

Last time: A closed subset of a Noetherian topological space is a union of finitely many irreducible ~~closed~~ subsets.

But any subset of a Noetherian space is itself Noeth. with the subspace topology. So all we need is:

Thm Every Noeth. space is a finite union of closed irred. subsets.

E.g. $V(xy) = V(x) \cup V(y) \cup V(x-l, y)$



Def The max. irred. closed subsets of a Noeth. space X are called ~~the~~ (irreducible) components of X .

Exercise (1) X has finitely many components.

(2) X is the union of its components.

(3) X is not the union of a proper subset of its components.

Exercise X is Noeth. $\Leftrightarrow$ every open subset is quasi-compact.

Exercise A Noetherian Hausdorff space has finitely many points.

Recall $X \subseteq \mathbb{A}^n$ closed set.

$A(X) = k[x_1, \dots, x_n] / I(X)$ polynomial functions on X .

Def If $f \in A(X)$ set $D(f) = \{a \in X \mid f(a) \neq 0\}$.

$D(f) \subseteq X$ is open.

Prop The sets $D(f)$ form a basis for the topology on X .

Proof Let $p \in U \subseteq X$, U open.

Show $\exists f \in A(X)$ s.t. $p \in D(f) \subseteq U$.

$Z = X \setminus U$ is closed.

$Z \subsetneq Z \cup \{p\} \Rightarrow I(Z \cup \{p\}) \subsetneq I(Z)$.

Take any $f \in I(Z) \setminus I(Z \cup \{p\})$.

□

Regular functions.

$X \subseteq A^n$ algebraic set, $U \subseteq X$ open subset.

A function $f: U \rightarrow k$ is called a regular fcn. if f locally rational.

I.e.: $\exists$ open cover $U = \bigcup_{\alpha} U_{\alpha}$ and fcn. $p_{\alpha}, q_{\alpha} \in A(X)$ s.t.
 $q_{\alpha}(a) \neq 0$ and $f(a) = \frac{p_{\alpha}(a)}{q_{\alpha}(a)} \quad \forall a \in U_{\alpha}$.

Def $k[U] = \{ \text{regular fcn. } U \rightarrow k \}$.

Note 1) $k[U]$ is a k -algebra. Elts. in k are the constant fcn.

2) $A(X) \subseteq k[X]$.

EXAMPLE HERE!

Lemma Let $q_1, \dots, q_m \in A(X)$. $D(q_1) \cup \dots \cup D(q_m) = X \Leftrightarrow (q_1, \dots, q_m) = A(X)$.

Pf $\Leftarrow$: If $1 = \sum u_i q_i$ then at least one fcn. q_i must be non-zero at any point.

$\Rightarrow$: Take $Q_i \in k[x_1, \dots, x_n]$ s.t. $q_i = \bar{Q}_i \in A(X)$.

$$D(q_1) \cup \dots \cup D(q_m) = X \Rightarrow \emptyset$$

$$\emptyset = X \cap V(Q_1) \cap \dots \cap V(Q_m) = V(I(X), Q_1, \dots, Q_m).$$

$$\text{Weak Nullstellensatz} \Rightarrow (I(X), Q_1, \dots, Q_m) = k[x_1, \dots, x_n]$$

$$\Rightarrow (q_1, \dots, q_m) = A(X).$$

□

Thm $k[X] = A(X)$.

Proof $f \in k[X]$ reg. fcn.

$X = U_1 \cup \dots \cup U_m$, $p_i, q_i \in A(X)$: $q_i \neq 0$ and $f = \frac{p_i}{q_i}$ on U_i .

By refining this cover we can assume $U_i = D(q_i)$, $q_i \in A(X)$.

Note: $f = \frac{p_i}{q_i} = \frac{p_i q_i}{q_i^2}$ on U_i and $D(q_i) = D(q_i q_i)$.

Replace p_i with $p_i q_i$ and q_i with $q_i q_i$.

(3)

May assume $U_i = D(q_i) = D(q_i^2)$.

~~Since~~ $X = D(q_1^2) \cup \dots \cup D(q_m^2) \Rightarrow$

$$1 = \sum_{i=1}^m h_i q_i^2, \quad h_i \in A(X).$$

Now: $q_i^2 f = q_i p_i$. (True inside U_i ; outside U_i $q_i = 0$.)

