ALGEBRAIC GEOMETRY PROBLEMS

Problem 1. Show that \( I(\mathbb{A}^n) = (0) \).

Problem 2. If \( I \subset R \) is any ideal, show that \( \sqrt{I} \) is a radical ideal.

Problem 3. (a) \( S \subset I(V(S)) \).
(b) \( W \subset V(I(W)) \).
(c) If \( W \) is an algebraic set then \( W = V(I(W)) \).
(d) If \( I \subset k[x_1, \ldots, x_n] \) is any ideal then \( V(I) = V(\sqrt{I}) \) and \( \sqrt{I} \subset I(V(I)) \).

Problem 4. (Hartshorne I.1.2 and I.1.11)
(a) Show that the set \( X = \{ (t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k \} \) is closed in \( \mathbb{A}^3 \) and find \( I(X) \).
(b) Same for the subset \( Y = \{ (t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k \} \) of \( \mathbb{A}^3 \).
(c) Show that \( I(Y) \) can’t be generated by less than three polynomials.

Problem 5. Let \( R \) be a commutative ring. The following are equivalent:
(a) \( R \) is Noetherian.
(b) Every ascending chain of ideals in \( R \) stabilizes.
(c) Every non-empty collection of ideals of \( R \) has a maximal element.

Problem 6. Show that \( W = \{ (x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3 \} \) is an irreducible closed subset of \( \mathbb{A}^3 \) and find \( I(W) \).

Problem 7. Find \( \sqrt{(y^2 + 2xy^2 + x^4, x^2 - x^3)} \).

Problem 8. Let \( X \) be a Noetherian topological space.
(a) If an irreducible closed set \( Y \) is contained in a union \( \bigcup X_i \) of finitely many closed sets \( X_i \), then \( Y \subset X_i \) for some \( i \).
(b) \( X \) has finitely many components.
(c) \( X \) is the union of its components.
(d) \( X \) is not the union of any proper subset of its components.

Problem 9. Let \( X \) be any space with functions and \( Y \subset \mathbb{A}^n \) an affine variety. Show that a function \( f : X \rightarrow Y \) is a morphism if and only if each coordinate function \( f_i : X \rightarrow k \) is regular for \( 1 \leq i \leq n \).

Problem 10. Let \( X = V(xy - zw) \subset \mathbb{A}^4 \) and \( U = V(y) \cup V(w) \subset X \). Define a regular function \( f : U \rightarrow k \) by \( f = x/w \) on \( V(w) \) and \( f = z/y \) on \( V(y) \).
Show that there are no polynomial functions \( p, q \in A(X) \) such that \( q(a) \neq 0 \) and \( f(a) = p(a)/q(a) \) for all \( a \in U \).

Problem 11. Let \( X \) be an affine variety such that the affine coordinate ring \( A(X) \) is a unique factorization domain. Let \( U \subset X \) be an open subset. Show that if \( f : U \rightarrow k \) is any regular function, then there exist \( p, q \in A(X) \) such that \( q(x) \neq 0 \) and \( f(x) = p(x)/q(x) \) for all \( x \in U \).
Problem 12. (a) \( k[\mathbb{A}^n \setminus \{0\}] = k[x_1, \ldots, x_n] \) for \( n \geq 2 \).
(b) \( \mathbb{A}^n \setminus \{0\} \) is not an affine variety for \( n \geq 2 \).
(c) Every global regular function on \( \mathbb{P}^n \) is constant, i.e. \( k[\mathbb{P}^n] = k \).
(d) \( \mathbb{P}^n \) is not quasi-affine for \( n \geq 1 \).

Problem 13. Let \( \varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2 \) be the morphism given by \( \varphi(t) = (t^2, t^3) \). Show that \( \varphi \) is bijective, but not an isomorphism.

Problem 14. Define the homogenization of a polynomial \( f \in k[x_1, \ldots, x_n] \) to be \( f^* = x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0) \). Equivalently, if we write \( f = f_0 + f_1 + \cdots + f_d \), with \( f_i \) a form of degree \( i \) and \( f_d \neq 0 \), then \( f^* = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + f_d \in k[x_0, x_1, \ldots, x_n] \).

