Note that no books, notes, or calculators may used during the exam.

These review problems all address material covered after the second midterm exam. Use the review sheets for the first two exams and the exams themselves to review earlier material. Of course, homework problems are also a good source of material for review.

1. Consider the initial/boundary value problem

$$\begin{gathered} 4u_{xx} = u_t + 8, \quad 0 < x < 3,\ t > 0; \\ u_x(0,t) = 4, \quad u(3,t) = 10, \quad t > 0; \\ u(x,0) = f(x), \quad 0 < x < 3. \end{gathered}$$

(a) Find the steady-state solution of the PDE with the given boundary conditions.

(b) Find the solution of the full problem; this will involve an infinite series, whose coefficients should be given as explicit integrals involving $f(x)$.

(c) Show that there is no steady-state solution as in (a) if the boundary condition at $x = 3$ is changed to $u_x(3,t) = 8$, and explain on physical grounds why this is so.

2. Consider the initial/boundary value problem

$$\begin{gathered} 9u_{xx} = u_{tt}, \quad 0 < x < 4,\ t > 0; \\ u(0,t) = 2, \quad u(4,t) = -10, \quad t > 0; \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < 4. \end{gathered}$$

(a) Find the steady-state solution of the PDE with the given boundary conditions.

(b) Find the solution of the full problem; this will involve an infinite series, whose coefficients should be given as explicit integrals involving $f(x)$ and $g(x)$.

3. Solve the following boundary value problem for the function $u(x,y,t)$:

$$9(u_{xx} + u_{yy}) = u_t, \qquad 0 < x < 2, \quad 0 < y < 1, \quad t > 0; \tag{PDE}$$

$$\left.\begin{aligned} u(0,y,t) = 0, \quad u_x(2,y,t) = 0, \qquad 0 < y < 1, \quad t > 0; \\ u_y(x,0,t) = 0, \quad u(x,1,t) = 0, \qquad 0 < x < 2, \quad t > 0; \end{aligned}\right. \tag{BC}$$

$$u(x,y,0) = f(x,y), \qquad 0 < x < 2, \quad 0 < y < 1. \tag{IC}$$

4. (a) Find the solution of Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ in the cube $0 \le x, y, z \le L$ if the four lateral sides and bottom ($x = 0$, $x = L$, $y = 0$, $y = L$, and $z = 0$) are held at zero temperature and the top is held at temperature 1.

(b) How would you treat the problem if all six sides of the cube were held at different temperatures? Hint: this is similar to the problem in a rectangle, discussed in Section 13.5 of Zill and Wright, and uses the superposition principle.

Short answers. I have tried to check these with reasonable care, but they are not guaranteed. If you get something different, send me an email and tell me what you get.

1. (a) $\psi(x) = x^2 + 4x - 11$; (b) $u(x,t) = \psi(x) + \sum_{n \text{ odd}} a_n \cos(n\pi x/6) \exp(-n^2\pi^2 t/9)$,
   with $a_n = \frac{2}{3}\int_0^3 (f(x) - \psi(x)) \cos(n\pi x/6)\, dx$.

2. (a) $\psi(x) = 2 - 3x$.

(b) $u(x,t) = \psi(x) + \sum_{n=1}^{\infty} \sin(n\pi x/4)(a_n \cos(3n\pi t/4) + b_n \sin(3n\pi t/4))$,
   $a_n = \frac{1}{2}\int_0^4 (f(x) - \psi(x)) \sin(n\pi x/4)\, dx, \qquad b_n = \frac{2}{3n\pi}\int_0^4 g(x) \sin(n\pi x/4)\, dx.$

3. $u(x,y,t) = \sum_{n,m \text{ odd}} C_{n,m} \sin(n\pi x/4) \cos(m\pi y/2) \exp(-(3\pi/4)^2 (n^2 + 4m^2)\, t)$,
   with $C_{n,m} = 2\int_0^2 \int_0^1 f(x,y) \sin(n\pi x/4) \cos(m\pi y/2)\, dy\, dx$.

4. (a) $$u(x,y,z) = \sum_{n,m \text{ odd}} \frac{16 \sin(n\pi x/L) \sin(m\pi y/L) \sinh(\pi\sqrt{n^2+m^2}\, z/L)}{nm\, \pi^2 \sinh(\pi\sqrt{n^2+m^2})}$$

(b) Split into six problems, each similar to (a); add the results.