

ALGEBRAIC GEOMETRY I, PROBLEM SET 3

Problem 1. Let X be a topological space and $W \subset X$ any subset.

- (a) $\overline{W} = X$ holds if and only if $\forall \emptyset \neq U \subset X$ open : $W \cap U \neq \emptyset$.
- (b) If $f : X \rightarrow Y$ is continuous, $\overline{f(X)} = Y$, and $\overline{W} = X$, then $\overline{f(W)} = Y$.
- (c) If X is irreducible and $\emptyset \neq U \subset X$ is open, then U is irreducible and $\overline{U} = X$.

Problem 2. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 3. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of $\mathbb{P}^N$, where $N = nm + n + m$.

Problem 4. Let X and Y be spaces with functions and let (P, π_X, π_Y) and (P', π'_X, π'_Y) be two products of X and Y . Show that there is a unique isomorphism $\varphi : P \xrightarrow{\sim} P'$ such that $\pi_X = \pi'_X \circ \varphi$ and $\pi_Y = \pi'_Y \circ \varphi$.

Problem 5. (a) Any subspace of a separated space with functions is separated.

- (b) A product of separated spaces with functions is separated.

Problem 6. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y .

- (a) Show that X is separated.
- (b) Show that $\mathbb{P}^n$ has this property.

Problem 7. [Hartshorne II.2.16 and II.2.17]

Let X be any pre-variety and $f \in k[X]$ a regular function.

(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \dots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i .]

- (b) $k[D(f)] = k[X]_f$.

(c) Let R be a k -algebra and let $f_1, \dots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \dots, f_r) = R$. If R_{f_i} is a finitely generated k -algebra for each i , then R is a finitely generated k -algebra.

(d) Suppose $f_1, \dots, f_r \in k[X]$ satisfy $(f_1, \dots, f_r) = k[X]$ and $D(f_i)$ is affine for each i . Then X is affine.