Given any ideal \( I \subset k[x_1, \ldots, x_n] \), let \( I^* \subset k[x_0, x_1, \ldots, x_n] \) be the homogeneous ideal generated by \( \{f^* \mid f \in I\} \).
(a) Find an example where \( I = (h_1, \ldots, h_m) \) and \( I^* \neq \langle h_1^*, \ldots, h_m^* \rangle \).
(b) Let \( X \subset \mathbb{A}^n \) be a closed subvariety. Identify \( \mathbb{A}^n \) with \( D_+(x_0) \subset \mathbb{P}^n \) and let \( \overline{X} \) be the closure of \( X \) in \( \mathbb{P}^n \). Show that \( I(\overline{X}) = I(X)^* \subset k[x_0, \ldots, x_n] \).

Problem 15. Let \( X \subset \mathbb{P}^n \) be a projective variety with projective coordinate ring \( R = k[x_0, \ldots, x_n]/I(X) \). Let \( f \in R \) be a non-constant homogeneous element. Show that \( D_+(f) \subset X \) is an open affine subvariety with affine coordinate ring \( k[D_+(f)] = R(f) \).

Problem 16. Show that if \( R \) is a finitely generated reduced \( k \)-algebra then the space with functions \( \text{Spec-m}(R) \) is an affine variety.

Problem 17. Let \( X \) be any space with functions. A map \( \varphi : \mathbb{P}^n \to X \) is a morphism if and only if \( \varphi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X \) is a morphism.

Problem 18. Prove that the Segre map \( s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m} \) gives an isomorphism of \( \mathbb{P}^n \times \mathbb{P}^m \) with a closed subvariety of \( \mathbb{P}^{n+m} \), where \( N = nm + n + m \).

Problem 19. Let \( \varphi : X \to Y \) be a morphism of spaces with functions and suppose \( Y = \bigcup V_i \) is an open covering such that each restriction \( \varphi : \varphi^{-1}(V_i) \to V_i \) is an isomorphism. Then \( \varphi \) is an isomorphism.

Problem 20. Assume that the characteristics of \( k \) is not 2. If \( C = V_+(f) \subset \mathbb{P}^2 \) is any curve defined by an irreducible homogeneous polynomial \( f \in k[x, y, z] \) of degree 2, then \( C \cong \mathbb{P}^1 \).

Problem 21. Let \( X \) and \( Y \) be spaces with functions and let \((P, \pi_X, \pi_Y)\) and \((P', \pi'_X, \pi'_Y)\) be two products of \( X \) and \( Y \). Show that there is a unique isomorphism \( \varphi : P \isom P' \) such that \( \pi_X = \pi'_X \circ \varphi \) and \( \pi_Y = \pi'_Y \circ \varphi \).

Problem 22. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 23. Let \( X \) be a pre-variety such that for each pair of points \( x, y \in X \) there is an open affine subvariety \( U \subset X \) containing both \( x \) and \( y \).
(a) Show that \( X \) is separated.
(b) Show that \( \mathbb{P}^n \) has this property.
Problem 24. [Hartshorne II.2.16 and II.2.17]
Let $X$ be any pre-variety and $f \in k[X]$ a regular function.
(a) If $h$ is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of $X$ for some $n > 0$. [Hint: Let $X = U_1 \cup \cdots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to $U_i$ for each $i$.]
(b) $k[D(f)] = k[X]_f$.
(c) Let $R$ be a $k$-algebra and let $f_1, \ldots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \ldots, f_r) = R$. If $R_{f_i}$ is a finitely generated $k$-algebra for each $i$, then $R$ is a finitely generated $k$-algebra.
(d) Suppose $f_1, \ldots, f_r \in k[X]$ satisfy $(f_1, \ldots, f_r) = k[X]$ and $D(f_i)$ is affine for each $i$. Then $X$ is affine.

Problem 25. Let $E$ be the elliptic curve $V_+(y^2 z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$. (These are just projections to the $x$ and $y$ axis on the open subset $D_+(z)$.)
(a) Find the maximal open sets in $E$ where $f$ and $g$ are defined as morphisms.
(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by $f$ and $g$.
(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what “typical” means.)

