REVIEW OF VARIETIES

1. AFFINE VARIETIES

$k = \overline{k}$ alg closed field.
$R$ f.g. reduced $k$-algebra.
$\text{Spec-m}(R) = \{\text{max. ideals } m \subset R\}$

Topology: Zariski closed sets are $Z(I) = \{m \supset I\}$

Let $f \in R$. Def. $f : \text{Spec-m}(R) \to k$, $f(m) =$ image of $f$ by $R \to R/m = k$.
Def: Let $U \subset \text{Spec-m}(R)$ be open, $f : U \to k$ a function.
$f$ is **regular** if it is locally of the form $f(m) = p(m)/q(m)$, $p, q \in R$.
$\mathcal{O}(U) = \{\text{regular } f : U \to k\}$.

Exercise $\ast$: $\mathcal{O}(\text{Spec-m}(R)) = R$

Coordinate ring: $\mathcal{A}(\text{Spec-m}(R)) = R$ (only for affine varieties)
Example: $R = k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n]/I$. $(f_1, \ldots, f_n) : X \sim Z(I) \subset \mathbb{A}^n$

2. SPACES WITH FUNCTIONS

Def: A **space with functions** is a top space $X$ with assignment $U \mapsto O_X(U) = \{\text{all fcns } U \to k\} (k$-subalgebra) such that

1) $U = \bigcup \alpha U_\alpha : f \in O_X(U) \iff f|_{U_\alpha} \in O_X(U_\alpha) \forall \alpha$.
2) $f \in O_X(U) \Rightarrow D(f) \subset U$ open and $1/f \in O_X(D(f))$.

Def: A **morphism** of SWFs is a cont. map $\varphi : X \to Y$ such that pullback of regular functions are regular.
I.e. if $V \subset Y$ is open and $f \in O_Y(V)$, then $\varphi^*(f) = f \circ \varphi \in O_X(\varphi^{-1}(V))$.

3. SUBSPACE OF SWF

$X$ SWF, $Y \subset X$ any subset. Give $Y$ structure of SWF as follows:

* Subspace topology.

* If $U \subset Y$ is open, $f : U \to k$ function, then $f$ is regular iff $f$ can locally be extended to regular func on $X$.

I.e. $\forall y \in U \exists U' \subset X$ and $F \in O_X(U')$ s.t. $y \in U'$ and $f(x) = F(x) \forall x \in U \cap U'$.
Def: A **prevariety** is a SWF $X$ s.t. $\exists$ open cover $X = U_1 \cup \cdots \cup U_m$, with $U_i \cong \text{Spec-m}(R_i)$ affine variety for each $i$.

Exercise: Let $X = \text{Spec-m}(R)$ be affine and $f \in R$. Then $X_f := D(f) \cong \text{Spec-m}(R_f)$.

Exercise: $X$ SWF and $Y$ affine variety.

1-1 correspondence $\{\text{morphisms } X \to Y\} \leftrightarrow \{\text{k-alg homs } A(Y) \to \mathcal{O}(X)\}$.

Cor: Two affine varieties isomorphic iff coordinate rings isomorphic.

Exercise: $\mathbb{A}^n \setminus \{0\}$ is not affine for $n \geq 2$.

Exercise: An open subset of a prevariety is a prevariety.

Exercise: A closed subset of a prevariety is a prevariety.

Def: $X$ top space. A subset $W \subset X$ is **locally closed** if it is an intersection of an open set and a closed set.

Cor: A locally closed subset of a prevariety is a prevariety.
4. Projective space

Def: \( \mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^* \) = lines through origin in \( \mathbb{A}^{n+1} \).
\( \pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) projection.

Top: \( U \subset \mathbb{P}^n \) open \( \iff \pi^{-1}(U) \subset \mathbb{A}^{n+1} \) open.

Regular fcns: \( f : U \to k \) is regular \( \iff \pi^*(f) = f \circ \pi : \pi^{-1}(U) \to k \) regular.

Notation: \( (a_0 : \cdots : a_n) = \pi(a_0, \ldots, a_n) \).

Projective coord ring: \( \mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \ldots, x_n] \).