So $f = \sum_{i=1}^m h_i q_i^2 f = \sum_{i=1}^m h_i q_i p_i \in A(X).$

□

Example DO ABOVE!

$$X = V(xy - zw) \subseteq A^4.$$

$$U = D(y) \cup D(w) = X \setminus V(y, w) \text{ open subset.}$$

Def. $f: U \rightarrow k$ by

$$f = \frac{x}{w} \text{ on } D(w) \text{ and } f = \frac{z}{y} \text{ on } D(y).$$

This fcn. is regular.

Exercise $\nexists p, q \in A(X)$ s.t. $f(a) = \frac{p(a)}{q(a)} \quad \forall a \in U.$

Spaces with functions

A space with functions is a top. space X together with an assignment to each open subset $U \subseteq X$ of a k -algebra $k[U]$ of k -valued functions on U , called regular fcn., s.t.

(a) If $U = \bigcup_{\alpha} U_{\alpha}$ open cover and $f: U \rightarrow k$ any fcn. ~~then~~

$f|_{U_{\alpha}}$ is regular for each $\alpha \iff$ ~~then~~ f is regular on U .

(b) If $f \in k[U]$ then $D(f) = \{a \in U \mid f(a) \neq 0\}$ is open and $\forall f \in k[D(f)]$.

Note ~~the algebra of regular functions on U is denoted $\mathcal{O}_X(U)$.~~

$k[U]$ is sometimes called $\mathcal{O}_X(U)$.

Examples

1) Alg. sets. are spaces with functions.

An alg. set considered as a space with functions is called an algebraic variety.

2) If M is a manifold, take $k = \mathbb{R}$ and let

$$k[U] = \{ C^\infty\text{-funs. } U \rightarrow \mathbb{R} \}.$$

3) X surf. $U \subseteq X$ open. Then U surf. by setting $\mathcal{O}_U(V) = \mathcal{O}_X(V)$, $V \subseteq U$.

Def ~~morphism~~

X and Y spaces with fcs. A morphism $\varphi: X \rightarrow Y$ is a continuous fcn. such that reg. fcs. are pulled back to reg. fcs.

I.e. If $V \subseteq Y$ open and $f \in \mathcal{O}_Y(V)$ and $U \subseteq X$ open s.t. $\varphi(U) \subseteq V$ then $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(U)$.

Def $\varphi: X \rightarrow Y$ is an isomorphism if $\exists$ other morphism $\psi: Y \rightarrow X$ s.t. $\varphi \circ \psi = \text{id}_Y$ and $\psi \circ \varphi = \text{id}_X$.

Exercise In example 2, morphism = C^∞ -map. of mfd's.

Exercise Id. map of surf. is a morphism.

Compositions of morphisms are morphisms.

Exercise $\varphi: X \rightarrow Y$ map of spaces w. fcs.

$X = \bigcup U_\alpha$ and $Y = \bigcup V_\alpha$ open covers. $\varphi(U_\alpha) \subseteq V_\alpha$.

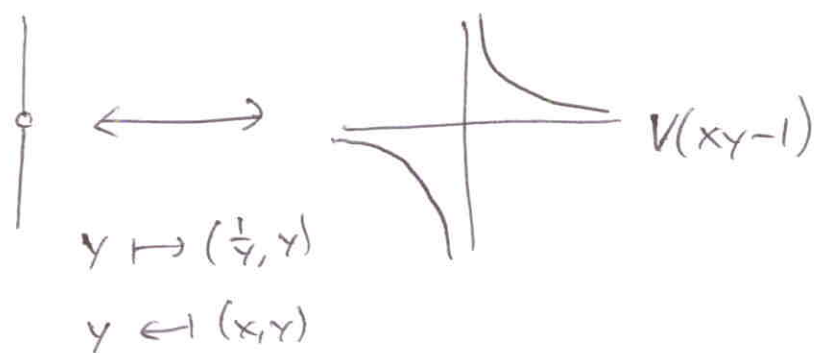
φ morphism $\Leftrightarrow$ each restriction $\varphi: U_\alpha \rightarrow V_\alpha$ morphism.

Exercise X surf. ~~surf~~ $Y \subseteq \mathbb{A}^n$ closed, affine variety.

$f = (f_1, \dots, f_n): X \rightarrow Y$ function.

f morphism $\Leftrightarrow f_i: X \rightarrow k$ regular $\forall i$.

Example $A^1 \setminus \{0\}$ is (isomorphic to) an affine variety.



Localization

R ring.