Problem 26. Let $X$ be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that $\varphi$ is defined as a morphism on all of $\mathbb{P}^3$.

Problem 27. (a) If $X$ has components $X_1, \ldots, X_m$ then $\dim(X) = \max \dim(X_i)$.
(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 28. The commutative algebra result lying over states that, if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.
(a) Use lying over to show that if $\varphi : X \rightarrow Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of $Y$.
(b) If $\varphi : X \rightarrow Y$ is any morphism of varieties, then its image $\varphi(X)$ is constructible, i.e. a finite union of locally closed subsets of $Y$.

Problem 29. [Hartshorne I.5.2]
Assume $\text{char}(k) \neq 2$. Locate the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$, and $Z = V(xy + x^3 + y^3)$ in $\mathbb{A}^3$. (Take a look at the nice pictures in Hartshorne!)

Problem 30. Assume $\text{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \ldots, x_n]$.
(a) Show that $X_{\text{sing}} = V_+(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$.
(b) Show that $X_{\text{sing}} \neq X$.

Problem 31. [Shafarevich II.1.13]
(a) Show that an intersection of $r$ hypersurfaces in $\mathbb{P}^r$ is never empty.
(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that $X$ contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that $X$ is singular. [Hint: Choose the coordinates on $\mathbb{P}^n$ such that $L = V_+(x_{r+1}, x_{r+2}, \ldots, x_n) \subset \mathbb{P}^n$.]
Problem 32. [Shafarevich II.1.10].
Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If $X$ has two different singular points, then $X$ contains the line joining them.

Problem 33. If $X$ is a variety and $x \in X$, we define the Zariski cotangent space to $X$ at $x$ to be $\mathfrak{m}_x/\mathfrak{m}_x^2$. The Zariski tangent space is the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Show that if $f : X \to Y$ is a morphism of varieties with $f(x) = y$, then $f$ induces linear maps $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ and $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$.

Problem 34. [Mostly Hartshorne I.6.3]
Give examples of varieties $X$ and $Y$, a point $P \in X$, and a morphism $\varphi : X \setminus \{P\} \to Y$ such that $\varphi$ can’t be extended to a morphism on all of $X$ in each of the cases:
(a) $X$ is a non-singular curve and $Y$ is not projective.
(b) $X$ is a curve, $P$ is a singular point on $X$, $Y$ is projective.
(c) $X$ is non-singular of dimension at least two, $Y$ is projective.

Problem 35. Let $X$ and $Y$ be curves and $\varphi : X \to Y$ a birational morphism.
(a) $X_{\text{sing}}$ is a proper closed subset of $X$.
(b) $\varphi(X_{\text{sing}}) \subset Y_{\text{sing}}$.
(c) If $y \in Y$ is a non-singular point, then $\varphi^{-1}(y)$ contains at most one point.

Problem 36. Two non-singular projective curves are isomorphic if and only if they have the same function field.

Problem 37. Resolution of singularities for curves.
Let $X$ be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a “proper” morphism $\varphi$.)

Problem 38. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \setminus \{P\}$ is affine.

Problem 39. [Hartshorne I.6.2]
Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, char$(k) \neq 2$.
(a) $E$ is a non-singular curve.
(b) The units in $k[E]$ are the non-zero elements of $k$. [Hints: Define an automorphism $\sigma : k[E] \to k[E]$ fixing $x$ and sending $y$ to $-y$. Then define a norm $N : k[E] \to k[x]$ by $N(a) = a \sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$,]
(c) $k[E]$ is not a unique factorization domain.
(d) Show that $E$ is not rational.

Problem 40. Let $m_0, m_1, \ldots, m_N \in k[x_0, \ldots, x_n]$ be all the monomials of degree $d$. The Veronese embedding is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by
$$v_d(x_0 : \cdots : x_n) = (m_0(x_0, \ldots, x_n) : \cdots : m_N(x_0, \ldots, x_n)).$$
(a) Show that $v_d$ is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.
(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree $d$, i.e. $S = V_{\mathbb{P}}(f)$ where $f \in k[x_0, \ldots, x_n]$ is an irreducible form of degree $d$. Show that $S = v_d^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$. 

