ALGEBRAIC GEOMETRY I, PROBLEM SET 4 SOLUTIONS

Problem 7. (a) Let $F, G, H \in k[x,y,z]$ be forms such that $V_+(G,H,z) = \emptyset$ in $\mathbb{P}^2$. Show that if $zF \in (G,H)$ then $F \in (G,H)$. [Hint: Use that $G_0 = G(x,y,0)$ and $H_0 = H(x,y,0)$ are relatively prime.]

(b) Let $C \subset \mathbb{P}^2$ be a curve, and set $\mathcal{O}_C(n) = \mathcal{O}_{\mathbb{P}^2}(n)|_C$. Then $\Gamma(C, \mathcal{O}_C(n)) = (k[x,y,z]/I(C))_n$ for all $n \geq 0$. [Hint: If $C = V_+(H) \subset D_+(y) \cup D_+(z)$ and if σ is a global section of $\mathcal{O}_C(n)$ then $\sigma/y^n = F(x,y,z)/y^m$ and $\sigma/z^n = A(x,y,z)/z^m$ for forms $F, A \in k[x,y,z]$ of degree $m \geq n$. Now use part (a).]

(c) Define the *arithmetic genus* of C to be $1 - P_C(0)$ where $P_C(m)$ is the Hilbert polynomial of $C \subset \mathbb{P}^2$. Show that $p_a = \frac{(d-1)(d-2)}{2}$ where d is the degree of C and that $\dim_k \Gamma(C, \mathcal{O}_C(n)) = nd + 1 - p_a$ for all large integers n.

Solution: (a) For any polynomial $P \in k[x,y,z]$ we set $P_0 = P(x,y,0) \in k[x,y]$. Since $V_+(H_0, G_0) = \emptyset \subset \mathbb{P}^1$, H_0 and G_0 are relatively prime. Assume $zF = AG + BH$. Then $A_0G_0 = -B_0H_0$ so $A_0 = CH_0$ and $B_0 = -CG_0$ for some form $C \in k[x,y]$. Set $A_1 = A - CH$ and $B_1 = B + CG$. Then $A_1(x,y,0) = B_1(x,y,0) = 0$ so $A_1 = zA'$ and $B_1 = zB'$ where $A', B' \in k[x,y,z]$. But then $z(A'G + B'H) = (A - CH)G + (B + CG)H = zF$ which implies that $F = A'G + B'H$.

(b) Let $i : C \to \mathbb{P}^2$ be the inclusion, $\mathcal{O}_C(n) = i^*\mathcal{O}_{\mathbb{P}^2}(n)$, and $S = k[x,y,z]$. We may assume that $C = V_+(H) \subset D_+(y) \cup D_+(z)$. Notice that $\mathcal{O}_{\mathbb{P}^2}(n)|_{D_+(z)}$ is trivial and generated by z^n. It follows that $\mathcal{O}_C(n)|_{C \cap D_+(z)}$ is trivial and generated by $i^*(z^n)$. We will denote this section of $\mathcal{O}_C(n)$ by z^n. Consider the map $S_n = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \to \Gamma(C, \mathcal{O}_C(n))$ defined by $F \mapsto i^*(F)$. The composition with the (injective) restriction map $\Gamma(C, \mathcal{O}_C(n)) \to \Gamma(C \cap D_+(Z), \mathcal{O}_C(n)) \cong \mathcal{O}_C(C \cap D_+(z))$ is given by $S_n \to (S/I(C))_{(z)}$; $F \mapsto F/z^n$, which shows that the kernel is $I(C)_n$, i.e. $(S/I(C))_n \subset \Gamma(C, \mathcal{O}_C(n))$.

Let $\sigma \in \Gamma(C, \mathcal{O}_C(n))$. Then $\sigma/y^n \in \mathcal{O}_C(C \cap D_+(y))$, so $\sigma/y^n = F(x,y,z)/y^m$ for some $F \in S_m$, $m \geq n$. Similarly $\sigma/z^n = A(x,y,z)/z^m$, where $A \in S_m$. Now $z^{m-n}F = y^{m-n}A \in \Gamma(C, \mathcal{O}_C(2m-n))$ so $z^{m-n}F = Ay^{m-n} + BH$ for some form $B \in S$. Using (a) this implies that $F = A'y^{m-n} + B'H$ for forms A' and B'. We conclude that $F = A'y^{m-n} \in \Gamma(C, \mathcal{O}_C(m))$ so $\sigma = A'$ is contained in $(S/I(C))_n$.

(c) The short exact sequence $0 \to S_{n-d} \to S_n \to (S/I(C))_n \to 0$ shows that $P_C(n) = \binom{n+2}{2} - \binom{n+2-d}{2} = nd + \frac{3d-d^2}{2}$. Therefore $p_a = 1 - P_C(0) = \frac{(d-1)(d-2)}{2}$ and $\dim_k \Gamma(C, \mathcal{O}_C(n)) = \dim_k(S/I(C))_n = \mathbb{P}_C(n) = nd + 1 - p_a$ for large n.