ALGEBRAIC GEOMETRY I, PROBLEM SET 3

Problem 1. Resolution of singularities for curves.

Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve $\tilde{X}$ with a finite morphism $\varphi : \tilde{X} \to X$ such that the restriction $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a "proper" morhism φ.)

Problem 2. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \to G$ and $i : G \to G$, and an identity element $e \in G$, such that G is a group in the usual sense when m is used to define multiplication and i maps any element to its inverse element.

(a) Show that $\mathrm{GL}_n(k)$ is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that $\mathbb{P}^1$ is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and $i : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying the group axioms.

(d) Challenge: How about $\mathbb{P}^n$ for $n \geq 2$?

Problem 3. Let G be an irreducible algebraic group acting on a variety X, i.e. we have a morphism $G \times X \to X$ such that the axioms for a group action are satisfied.

(a) Show that each orbit in X is locally closed.

(b) Each orbit is a non-singular variety.

Problem 4. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

has rank one. Show that X is a non-singular rational closed subvariety of $\mathbb{P}^5$, and find its dimension and degree.

Problem 5. [Hartshorne I.5.3 and I.5.4]

Let $X \subset \mathbb{P}^2$ be a curve and $P \in \mathbb{P}^2$ any point. Let $I_{X,P} \subset \mathcal{O}_{\mathbb{P}^2,P}$ be the ideal of functions $f \in \mathcal{O}_{\mathbb{P}^2,P}$ such that $f|_{U \cap X} = 0$ for some open set U containing P. The multiplicity $\mu_P(X)$ of X at P is the largest number r such that $I_{X,P} \subset \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{\mathbb{P}^2,P}$ is the maximal ideal.

(a) $P \in X \Leftrightarrow \mu_P(X) \geq 1$.

(b) P is a non-singular point of X iff $\mu_P(X) = 1$.

(c) Let $Y \subset \mathbb{P}^2$ be another curve such that $X \cap Y$ is a finite set. Show that if $P \in X \cap Y$ then $I(X \cdot Y; P) = \dim_k \mathcal{O}_{\mathbb{P}^2,P}/(I_{X,P} + I_{Y,P})$.

(d) $I(X \cdot Y; P) = 1$ iff P is a non-singular point of both X and Y, and the tangent directions at P are different.

(e) $I(X \cdot Y; P) \geq \mu_P(X) \cdot \mu_P(Y)$.

(f) For all but a finite number of lines $L \subset \mathbb{P}^2$ through P we have $\mu_P(X) = I(X \cdot L; P)$.

Problem 6. Let $\mathcal{F}$ be a sheaf on X and $p \in X$ a point. Prove the following from the definition of the stalk $\mathcal{F}_p$:

(a) Each element of $\mathcal{F}_p$ has the form s_p for some section $s \in \mathcal{F}(U)$, $p \in U$.

(b) Let $s \in \mathcal{F}(U)$, $p \in U$. Then $s_p = 0 \Leftrightarrow s|_V = 0$ for some $p \in V \subset U$.

(c) Let $s \in \mathcal{F}(U)$. Prove that $s = 0$ if and only if $s_p = 0 \;\forall\; p \in U$.

Problem 7. [Hartshorne II.1.2]

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $U = \bigcup V_i$ of U and sections $t_i \in \mathcal{F}(V_i)$ such that $\varphi_{V_i}(t_i) = s|_{V_i}$ for all i.

Problem 8. [Hartshorne II.1.14]

Let $\mathcal{F}$ be a sheaf on X and $s \in \mathcal{F}(X)$ a global section. Show that the set $\{p \in X \mid s_p \neq 0\}$ is a closed subset of X.

Problem 9. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on X. Show that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\mathrm{Im}(\varphi)_p = \mathrm{Im}(\varphi_p)$ for all $p \in X$.

Problem 10. Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ a sheaf on Y. Show that $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$.

Problem 11. Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Show that the map $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ constructed in class is bijective.

Problem 12. (a) Let X be an affine variety, M a $k[X]$-module, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Show that $\mathrm{Hom}_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$.

(b) If X is affine and M and N are $k[X]$-modules then $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = (M \otimes_{k[X]} N)^{\sim}$.

(c) If $f : X \to Y$ is a morphism of varieties and $\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a (quasi-) coherent $\mathcal{O}_X$-module.

Problem 13. (a) X is a ringed space, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. Then the assignment $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines an $\mathcal{O}_X$-module. It is denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(b) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Show that $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 14. A morphism $f : X \to Y$ of varieties is called *affine* if for every open affine set $V \subset Y$ the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$-module for all open affine $V \subset Y$.

Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall\; i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$-module for all i then f is finite.

Problem 15. (a) Let X be a complete variety and $f : X \to Y = \text{Spec-m}(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety X and a birational morphism $f : X \to Y$ such that $f_*\mathcal{O}_X$ is not locally free on Y.

Problem 16. (a) $Y \subset \mathbb{P}^n$ is a hypersurface of degree d with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Show that $\mathcal{I}_Y \cong \mathcal{O}(-d)$.

(b) Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding, $N = \binom{n+d}{n} - 1$. Show that $(v_d)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.