Problem 41. Let $L_1$, $L_2$, and $L_3$ be lines in $\mathbb{P}^3$ such that none of them meet.
(a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing $L_1$, $L_2$, and $L_3$. [Hint: Start by applying an automorphism of $\mathbb{P}^3$ to make the lines nice.]
(b) $S$ is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting $L_1$, $L_2$, and $L_3$.
(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet $L_1$, $L_2$, or $L_3$. Then the number of lines meeting $L_1$, $L_2$, $L_3$, and $L_4$ is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.

Problem 42. An algebraic group is a pre-variety $G$ together with morphisms $m : G \times G \to G$ and $i : G \to G$, and an identity element $e \in G$, such that $G$ is a group in the usual sense when $m$ is used to define multiplication and $i$ maps any element to its inverse element.
(a) Show that $\text{GL}_n(k)$ is an algebraic group.
(b) Show that any algebraic group is separated.
(c) Show that $\mathbb{P}^1$ is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $i : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the group axioms.
(d) Challenge: How about $\mathbb{P}^n$ for $n \geq 2$?

Problem 43. Let $G$ be an irreducible algebraic group acting on a variety $X$, i.e. we have a morphism $G \times X \to X$ such that the axioms for a group action are satisfied.
(a) Show that each orbit in $X$ is locally closed.
(b) Each orbit is a non-singular variety.

Problem 44. Let $\text{GL}_n(k)$ act on $\text{Gr}(d,n)$ by $g.V = \{g(x) \mid x \in V\}$. Show that for any points $V_1, V_2 \in \text{Gr}(d,n)$ there exists an element $g \in \text{GL}_n(k)$ such that $g.V_1$ and $g.V_2$ are both in $U_{1,\ldots,d} \subset \text{Gr}(d,n)$. Conclude that $\text{Gr}(d,n)$ is separated.

Problem 45. (a) Let $0 < p < q < n$ be integers and $E = k^n$. Show that the set \{(V,W) \in \text{Gr}(p,E) \times \text{Gr}(q,E) \mid V \subset W\} is closed in $\text{Gr}(p,E) \times \text{Gr}(q,E)$.
(b) Let $0 < d_1 < d_2 < \cdots < d_m < n$ be integers and let $\text{Fl}(d_1,\ldots,d_m;E)$ be the set of flags of subspaces $V_1 \subset V_2 \subset \cdots \subset V_m \subset E$ such that $\dim V_i = d_i$. Give this set a structure of projective variety.

Problem 46. Set $E = k^n$, $X = \text{Gr}(d,E)$, and let $F_1 \subset F_2 \subset \cdots \subset F_n = E$ be a flag of subspaces such that $\dim F_i = i$. Given a sequence of integers $a = (0 < a_1 < a_2 < \cdots < a_d \leq n)$, let $\Omega_a(F_i)$ be the set of all $V \in X$ such that $\dim(V \cap F_p) = i$ whenever $a_i \leq p < a_{i+1}$, $0 \leq i \leq d$. (We set $a_0 = 0$ and $a_{d+1} = n + 1$.)
(a) Show that $\Omega_a(F_i) \cong \mathbb{A}^m$, where $m = \sum a_i - (d+1)$.
(b) Show that the orbits for the action of the upper triangular matrices on $X$ are the sets $\Omega_a(F_i)$ for all sequences $a$ where $F_i = \text{span}\{e_1,\ldots,e_i\}$.
(c) The Schubert varieties in $X$ are the closures $\Omega_a(F_i) = \overline{\Omega_a(F_i)}$. Find a singular Schubert variety in some Grassmannian.

Problem 47. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix
\[
\begin{pmatrix}
x_0 & x_1 & x_2 \\
x_3 & x_4 & x_5
\end{pmatrix}
\] has rank one. Show that $X$ is a non-singular rational closed subvariety of $\mathbb{P}^5$, and find its dimension and degree.
Problem 48. [Mostly Hartshorne I.7.1] In this problem, just find the numbers and give an argument why they are correct that could be expanded into a proof.