Def: Let \( f \in k[x_0, \ldots, x_n] \) homogeneous poly.
\( D_+(f) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) \neq 0\} \)
Exercise: \( D_+(x_i) \cong \mathbb{A}^n \).

Cor: \( \mathbb{P}^n = D_+(x_0) \cup \cdots \cup D_+(x_n) \) is a prevariety.

Exercise: \( X \) SWF and \( \phi : \mathbb{P}^n \to X \) function. Then \( \phi \) is a morphism iff \( \phi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X \) is a morphism.

Def: If \( W \subset \mathbb{P}^n \) subset, then \( I(W) = I(\pi^{-1}(W)) \subset k[x_0, \ldots, x_n] \).
Def: If \( I \subset k[x_0, \ldots, x_n] \) homogeneous ideal, then \( Z_+(I) = \pi(Z(I)) \subset \mathbb{P}^n \).

Projective Nullstellensatz: \( I \subset k[x_0, \ldots, x_n] \) homogeneous ideal. If \( Z_+(I) \neq \emptyset \) then \( I(Z_+(I)) = \sqrt{I} \).

5. Projective varieties

Def. A **projective variety** is a closed subset of \( \mathbb{P}^n \) (with SWF structure).

A **quasi-projective variety** is a locally closed subset of \( \mathbb{P}^n \).

An **affine variety** is a closed subset of \( \mathbb{A}^n \).

A **quasi-affine variety** is a locally closed subset of \( \mathbb{A}^n \).

Exercise: \( \mathbb{P}^n \) is not quasi-affine for \( n \geq 1 \).

Exercise*: If \( X \) is both projective and quasi-affine, then \( X \) is finite.

Def: If \( X \subset \mathbb{P}^n \) is closed, then proj. coord. ring of \( X \) is \( k[x_0, \ldots, x_n]/I(X) \).

DEPENDS ON EMBEDDING!!

Def: \( R \) graded ring, \( f \in R_d \).
\( R_f = \{ \text{homogeneous elts. in } R \text{ of degree zero} \} = \{g/f^m \mid g \in R_{dm}\} \).
Exercise: \( R \) f.g. reduced graded \( k \)-algebra \( \implies R_f \) f.g. reduced \( k \)-algebra.
Exercise: \( X \subset \mathbb{P}^n \) projective, \( R = k[x_0, \ldots, x_n]/I(X) \). \( f \in R_d \) with \( d > 0 \). Then \( X_f := X \cap D_+(f) \cong \text{Spec-m}(R_f) \).

Hints: Enough to assume \( X = \mathbb{P}^n \), \( R = k[x_0, \ldots, x_n] \).
Show that \( \mathcal{O}(D_+(f)) = R_f \).
Identity map \( R_f \to \mathcal{O}(D_+(f)) \) defines morphism \( D_+(f) \to \text{Spec-m}(R_f) \).
Show this is an isomorphism.

6. Products

Let \( X \) and \( Y \) be SWFs. A **product** of \( X \) and \( Y \) is a SWF called \( X \times Y \) with morphisms \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \), such that \( (X \times Y, \pi_X, \pi_Y) \) is universal.

Exercise: Show that products of SWFs exist and are unique.
Example: \( \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \). NOTE: \( \mathbb{A}^2 \) does not have the product topology!
Exercise: If \( X \) and \( Y \) are affine varieties, then \( X \times Y \cong \text{Spec-m}(A(X) \otimes_k A(Y)) \).
Cor: A product of prevarieties is a prevariety.
7. Separated SWFs

Def: A SWF $X$ is separated if $\forall$ SWFs $Y$ and morphisms $f, g : Y \to X$ the set \{ $y \in Y \mid f(y) = g(y)$ \} $\subset Y$ is closed.

(Algebraic version of Hausdorff.)

Non-example: $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{O_1, O_2\}$ = union of two copies of $\mathbb{A}^1$.

Def: An algebraic variety is a separated prevariety.

Exercise: Any subspace of a separated SWF is separated.

Exercise: A product of separated SWFs is separated.

Exercise: $\Delta : X \to X \times X, x \mapsto (x, x)$ is a morphism.

Def: $\Delta_X := \Delta(X) \subset X \times X$.

Exercise: $\Delta : X \to \Delta_X$ isomorphism.

