1. Affine varieties

$k = \overline{k}$ alg closed field.
$R$ f.g. reduced $k$-algebra.
Spec-$m(R) = \{ \text{max. ideals } m \subset R \}$
Topology: Zariski closed sets are $Z(I) = \{ m \supset I \}$
Let $f \in R$. Def. $f : \text{Spec-}m(R) \to k$, $f(m) =$ image of $f$ by $R \to R/m = k$.
Def: Let $U \subset \text{Spec-}m(R)$ be open, $f : U \to k$ a function.
$f$ is regular if it is locally of the form $f(m) = p(m)/q(m)$, $p, q \in R$.
$\mathcal{O}(U) = \{ \text{regular } f : U \to k \}$.
Exercise*: $\mathcal{O}($Spec-$m(R)) = R$

Coordinate ring: $\mathcal{A}($Spec-$m(R)) = R$ (only for affine varieties)
Example: $R = k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n]/I$. ($f_1, \ldots, f_n) : X \xrightarrow{\sim} Z(I) \subset A^n$

2. Spaces with functions

Def: A space with functions is a top space $X$ with assignment $U \mapsto \mathcal{O}_X(U) \subset \{ \text{all fcns } U \to k \}$ (k-subalgebra) such that
1. $U = \bigcup_\alpha U_\alpha : f \in \mathcal{O}_X(U) \iff f|_{U_\alpha} \in \mathcal{O}_X(U_\alpha) \ \forall \alpha$.
2. $f \in \mathcal{O}_X(U) \Rightarrow D(f) \subset U$ open and $1/f \in \mathcal{O}_X(D(f))$.
Def: A morphism of SWFs is a cont. map $\varphi : X \to Y$ such that pullback of regular functions are regular.
I.e. if $V \subset Y$ is open and $f \in \mathcal{O}_Y(V)$, then $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

3. Subspace of SWF

$X$ SWF, $Y \subset X$ any subset. Give $Y$ structure of SWF as follows:
* Subspace topology.
* If $U \subset Y$ is open, $f : U \to k$ function, then $f$ is regular iff $f$ can locally be extended to regular fn on $X$.
I.e. $\forall y \in U \exists U' \subset X$ and $F \in \mathcal{O}_X(U')$ s.t. $y \in U'$ and $f(x) = F(x)$, $\forall x \in U \cap U'$.
Def. A prevariety is a SWF $X$ s.t. $\exists$ open cover $X = U_1 \cup \cdots \cup U_m$, with $U_i \cong \text{Spec-}m(R_i)$ affine variety for each $i$.
Exercise: Let $X = \text{Spec-}m(R)$ be affine and $f \in R$. Then $X_f := D(f) \cong \text{Spec-}m(R_f)$.
Exercise: $X$ SWF and $Y$ affine variety.
1-1 correspondence $\{ \text{morphisms } X \to Y \} \leftrightarrow \{ k$-alg homs $A(Y) \to \mathcal{O}(X) \}$.
Cor: Two affine varieties isomorphic iff coordinate rings isomorphic.
Exercise: \( \mathbb{A}^n \setminus \{0\} \) is not affine for \( n \geq 2 \).

Exercise: An open subset of a prevariety is a prevariety.

Exercise: A closed subset of a prevariety is a prevariety.

Def: \( X \) top space. A subset \( W \subset X \) is **locally closed** if it is an intersection of an open set and a closed set.

Cor: A locally closed subset of a prevariety is a prevariety.

4. Projective space

Def: \( \mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^* \) = lines through origin in \( \mathbb{A}^{n+1} \).

\[ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n \ \text{projection}. \]

Topology: \( U \subset \mathbb{P}^n \) open \( \iff \pi^{-1}(U) \subset \mathbb{A}^{n+1} \) open.

Regular fns: \( f : U \to k \) is regular \( \iff \pi^*(f) = f \circ \pi : \pi^{-1}(U) \to k \) regular.

Notation: \( (a_0 : \cdots : a_n) = \pi(a_0, \ldots, a_n) \).

Projective coord ring: \( \mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \ldots, x_n] \).

Def: Let \( f \in k[x_0, \ldots, x_n] \) homogeneous poly.

\[ D_+(f) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) \neq 0\} \]

Exercise: \( D_+(x_i) \cong \mathbb{A}^n \).

Cor: \( \mathbb{P}^n = D_+(x_0) \cup \cdots \cup D_+(x_n) \) is a prevariety.

Exercise: \( X \) SWF and \( \phi : \mathbb{P}^n \to X \) function. Then \( \phi \) is a morphism iff \( \phi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \to X \) is a morphism.