(a) Find the degree of $v_3(\mathbb{P}^n)$ in $\mathbb{P}^N$ where $v_3$ is the Veronese embedding.
(b) Find the degree of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$.
(c) Challenge: Find the degree of $\text{Gr}(2, 5)$ in $\mathbb{P}^9$.

Problem 49. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X, P} \subset \mathcal{O}_{\mathbb{P}^2, P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2, P}$ such that $f|_{U \cap X} = 0$ for some open set $U$ containing $P$. The multiplicity $\mu_P(X)$ of $X$ at $P$ is the largest number $r$ such that $I_{X, P} \subset m_P^r$ where $m_P \subset \mathcal{O}_{\mathbb{P}^2, P}$ is the maximal ideal.

(a) $P \in X \iff \mu_P(X) \geq 1$.
(b) $P$ is a non-singular point of $X$ iff $\mu_P(X) = 1$.
(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2, P}/(I_{X, P} + I_{Y, P})$.
(d) $I(X \cdot Y; P) = 1$ iff $P$ is a non-singular point of both $X$ and $Y$, and the tangent directions at $P$ are different.
(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.
(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through $P$ we have $\mu_P(X) = I(X \cdot L; P)$.

Problem 50. Let $\mathcal{F}$ be a sheaf on $X$ and $p \in X$ a point. Prove the following from the definition of the stalk $\mathcal{F}_p$:

(a) Each element of $\mathcal{F}_p$ has the form $s_p$ for some section $s \in \mathcal{F}(U)$, $p \in U$.
(b) Let $s \in \mathcal{F}(U)$, $p \in U$. Then $s_p = 0 \iff s|_V = 0$ for some $p \in V \subset U$.
(c) Let $s \in \mathcal{F}(U)$. Prove that $s = 0$ if and only if $s_p = 0$ for all $p \in U$.

Problem 51. [Hartshorne II.1.2]

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X$. Show that $\varphi$ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $U = \bigcup V_i$ of $U$ and sections $t_i \in \mathcal{F}(V_i)$ such that $\varphi_{V_i}(t_i) = s|_{V_i}$ for all $i$.

Problem 52. [Hartshorne II.1.14]

Let $\mathcal{F}$ be a sheaf on $X$ and $s \in \mathcal{F}(X)$ a global section. Show that the set $\{p \in X \mid s_p \neq 0\}$ is a closed subset of $X$.

Problem 53. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $X$. Show that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\text{Im}(\varphi)_p = \text{Im}(\varphi_p)$ for all $p \in X$.

Problem 54. Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ a sheaf on $Y$. Show that $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$.

Problem 55. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on $X$, and $\mathcal{G}$ a sheaf on $Y$. Show that the map $\text{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \text{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ constructed in class is bijective.

Problem 56. (a) Let $X$ be an affine variety, $M$ a $k[X]$-module, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Show that $\text{Hom}_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong \text{Hom}_{\mathcal{O}_X}(M, \mathcal{F})$.
(b) If $X$ is affine and $M$ and $N$ are $k[X]$-modules then $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = (M \otimes_k N)^\sim$.
(c) If $f : X \to Y$ is a morphism of varieties and $\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_X$-module.
Problem 57. (a) $X$ is a ringed space, $F$ and $G$ are $\mathcal{O}_X$-modules. Then the assignment $U \mapsto \text{Hom}_{\mathcal{O}_X}(F|_U, G|_U)$ defines an $\mathcal{O}_X$-module. It is denoted $\text{Hom}_{\mathcal{O}_X}(F, G)$.

(b) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that $\mathcal{L}^{-1} = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 58. Let $X$ be a scheme of characteristic $p > 0$, $F : X \to X$ the Frobenius morphism, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Show that $F^* \mathcal{L} \cong L^p$.

Problem 59. A morphism $f : X \to Y$ of pre-varieties is called affine if, for every open affine subset $V \subset Y$, the inverse image $f^{-1}(V)$ is also affine. The morphism $f$ is called finite if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for every open affine $V \subset Y$.

