varepsilon < r$ and $f(B(z_0, \varepsilon)) \subseteq B(w_0, \delta)$.

Let $z \in B^*(z_0, \varepsilon)$. Then $w = f(z) \in B(w_0, \delta)$. $\boxed{w \neq w_0}$

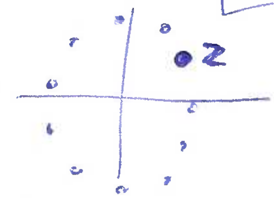
Rouché's Thm $\Rightarrow$ $f(\zeta) = w$ has exactly m solutions in $B(z_0, r)$ COUNTED WITH MULT!

But $f'(\zeta) \neq 0$ for $\zeta \in B^*(z_0, r)$, so all sols. have mult. 1.

□

<u>Example</u> $f(z) = z^m$ has zero of order m at $z_0 = 0$. ③

If $z \neq 0$ then $f(\zeta) = f(z)$ has exactly m solutions: $\boxed{\zeta = z e^{\frac{2\pi i}{m}}}$



More generally, $f(z) = z^m g(z)$, $g(0) \neq 0$.

Close to $z_0 = 0$, $g(z) \simeq g(z_0)$.

still m solutions to $f(\zeta) = f(z)$ for $z \neq 0$, z close to 0.
ζ close to 0.

<u>Thm</u> $D \subseteq \mathbb{C}$ open domain, $f: D \to \mathbb{C}$ non-const analytic.
Then $|f(z)|$ has no local max. on D.

<u>Proof</u>
Assume $|f(z)|$ attains local max at $z_0 \in D$. $M = |f(z_0)|$.

Choose $r > 0$ s.t. $M = \max$ of $|f(z)|$ on $B(z_0, r)$.
$f(B(z_0, r)) \subseteq \{ w \in \mathbb{C} \mid |w| \leq M \}$.
$f(z_0) \in \partial f(B(z_0, r)) = \emptyset$ ↯.
□

<u>Q:</u> Can $|f(z)|$ have local **min** on D ?

<u>Thm</u> $D \subseteq \mathbb{C}$ open domain, $f: D \to \mathbb{C}$ non-const. analytic.
Then $\operatorname{Re}(f(z))$ has no local max. or min. on D.

<u>Proof</u> $\exp(f(z))$ and $\exp(-f(z))$ are non-const analytic on D.
$$|\exp(f(z))| = e^{\operatorname{Re}(f(z))} \quad \text{and} \quad |\exp(-f(z))| = e^{-\operatorname{Re}(f(z))}$$
have no local max.
$\therefore \operatorname{Re}(f(z))$ has no local max or min on D.
□

Thm $D \subseteq \mathbb{C}$ bounded domain, $f: \overline{D} \to \mathbb{C}$ cont.,
f analytic on D.
Then $|f|$, $\mathrm{Re}(f)$, $-\mathrm{Re}(f)$ attain maxima on ∂D.

Proof The functions are cont. and $\overline{D}$ is compact.
And the maxima are not attained on D.
$\square$

Cor $D \subseteq \mathbb{C}$ domain, $f: D \to \mathbb{C}$ analytic.
$\gamma \subseteq D$ simple closed curve, inside$(\gamma) \subseteq D$.
$\mathrm{Re}(f)$ constant on γ $\Rightarrow$ f constant on D.

Pf Thm $\Rightarrow$ $\mathrm{Re}(f)$ constant inside γ.
So $f'(z) = 0$ inside γ, so f constant inside γ.
D connected $\Rightarrow$ f constant on D.
$\square$

Schwartz's Lemma
f analytic on $B(0,1)$, $f(0) = 0$, $|f(z)| \leq 1$ for $z \in B(0,1)$.
Then $|f(z)| \leq |z|$ for all $z \in B(0,1)$.
If $|f(z_0)| = |z_0|$ for some $z_0 \in B(0,1)$, then
$f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Proof $g(z) = \dfrac{f(z)}{z}$ is analytic on $B(0,1)$.
For $|z| = r$: $|g(z)| = \dfrac{|f(z)|}{r} \leq \dfrac{1}{r}$
Implies: $|g(z)| \leq 1$ for all $z \in B(0,1)$. So $|f(z)| \leq |z|$.
If $|f(z_0)| = |z_0|$ then $|g(z)|$ attains max in $B(0,1)$,
$\square$ So $g(z) = \lambda$ constant. $\therefore f(z) = \lambda z$.

Mean-Value Thm

$f : D \longrightarrow \mathbb{C}$ analytic, $\overline{B(z_0, r)} \subseteq D$.

Then $f(z_0) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(z_0 + r e^{it})\, dt$

~~Proof~~ $\cancel{\gamma(t) = z_0 + r e^{it}}$.

~~Cauchy's Formula:~~ $\cancel{f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz}$

$\cancel{f(z_0) = \int_\gamma}$

Proof $\gamma(t) = z_0 + r e^{it}$

Cauchy's formula:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0}\, \gamma'(t)\, dt$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r e^{it})}{r e^{it}}\, i r e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r e^{it})\, dt$$

$\square$