Exercise: $X$ is separated $\iff \Delta_X \subset X \times X$ is closed.

Exercise: $\mathbb{A}^n$ is separated, hence all (quasi-) affine varieties are algebraic varieties.

Exercise: $\mathbb{P}^n$ is separated, hence all (quasi-) projective varieties are varieties.

8. Rational maps

Def: A topological space $X$ is irreducible if $X$ is not a union of two proper closed subsets.

Let $X$ and $Y$ be irreducible varieties.

Consider pairs $(U, f)$ such that $\emptyset \neq U \subset X$ is open and $f : U \to Y$ is a morphism.

Relation: $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$.

Exercise: $\sim$ is an equiv. relation. (Since $X$ is irreducible and $Y$ is separated.)

Def: A rational map $f : X \dashrightarrow Y$ is an equivalence class for $\sim$.

Exercise: There is a unique maximal open subset of points in $X$ where $f$ is defined as a morphism.

Def: A rational function on $X$ is a rational map $f : X \dashrightarrow \mathbb{A}^1 = k$.

$f$ is given by a regular function $f : U \to k$, where $\emptyset \neq U \subset X$ is open.

Def: $k(X) = \{ f : X \dashrightarrow k \}$

Exercise: $k(X)$ is a field.

Exercise: $\emptyset \subset U \subset X$ open $\Rightarrow k(U) = k(X)$.

Exercise: $X$ irreducible affine variety $\Rightarrow k(X) = K(A(X))$ fraction field.

Def: $(U, f) : X \dashrightarrow Y$ is dominant if $f(U) = Y$.

Exercise: If $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are rational maps and $f$ is dominant, then $\exists$ well-defined composition $g \circ f : X \dashrightarrow Z$.

Exercise: Let $X$ and $Y$ be irreducible varieties. 1-1 correspondence:

{ dominant $f : X \dashrightarrow Y$ } $\leftrightarrow$ { field ext. $k(Y) \subset k(X)$ over $k$ }.

Def: $f : X \dashrightarrow Y$ is birational if $f$ is dominant and $\exists$ dominant $g : Y \dashrightarrow X$

s.t. $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$.

Def: $X$ and $Y$ are birationally equivalent (written $X \approx Y$) iff $\exists$ birational map $f : X \dashrightarrow Y$.

Example: $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$

Exercise: $X \approx Y \Leftrightarrow k(X) \cong k(Y)$ as $k$-algebras $\Leftrightarrow$

$\exists$ open subsets $U \subset X$ and $V \subset Y$ s.t. $U \cong V$.

Def: $X$ is rational if $X$ is birationally equivalent to $\mathbb{A}^n$ for some $n$.

Def: $X$ is unirational if $\exists$ dominant rational map $f : \mathbb{A}^n \dashrightarrow X$.

Exercise*: $E = Z(y^2 - x^3 + x) \subset \mathbb{A}^2$ is not rational.

Exercise**: If $C$ is a unirational curve, then $C$ is rational.
9. Complete varieties

Def: A variety $X$ is **complete** if for any variety $Y$, $\pi_Y : X \times Y \to Y$ is closed.
(Analogue of compact manifolds. Schemes: same as proper over $\text{Spec}(k)$.)

Note: 1) Closed subsets of complete varieties are complete.
2) Products of complete varieties are complete.
Example: Points are complete!
Example: $\mathbb{A}^1$ is not complete.

$W = \mathbb{Z}(xy - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ is closed but $\pi_2(W) = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$.

Exercise: Let $\varphi : X \to Y$ be a morphism of varieties. If $X$ is complete then $\varphi(X) \subset Y$ is closed and complete. (Use graph $\Gamma_f \subset X \times Y$.)

Exercise: $\varphi : X \to Y$ cont. map of top. spaces. Then $X$ irred. $\Rightarrow$ $\varphi(X)$ irred.
Cor: If $X$ is irreducible and complete then $\mathcal{O}(X) = k$.
Proof: If $f : X \to \mathbb{A}^1$ is any morphism then $f(X) \subset \mathbb{A}^1$ is closed, complete, and irreducible, hence a point.

Exercise: Any complete quasi-affine variety if finite.
Exercise* : $\mathbb{P}^n$ is complete, hence all projective varieties are complete.