

## REVIEW OF VARIETIES

### 1. AFFINE VARIETIES

$k = \bar{k}$  alg closed field.

$R$  f.g. reduced  $k$ -algebra.

$\text{Spec-m}(R) = \{ \text{max. ideals } \mathfrak{m} \subset R \}$

Topology: Zariski closed sets are  $Z(I) = \{ \mathfrak{m} \supset I \}$

Let  $f \in R$ . Def.  $f : \text{Spec-m}(R) \rightarrow k$ ,  $f(\mathfrak{m}) = \text{image of } f \text{ by } R \rightarrow R/\mathfrak{m} = k$ .

Def: Let  $U \subset \text{Spec-m}(R)$  be open,  $f : U \rightarrow k$  a function.

$f$  is **regular** if it is locally of the form  $f(\mathfrak{m}) = p(\mathfrak{m})/q(\mathfrak{m})$ ,  $p, q \in R$ .

$k[U] = \{ \text{regular } f : U \rightarrow k \}$ .

Exercise\*:  $k[\text{Spec-m}(R)] = R$

**Coordinate ring:**  $A(\text{Spec-m}(R)) = R$  (only for affine varieties)

Example:  $R = k[f_1, \dots, f_n] = k[x_1, \dots, x_n]/I$ .  $(f_1, \dots, f_n) : X \xrightarrow{\sim} Z(I) \subset \mathbb{A}^n$

### 2. SPACES WITH FUNCTIONS

Def: A **space with functions** is a top space  $X$  with functor

$U \mapsto k[U] \subset \{ \text{all fens } U \rightarrow k \}$  such that

(1)  $U = \bigcup_{\alpha} U_{\alpha} : f \in k[U] \Leftrightarrow f|_{U_{\alpha}} \in k[U_{\alpha}] \forall \alpha$ .

(2)  $f \in k[U] \Rightarrow D(f) \subset U$  open and  $1/f \in k[D(f)]$ .

Notation:  $\mathcal{O}_X(U) = k[U]$

Def: A **morphism** of SWFs is a cont. map  $\varphi : X \rightarrow Y$  such that pullback of regular functions are regular.

I.e. if  $V \subset Y$  is open and  $f \in \mathcal{O}_Y(V)$ , then  $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$ .

### 3. SUBSPACE OF SWF

$X$  SWF,  $Y \subset X$  any subset. Give  $Y$  structure of SWF as follows:

\* Subspace topology.

\* If  $U \subset Y$  is open,  $f : U \rightarrow k$  function, then  $f$  is regular iff  $f$  can locally be extended to regular fcn on  $X$ .

I.e.  $\forall y \in U \exists U' \subset X$  and  $F \in \mathcal{O}_X(U')$  s.t.  $y \in U'$  and  $f(x) = F(x) \forall x \in U \cap U'$ .

Def. A **prevariety** is a SWF  $X$  s.t.  $\exists$  open cover  $X = U_1 \cup \dots \cup U_m$ , with  $U_i \cong \text{Spec-m}(R_i)$  affine variety for each  $i$ .

Exercise: Let  $X = \text{Spec-m}(R)$  be affine and  $f \in R$ . Then  $X_f := D(f) \cong \text{Spec-m}(R_f)$ .

Exercise:  $X$  SWF and  $Y$  affine variety.

1-1 correspondence  $\{ \text{morphisms } X \rightarrow Y \} \leftrightarrow \{ k\text{-alg homs } A(Y) \rightarrow k[X] \}$ .

Cor: Two affine varieties isomorphic iff coordinate rings isomorphic.

Exercise:  $\mathbb{A}^n \setminus \{0\}$  is not affine for  $n \geq 2$ .

Exercise: An open subset of a prevariety is a prevariety.

Exercise: A closed subset of a prevariety is a prevariety.

Def:  $X$  top space. A subset  $W \subset X$  is **locally closed** if it is an intersection of an open set and a closed set.

Cor: A locally closed subset of a prevariety is a prevariety.

#### 4. PROJECTIVE SPACE

Def:  $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^* =$  lines through origin in  $\mathbb{A}^{n+1}$ .

$\pi : \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  projection.

Topology:  $U \subset \mathbb{P}^n$  open  $\Leftrightarrow \pi^{-1}(U) \subset \mathbb{A}^{n+1}$  open.

Regular fcn:  $f : U \rightarrow k$  is regular  $\Leftrightarrow \pi^*(f) = f \circ \pi : \pi^{-1}(U) \rightarrow k$  regular.

Notation:  $(a_0 : \dots : a_n) = \pi(a_0, \dots, a_n)$ .

Projective coord ring:  $k[\mathbb{A}^{n+1}] = k[x_0, \dots, x_n]$ .

Def: Let  $f \in k[x_0, \dots, x_n]$  homogeneous poly.