Def: If \( W \subset \mathbb{P}^n \) subset, then \( I(W) = I(\pi^{-1}(W)) \subset k[x_0, \ldots, x_n] \).

Def: If \( I \subset k[x_0, \ldots, x_n] \) homogeneous ideal, then \( Z_+(I) = \pi(Z(I)) \subset \mathbb{P}^n \).

Projective Nullstellensatz: \( I \subset k[x_0, \ldots, x_n] \) homogeneous ideal. If \( Z_+(I) \neq \emptyset \) then \( I(Z_+(I)) = \sqrt{I} \).

5. Projective varieties

Def. A **projective variety** is a closed subset of \( \mathbb{P}^n \) (with SWF structure).

A **quasi-projective variety** is a locally closed subset of \( \mathbb{P}^n \).

An **affine variety** is a closed subset of \( \mathbb{A}^n \).

A **quasi-affine variety** is a locally closed subset of \( \mathbb{A}^n \).

Exercise: \( \mathbb{P}^n \) is not quasi-affine for \( n \geq 1 \).

Exercise*: If \( X \) is both projective and quasi-affine, then \( X \) is finite.

Def: If \( X \subset \mathbb{P}^n \) is closed, then proj. coord. ring of \( X \) is \( k[x_0, \ldots, x_n]/I(X) \). Depends on embedding!!

Def: \( R \) graded ring, \( f \in R_d \).

\[ R_{(f)} = \{ \text{homogeneous elts. in } R_f \text{ of degree zero } \} = \{g/f^m \mid g \in R_{dm}\}. \]

Exercise: \( R \) f.g. reduced graded \( k \)-algebra \( \Rightarrow \) \( R_{(f)} \) f.g. reduced \( k \)-algebra.

Exercise: \( X \subset \mathbb{P}^n \) projective, \( R = k[x_0, \ldots, x_n]/I(X) \). \( f \in R_d \) with \( d > 0 \). Then \( X_f := X \cap D_+(f) \cong \text{Spec-m}(R_{(f)}). \)

Hints: Enough to assume \( X = \mathbb{P}^n \), \( R = k[x_0, \ldots, x_n] \).
Show that $O(D_+(f)) = R(f)$.
Identity map $R(f) \to O(D_+(f))$ defines morphism $D_+(f) \to \text{Spec-m}(R(f))$.
Show this is an isomorphism.

6. Products

Let $X$ and $Y$ be SWFs. A **product** of $X$ and $Y$ is a SWF called $X \times Y$ with morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, such that $(X \times Y, \pi_X, \pi_Y)$ is universal.

Exercise: Show that products of SWFs exist and are unique.

Example: $A^1 \times A^1 = A^2$. NOTE: $A^2$ does not have the product topology!

Exercise: If $X$ and $Y$ are affine varieties, then $X \times Y \cong \text{Spec-m}(A(X) \otimes_k A(Y))$.
Cor: A product of prevarieties is a prevariety.

7. Separated SWFs

Def: A SWF $X$ is **separated** if $\forall$ SWFs $Y$ and morphisms $f, g : Y \to X$ the set \{ $y \in Y \mid f(y) = g(y)$ \} $\subset Y$ is closed.

(Algebraic version of Hausdorff.)
Non-example: $X = (A^1 \setminus \{0\}) \cup \{O_1, O_2\} = \text{union of two copies of } A^1$.

Def: An **algebraic variety** is a separated prevariety.

Exercise: Any subspace of a separated SWF is separated.

Exercise: A product of separated SWFs is separated.

Exercise: $\Delta : X \to X \times X, x \mapsto (x, x)$ is a morphism.
Def: $\Delta_X := \Delta(X) \subset X \times X$.
Exercise: $\Delta : X \to \Delta_X$ isomorphism.

Exercise: $X$ is separated $\iff \Delta_X \subset X \times X$ is closed.

Exercise: $A^n$ is separated, hence all (quasi-) affine varieties are algebraic varieties.

Exercise: $P^n$ is separated, hence all (quasi-) projective varieties are varieties.

8. Rational maps

Def: A topological space $X$ is **irreducible** if $X$ is not a union of two proper closed subsets.

Let $X$ and $Y$ be irreducible varieties.
Consider pairs $(U, f)$ such that $\emptyset \neq U \subset X$ is open and $f : U \to Y$ is a morphism.
Relation: $(U, f) \sim (V, g)$ iff $f = g$ on $U \cap V$.
Exercise: $\sim$ is an equiv. relation. (Since $X$ is irreducible and $Y$ is separated.)

