ALGEBRAIC GEOMETRY I, PROBLEM SET 2

Problem 1. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \to Y$ is a morphism if and only if each coordinate function $f_i : X \to k$ is regular for $1 \leq i \leq n$.

Problem 2. (a) $k[\mathbb{A}^n \smallsetminus \{0\}] = k[x_1, \ldots, x_n]$ for $n \geq 2$.
(b) $\mathbb{A}^n \smallsetminus \{0\}$ is not an affine variety for $n \geq 2$.
(c) Every global regular function on $\mathbb{P}^n$ is constant, i.e. $k[\mathbb{P}^n] = k$.
(d) $\mathbb{P}^n$ is not quasi-affine for $n \geq 1$.

Problem 3. Let $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 4. Define the *homogenization* of a polynomial $f \in k[x_1, \ldots, x_n]$ to be $f^* = x_0^{\deg(f)} f(x_1/x_0, \ldots, x_n/x_0)$. Equivalently, if we write $f = f_0 + f_1 + \cdots + f_d$, with f_i a form of degree i and $f_d \neq 0$, then $f^* = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + f_d \in k[x_0, x_1, \ldots, x_n]$.

Given any ideal $I \subset k[x_1, \ldots, x_n]$, let $I^* \subset k[x_0, x_1, \ldots, x_n]$ be the homogeneous ideal generated by $\{f^* \mid f \in I\}$.

(a) Find an example where $I = \langle h_1, \ldots, h_m \rangle$ and $I^* \neq \langle h_1^*, \ldots, h_m^* \rangle$.

(b) Let $X \subset \mathbb{A}^n$ be a closed subvariety. Identify $\mathbb{A}^n$ with $D_+(x_0) \subset \mathbb{P}^n$ and let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Show that $I(\overline{X}) = I(X)^* \subset k[x_0, \ldots, x_n]$.

Problem 5. Show that if R is a finitely generated reduced k-algebra then the space with functions Spec-m(R) is an affine variety.

Problem 6. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \to X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \smallsetminus \{0\} \to X$ is a morphism.

Problem 7. Let $\varphi : X \to Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 8. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.