(a) Let $Y = \bigcup V_i$ be an open affine covering of $Y$ such that $f^{-1}(V_i)$ is affine $\forall$ $i$. Show that $f$ is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all $i$ then $f$ is finite.

(b) If $f$ is affine and $Y$ is separated, then $X$ is separated.

Problem 60. (a) Let $X$ be a complete variety and $f : X \to Y = \text{Spec} \text{-} \text{m}(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety $X$ and a birational morphism $f : X \to Y$ such that $f_* \mathcal{O}_X$ is not locally free on $Y$.

Problem 61. (a) $Y \subset \mathbb{P}^n$ is a hypersurface of degree $d$ with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Show that $\mathcal{I}_Y \cong \mathcal{O}(-d)$.

(b) Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding, $N = \binom{n+d}{n} - 1$. Show that $(v_d)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.

Problem 62. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ be any non-constant morphism. Then $\dim \varphi(\mathbb{P}^n) = n$. Furthermore, $\varphi$ is the composition of a Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^{n-1}$, a projection $\mathbb{P}(k^N) \to \mathbb{P}(L) \to \mathbb{P}(k^N/L)$ for some linear subspace $L \subset k^N$, and an inclusion of a linear subspace $\mathbb{P}(k^N/L) \subset \mathbb{P}^n$.

Problem 63. (a) Let $\varphi : X \to Y$ be an affine morphism of pre-varieties. Show that if $Y$ is separated then so is $X$.

(b) $X$ is an irreducible affine variety, $U \subset X$ an open affine subset, $U \subset \bar{X}$ their normalizations, and $\pi : \bar{X} \to X$ the normalization map. Show that $\pi^{-1}(U) = U$.

(c) If $X$ is any irreducible variety then $\pi : \bar{X} \to X$ is a finite morphism. Conclude that $\bar{X}$ is separated.

Problem 64. (a) If $Y$ is a normal variety and $f : Y \to X$ a dominant morphism, then there exists a unique morphism $\bar{f} : Y \to X$ such that $f = \pi \circ \bar{f}$.

(b) Give a counter example to (a) when $f$ is not dominant.

Problem 65. $X = V(xy-z^2) \subset \mathbb{A}^3$ is normal. [Hint: $k[X] = k[x, xt, xt^2] \subset k(x, t)$ where $t = z/x$.]

Problem 66. If $X$ is any normal rational variety then $\text{Cl}(X)$ is a finitely generated Abelian group.

Problem 67. (a) Let $X \subset \mathbb{P}^2$ be a non-singular curve of degree 3 and $P \in X$ a point. Show that $\dim k \Gamma(X, \mathcal{L}(n[P])) \geq n$ for all $n$.

(b) Any proper open subset of $X$ is affine.
Problem 68. (a) Let $F, G, H \in k[x, y, z]$ be forms such that $V_0(G, H, z) = \emptyset$ in $\mathbb{P}^2$. Show that if $zF \in (G, H)$ then $F \in (G, H)$. [Hint: Use that $G_0 = G(x, y, 0)$ and $H_0 = H(x, y, 0)$ are relatively prime.]

(b) Let $C \subset \mathbb{P}^2$ be a curve, and set $O_C(n) = \mathcal{O}_{\mathbb{P}^2}(n)|_C$. Then $\Gamma(C, O_C(n)) = (k[x, y, z]/I(C))_n$ for all $n \geq 0$. [Hint: If $C = V_+(H) \subset D_+(y) \cup D_+(z)$ and if $\sigma$ is a global section of $O_C(n)$ then $\sigma/y^n = F(x, y, z)/y^m$ and $\sigma/z^n = A(x, y, z)/z^m$ for forms $F, A \in k[x, y, z]$ of degree $m \geq n$. Now use part (a).]

(c) Define the arithmetic genus of $C$ to be $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbb{P}^2$. Show that $p_a = \frac{(d-1)(d-2)}{2}$ where $d$ is the degree of $C$ and that $\dim_k \Gamma(C, O_C(n)) = nd + 1 - p_a$ for all large integers $n$.