$D_+(f) = \{(a_0 : \dots : a_n) \in \mathbb{P}^n \mid f(a_0, \dots, a_n) \neq 0\}$

Exercise:  $D_+(x_i) \cong \mathbb{A}^n$ .

Cor:  $\mathbb{P}^n = D_+(x_0) \cup \dots \cup D_+(x_n)$  is a prevariety.

Exercise:  $X$  SWF and  $\phi : \mathbb{P}^n \rightarrow X$  function. Then  $\phi$  is a morphism iff  $\phi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \rightarrow X$  is a morphism.

Def: If  $W \subset \mathbb{P}^n$  subset, then  $I(W) = I(\pi^{-1}(W)) \subset k[x_0, \dots, x_n]$ .

Def: If  $I \subset k[x_0, \dots, x_n]$  homogeneous ideal, then  $Z_+(I) = \pi(Z(I)) \subset \mathbb{P}^n$ .

Projective Nullstellensatz:  $I \subset k[x_0, \dots, x_n]$  homogeneous ideal. If  $Z_+(I) \neq \emptyset$  then  $I(Z_+(I)) = \sqrt{I}$ .

#### 5. PROJECTIVE VARIETIES

Def. A **projective variety** is a closed subset of  $\mathbb{P}^n$  (with SWF structure).

A **quasi-projective variety** is a locally closed subset of  $\mathbb{P}^n$ .

An **affine variety** is a closed subset of  $\mathbb{A}^n$ .

A **quasi-affine variety** is a locally closed subset of  $\mathbb{A}^n$ .

Exercise:  $\mathbb{P}^n$  is not quasi-affine for  $n \geq 1$ .

Exercise\*: If  $X$  is both projective and quasi-affine, then  $X$  is finite.

Def: If  $X \subset \mathbb{P}^n$  is closed, then proj. coord. ring of  $X$  is  $k[x_0, \dots, x_n]/I(X)$ .

DEPENDS ON EMBEDDING!!

Def:  $R$  graded ring,  $f \in R_d$ .

$R_{(f)} = \{ \text{homogeneous elts. in } R_f \text{ of degree zero} \} = \{g/f^m \mid g \in R_{dm}\}$ .

Exercise:  $R$  f.g. reduced graded  $k$ -algebra  $\Rightarrow R_{(f)}$  f.g. reduced  $k$ -algebra.

Exercise:  $X \subset \mathbb{P}^n$  projective,  $R = k[x_0, \dots, x_n]/I(X)$ .  $f \in R_d$  with  $d > 0$ . Then  $X_f := X \cap D_+(f) \cong \text{Spec-m}(R_{(f)})$ .

Hints: Enough to assume  $X = \mathbb{P}^n$ ,  $R = k[x_0, \dots, x_n]$ .

Show that  $k[D_+(f)] = R_{(f)}$ .

Identity map  $R_{(f)} \rightarrow k[D_+(f)]$  defines morphism  $D_+(f) \rightarrow \text{Spec-m}(R_{(f)})$ .

Show this is an isomorphism.

#### 6. PRODUCTS

Let  $X$  and  $Y$  be SWFs. A **product** of  $X$  and  $Y$  is a SWF called  $X \times Y$  with morphisms  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$ , such that  $(X \times Y, \pi_X, \pi_Y)$  is universal.

Exercise: Show that products of SWFs exist and are unique.

Example:  $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ . NOTE:  $\mathbb{A}^2$  does not have the product topology!

7. SEPARATED SWFs

Def: A SWF  $X$  is **separated** if  $\forall$  SWFs  $Y$  and morphisms  $f, g : Y \rightarrow X$  the set  $\{y \in Y \mid f(y) = g(y)\} \subset Y$  is closed.

(Algebraic version of Hausdorff.)

Non-example:  $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{O_1, O_2\} =$  union of two copies of  $\mathbb{A}^1$ .

Def: An **algebraic variety** is a separated prevariety.

Exercise: Any subspace of a separated SWF is separated.

Exercise: A product of separated SWFs is separated.

Exercise:  $\Delta : X \rightarrow X \times X, x \mapsto (x, x)$  is a morphism.

Def:  $\Delta_X := \Delta(X) \subset X \times X$ .

Exercise:  $\Delta : X \rightarrow \Delta_X$  isomorphism.

Exercise:  $X$  is separated  $\Leftrightarrow \Delta_X \subset X \times X$  is closed.

Exercise:  $\mathbb{A}^n$  is separated, hence all (quasi-) affine varieties are algebraic varieties.

Exercise:  $\mathbb{P}^n$  is separated, hence all (quasi-) projective varieties are varieties.

Exercise: If  $X$  and  $Y$  are affine varieties, then  $X \times Y \cong \text{Spec-m}(A(X) \otimes_k A(Y))$ .