Def: A **rational map** $f : X \dashrightarrow Y$ is an equivalence class for $\sim$.
Exercise: There is a unique maximal open subset of points in $X$ where $f$ is defined as a morphism.

Def: A **rational function** on $X$ is a rational map $f : X \dashrightarrow A^1 = k$.
$f$ is given by a regular function $f : U \to k$, where $\emptyset \neq U \subset X$ is open.
Def: $k(X) = \{ f : X \dashrightarrow k \}$
Exercise: \( k(X) \) is a field.
Exercise: \( \emptyset \neq U \subset X \) open \( \Rightarrow k(U) = k(X) \).
Exercise: \( X \) irreducible, affine variety \( \Rightarrow k(X) = K(A(X)) \) fraction field.

Def: For \( V \subset X \) irreducible, closed, the local ring of \( V \) along \( V \) is the subring \( O_{X,V} \subset k(X) \) of rational functions that are defined in at least one point of \( V \):

\[
O_{X,V} = \{(U,f) \in k(X) \mid U \cap V \neq \emptyset\}.
\]

Unique max. ideal: \( m_{X,V} = \{(f,U) \in O_{X,V} \mid f(x) = 0 \forall x \in V \cap U\} \).

Exercise: \( X \) irreducible, affine, \( V \subset X \) irreducible closed \( \Rightarrow O_{X,V} = A(X)/I(V) \).

Def: \( (U,f) : X \to Y \) is dominant if \( f(U) = Y \).

Exercise: If \( f : X \to Y \) and \( g : Y \to Z \) are rational maps and \( f \) is dominant, then \( \exists \) well-defined composition \( g \circ f : X \to Z \).

Exercise: Let \( X \) and \( Y \) be irreducible varieties. 1-1 correspondence:

\[
\{ \text{dominant } f : X \to Y \} \leftrightarrow \{ \text{field ext. } k(Y) \subset k(X) \text{ over } k \}.
\]

Def: \( f : X \to Y \) is birational if \( f \) is dominant and \( \exists \) dominant \( g : Y \to X \) s.t.

\[
f \circ g = \text{id}_Y \text{ and } g \circ f = \text{id}_X.
\]

Def: \( X \) and \( Y \) are birationally equivalent (written \( X \cong Y \)) iff \( \exists \) birational map \( f : X \to Y \).

Example: \( \mathbb{A}^2 \cong \mathbb{P}^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \)

Exercise: \( X \cong Y \Leftrightarrow k(X) \cong k(Y) \) as \( k \)-algebras \( \Leftrightarrow \)

\( \exists \) open subsets \( U \subset X \) and \( V \subset Y \) s.t. \( U \cong V \).

Def: \( X \) is rational if \( X \) is birationally equivalent to \( \mathbb{A}^n \) for some \( n \).

Def: \( X \) is unirational if \( \exists \) dominant rational map \( f : \mathbb{A}^n \to X \).

Exercise*: \( E = \mathbb{Z}(y^2 - x^3 + x) \subset \mathbb{A}^2 \) is not rational.

Exercise**: If \( C \) is a unirational curve, then \( C \) is rational.

9. Complete Varieties

Def: A variety \( X \) is complete if for any variety \( Y, \pi_Y : X \times Y \to Y \) is closed.
(Analogue of compact manifolds. Schemes: same as proper over \( \text{Spec}(k) \).)

Note: 1) Closed subsets of complete varieties are complete.
2) Products of complete varieties are complete.

Example: Points are complete!

Example: \( \mathbb{A}^1 \) is not complete.

\( W = \mathbb{Z}(xy - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1 \) is closed but \( \pi_2(W) = \mathbb{A}^1 \setminus \{0\} \) is not closed in \( \mathbb{A}^1 \).

Exercise: Let \( \varphi : X \to Y \) be a morphism of varieties. If \( X \) is complete then \( \varphi(X) \subset Y \) is closed and complete. (Use graph \( \Gamma_f \subset X \times Y \).)

Exercise: \( \varphi : X \to Y \) cont. map of top. spaces. Then \( X \) irreducible. \( \Rightarrow \varphi(X) \) irreducible.

Cor: If \( X \) is irreducible and complete then \( O(X) = k \).

Proof: If \( f : X \to \mathbb{A}^1 \) is any morphism then \( f(X) \subset \mathbb{A}^1 \) is closed, complete, and irreducible, hence a point.

Exercise: Any complete quasi-affine variety if finite.

Exercise*: \( \mathbb{P}^n \) is complete, hence all projective varieties are complete.