$S \subseteq R$ mult. closed subset, i.e. $s, t \in S \Rightarrow st \in S; 1 \in S$.

$$S^{-1}R = \{ f/s \mid f \in R, s \in S \}.$$

$$f/s = g/t \Leftrightarrow \exists u \in S : u(tf - sg) = 0.$$

$$\text{This is a ring. } f/s \cdot g/t = fg/st. \quad f/s + g/t = \frac{tf + sg}{st}.$$

If $f \in R$ we set $R_f = S^{-1}R$, $S = \{ f^n \mid n \geq 0 \}$.

Facts 1) R reduced $\Rightarrow S^{-1}R$ reduced. $R_f = R[y]/(yf-1)$

$$2) R/I \text{ reduced} \Leftrightarrow I = \sqrt{I}.$$

Prop

$X \subseteq A^n$ closed affine variety, $f \in A(X)$. Then $D(f) \subseteq X$ is an affine variety with affine coord. ring $A(X)_f$.

Pf. $I = I(X) \subseteq k[x_1, \dots, x_n]$. $J = (I, yf-1) \subseteq k[x_1, \dots, x_n, y]$.

$$\varphi: D(f) \longrightarrow V(J) \subseteq A^{n+1}$$

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n)^{-1})$$

is an isomorphism.

$$A(X) \text{ reduced} \Rightarrow A(X)_f = k[x_1, \dots, x_n, y]/J \text{ reduced.}$$

$$\Rightarrow J = \sqrt{J} = I(V(J)). \quad \text{So } k[D(f)] = k[V(J)] = k[x_1, \dots, x_n, y]/J = A(X)_f.$$

□

Prevarieties

(6)

Def A pre-variety is a space with fct. X such that X has a finite open cover $X = U_1 \cup \dots \cup U_m$ where each U_i is (isomorphic to) an affine variety.

Example Affine varieties are pre-varieties.

Exercise A pre-variety is a Noetherian top. space.

Cor (to Prop). An open subset of a pre-variety is a pre-variety.

Question Is the pre-variety $\mathbb{A}^2 - \{(0,0)\}$ an affine variety?

Prop X surfs, $Y \subseteq \mathbb{A}^n$ affine variety. 1-1 correspondence:

$$\{\text{morphisms } X \rightarrow Y\} \longleftrightarrow \{k\text{-alg. hom. } A(Y) \rightarrow k[X]\}$$

$$\varphi \longleftrightarrow \varphi^*$$

Pf. The map $\varphi \mapsto \varphi^*$ well def. (and φ^* k -alg. hom.)

$$I = I(Y) \subseteq k[x_1, \dots, x_n].$$

$\bar{y}_i$ = image of y_i in $A(Y)$.

If $\alpha: A(Y) \rightarrow k[X]$ k -alg. hom, set $\varphi_i = \alpha(\bar{y}_i) \in k[X]$.

Def ~~$\varphi: X \rightarrow \mathbb{A}^n$~~ by $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

~~Since~~ If $f \in I(Y)$ then $f(\bar{y}_1, \dots, \bar{y}_n) = 0 \Rightarrow f(\varphi_1, \dots, \varphi_n) = 0$

$\therefore \varphi(X) \subseteq V(I) = Y$. since α k -alg. hom.

φ morphism because φ_i regular.

$$\varphi^*(\bar{y}_i) = y_i \circ \varphi = \varphi_i = \alpha(\bar{y}_i) \Rightarrow \varphi^* = \alpha.$$

~~Conversely~~ If $\varphi: X \rightarrow Y$ morphism then

$$\varphi_i = y_i \circ \varphi = \varphi^*(\bar{y}_i) \Rightarrow \varphi = \text{morphism constructed from } \varphi^*.$$

Corollary Two affine varieties are isomorphic $\Leftrightarrow$ the ~~the~~ affine coord. rings are iso. as k -algebras.

Exercise $\mathbb{A}^n - \{(0, 0, \dots, 0)\}$ is not affine for $n \geq 2$.

R f.g. reduced k -algebra, generated by $v_1, \dots, v_n \in R$ over k . (2)

Have $k[x_1, \dots, x_n] \twoheadrightarrow R$, $x_i \mapsto v_i$.

Set $I = \ker$. Then $X = V(I) \subseteq \mathbb{A}^n$ affine var. and

I radical $\Rightarrow A(X) = k[x_1, \dots, x_n]/I = R$.

$$\left\{ \begin{array}{l} \text{Reduced f.g.} \\ k\text{-algebras} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Affine} \\ \text{varieties} \end{array} \right\} \quad (\text{up to iso.})$$

Note: $\mathfrak{m} \subseteq R$ max. ideal $\Rightarrow$ inv. image $M \subseteq k[x_1, \dots, x_n]$ max. l.