Cor: A product of pre-varieties is a pre-variety.

8. RATIONAL MAPS

Def: A topological space  $X$  is **irreducible** if  $X$  is not a union of two proper closed subsets.

Let  $X$  and  $Y$  be irreducible varieties.

Consider pairs  $(U, f)$  such that  $\emptyset \neq U \subset X$  is open and  $f : U \rightarrow Y$  is a morphism.

Relation:  $(U, f) \sim (V, g)$  iff  $f = g$  on  $U \cap V$ .

Exercise:  $\sim$  is an equiv. relation. (Since  $X$  is irreducible and  $Y$  is separated.)

Def: A **rational map**  $f : X \dashrightarrow Y$  is an equivalence class for  $\sim$ .

Exercise: There is a unique maximal open subset of points in  $X$  where  $f$  is defined as a morphism.

Def: A **rational function** on  $X$  is a rational map  $f : X \dashrightarrow \mathbb{A}^1 = k$ .

$f$  is given by a regular function  $f : U \rightarrow k$ , where  $\emptyset \neq U \subset X$  is open.

Def:  $k(X) = \{f : X \dashrightarrow k\}$

Exercise:  $k(X)$  is a field.

Exercise:  $\emptyset \subset U \subset X$  open  $\Rightarrow k(U) = k(X)$ .

Exercise:  $X$  irred. affine variety  $\Rightarrow k(X) = K(A(X))$  fraction field.

Def:  $(U, f) : X \dashrightarrow Y$  is **dominant** if  $\overline{f(U)} = Y$ .

Exercise: If  $f : X \dashrightarrow Y$  and  $g : Y \dashrightarrow Z$  are rational maps and  $f$  is dominant, then  $\exists$  well-defined composition  $g \circ f : X \dashrightarrow Z$ .

Exercise: Let  $X$  and  $Y$  be irreducible varieties. 1-1 correspondence:

$\{ \text{dominant } f : X \dashrightarrow Y \} \leftrightarrow \{ \text{field ext. } k(Y) \subset k(X) \text{ over } k \}$ .

Def:  $f : X \dashrightarrow Y$  is **birational** if  $f$  is dominant and  $\exists$  dominant  $g : Y \dashrightarrow X$  s.t.  $f \circ g = \text{id}_Y$  and  $g \circ f = \text{id}_X$ .

Def:  $X$  and  $Y$  are **birationally equivalent** (written  $X \approx Y$ ) iff  $\exists$  birational map  $f : X \dashrightarrow Y$ .

Example:  $\mathbb{A}^2 \approx \mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$

Exercise:  $X \approx Y \Leftrightarrow k(X) \cong k(Y)$  as  $k$ -algebras  $\Leftrightarrow$

$\exists$  open subsets  $U \subset X$  and  $V \subset Y$  s.t.  $U \cong V$ .

Def:  $X$  is **rational** if  $X$  is birationally equivalent to  $\mathbb{A}^n$  for some  $n$ .

Def:  $X$  is **unirational** if  $\exists$  dominant rational map  $f : \mathbb{A}^n \dashrightarrow X$ .

Exercise\*:  $E = Z(y^2 - x^3 + x) \subset \mathbb{A}^3$  is not rational.

Exercise\*\*: If  $C$  is a unirational curve, then  $C$  is rational.

### 9. COMPLETE VARIETIES

Def: A variety  $X$  is **complete** if for any variety  $Y$ ,  $\pi_Y : X \times Y \rightarrow Y$  is closed.  
(Analogue of compact manifolds. Schemes: same as proper over  $\text{Spec}(k)$ .)

Note: 1) Closed subsets of complete varieties are complete.

2) Products of complete varieties are complete.

Example: Points are complete!

Example:  $\mathbb{A}^1$  is not complete.

$W = Z(xy - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$  is closed but  $\pi_2(W) = \mathbb{A}^1 \setminus \{0\}$  is not closed in  $\mathbb{A}^1$ .

Exercise: Let  $\varphi : X \rightarrow Y$  be a morphism of varieties. If  $X$  is complete then  $\varphi(X) \subset Y$  is closed and complete. (Use graph  $\Gamma_f \subset X \times Y$ .)

Exercise:  $\varphi : X \rightarrow Y$  cont. map of top. spaces. Then  $X$  irred.  $\Rightarrow \varphi(X)$  irred.

Cor: If  $X$  is irreducible and complete then  $k[X] = k$ .

Proof: If  $f : X \rightarrow \mathbb{A}^1$  is any morphism then  $f(X) \subset \mathbb{A}^1$  is closed, complete, and irreducible, hence a point.

Exercise: Any complete quasi-affine variety is finite.

Exercise\*:  $\mathbb{P}^n$  is complete, hence all projective varieties are complete.