Problem 69. (a) Let $C \subset \mathbb{P}^2$ be a non-singular curve and $Y \subset \mathbb{P}^2$ an irreducible curve different from $C$. Set $Y.C = \sum P I(Y.C; P) \in \text{Div}(C)$. Show that $\mathcal{L}(Y)|_C \cong \mathcal{L}(Y.C)$ on $C$.

(b) Let $L = V_+(f)$ and $M = V_+(g) \subset \mathbb{P}^2$ be lines (not equal to $C$) where $f, g \in k[x, y, z]$ are linear forms. Then the divisor of $f/g \in k(C)$ is $(f/g) = L.C - M.C$.

Problem 70. Let $E \subset \mathbb{P}^2$ be an elliptic curve and $P_0 \in E$ any point. Show that the map $E \rightarrow \mathcal{O}_E^\times(E)$ given by $P \mapsto P - P_0$ is bijective.

Problem 71. Let $D : S \rightarrow M$ be an $R$-derivation and $p(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ a polynomial. Then $D(p(a_1, \ldots, a_n)) = \sum_{i=1}^n \frac{\partial p}{\partial x_i}(a_1, \ldots, a_n) D(a_i)$ for all elements $a_1, \ldots, a_n \in S$.

Problem 72. Let $E = V_+(zy^2 - x^3 + z^2x) \subset \mathbb{P}^2$, $\text{char}(k) \neq 2$. Show that $\Omega_E \cong \mathcal{O}_E$. [Hint: Compute the divisor of the section $d(x/z)$ of $\Omega_E$.]

Problem 73. Let $\phi : X \rightarrow Y$ be a morphism of varieties and let $P \in X$ be a point such that $\phi^* : \mathcal{O}_Y(\phi(P)) \rightarrow \mathcal{O}_{X,P}$ is an isomorphism. Then there exists an open subset $U \subset Y$ such that $\phi(P) \in U$ and $\phi : \phi^{-1}(U) \cong U$ is an isomorphism.

Problem 74. Let $X$ be a variety and $V \subset X$ any subset. Then $V$ inherits a structure of space with functions from $X$. Assume that $V$ is a variety with this structure. Show that $V$ is locally closed in $X$.

Problem 75. Let $f : X \rightarrow Y$ be a rational map of algebraic varieties. Show that there exists a birational morphism $\pi : \hat{X} \rightarrow X$ such that the rational map $f \pi$ extends to a morphism of varieties $f \pi : \hat{X} \rightarrow Y$.

Problem 76. Set $E = V(y^2 - x^3 - 3) \subset \mathbb{C}^2$, $P = (1, 2) \in E$, and $U = E \setminus \{P\}$.

(a) Show that $U$ is an open affine subvariety of $E$.

(b) Challenge: $U$ is not of the form $D(f)$ for any regular function $f \in \mathcal{O}_E(E)$.

Problem 77. Set $E = k^{n+1}$ and recall that $\mathbb{P}^n = \{l \subset E \mid l$ is a line through the origin of $E \}$. Define $S = \{ (l, v) \in \mathbb{P}^n \times E \mid v \in l \}$, and let $\rho : S \rightarrow \mathbb{P}^n$ be the projection.

(a) $S$ is a subbundle of rank 1 of the trivial vector bundle $\mathbb{P}^n \times E$.

Define an $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{L}$ by $\Gamma(U, \mathcal{L}) = \{ \text{morphisms } s : U \rightarrow L \mid s \rho = 1_U \}$.

(b) $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.

Let $\pi : E \setminus \{0\} \rightarrow \mathbb{P}^n$ be the projection. For $d \in \mathbb{Z}$ we define an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^n}(d)$ by $\Gamma(U, \mathcal{O}(d)) = \{ s \in \mathcal{O}_E(\pi^{-1}(U)) \mid s(\lambda v) = \lambda^d s(v) \forall \lambda \in k, v \in E \}$.

(c) The sheaf $\mathcal{O}(d)$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank 1.
(d) Find an integer \( d \in \mathbb{Z} \) such that \( \mathcal{L} \cong \mathcal{O}(d) \) as an \( \mathcal{O}_P \)-module.