So $M = (x_1 - a_1, \dots, x_n - a_n)$ and $R/\mathfrak{m} = k[x_1, \dots, x_n]/M = k$.

Canonical construction

$$\text{Spec-}\mathfrak{m}(R) = \{ \text{max. ideals } \mathfrak{m} \subseteq R \}.$$

Topology: Closed sets are those of form $V(I) = \{ \mathfrak{m} \supseteq I \}$.

$f \in R$. Def. fcn. $f: \text{Spec-}\mathfrak{m}(R) \rightarrow k$ by

$$f(\mathfrak{m}) = \text{image of } f \text{ under } R \rightarrow R/\mathfrak{m} = k.$$

If $U \subseteq \text{Spec-}\mathfrak{m}(R)$ is open, a fcn $f: U \rightarrow k$ is regular if it locally has form $f(\mathfrak{m}) = p(\mathfrak{m})/q(\mathfrak{m})$ for $p, q \in R$.

Exercise $\text{Spec-}\mathfrak{m}(R) \cong X$ where $X \subseteq \mathbb{A}^n$ constr. above.

Subspaces of spaces with functions

X SWFs, $Y \subseteq X$ any subset.

Make Y a SWF as follows:

Use subspace topology.

If $U \subseteq Y$ open then a fcn. $f: U \rightarrow k$ is regular if it can locally be extended to regular fcn. on X .

I.e. $\forall y \in U \exists U' \subseteq X$ open, $y \in U'$ and $\exists F \in \mathcal{O}_X(U')$ s.t. $f(x) = F(x) \forall x \in U \cap U'$.

Exercise (1) Y is a SWF.

(2) Inclusion $i: Y \rightarrow X$ is a morphism.

(3) If Z SWF and $\varphi: Z \rightarrow Y$ any map then
 φ morphism $\Leftrightarrow i \circ \varphi: Z \rightarrow X$ morphism.

(4) The SWF structure on Y is determined uniquely by (2)+(3).

(5) If $Z \subseteq Y \subseteq X$ then Z inherits same structure from Y and X .

Example $X \subseteq \mathbb{A}^n$ affine var. inherits its structure from $\mathbb{A}^n$.

If $Y \subseteq X$ closed then Y 's structure is inherited from X (or $\mathbb{A}^n$).

Prop A closed subset of a pre-variety is a pre-variety.

PF. X pre-var., $Y \subseteq X$ closed.

$X = U_1 \cup \dots \cup U_m$, U_i open affine.

$Y \cap U_i$ closed in $U_i \Rightarrow Y \cap U_i$ affine.

These sets cover $Y \Rightarrow Y$ pre-variety.

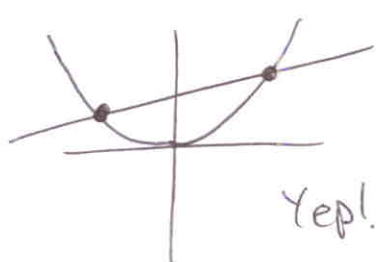
□

Projective Space

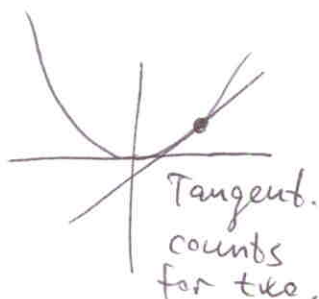
(4)

Thm Two distinct lines in the plane intersect in exactly one point (if not parallel).

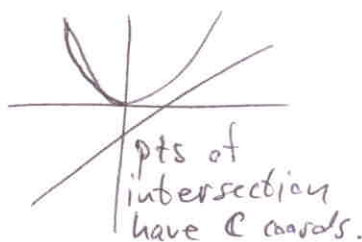
Thm A line and a parabola have exactly two points in common. (when not false).



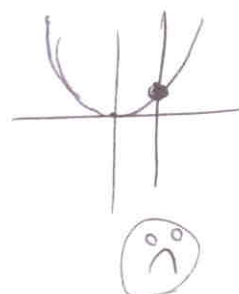
Yep!



Tangent.
counts
for two.



pts of
intersection
have $\mathbb{C}$ coords.



Solution: Add extra points at ∞ and make figures intersect there!