Working with tensor products

R commutative ring.
M, N R-modules.

There exists a universal R-bilinear map \( \mu : M \times N \to M \otimes_R N \).

If \( \phi : M \times N \to P \) is any R-bilinear map, then \( \exists ! R \)-linear map
\( \hat{\phi} : M \otimes_R N \to P \) such that \( \phi = \hat{\phi} \circ \mu \).

\[
\begin{array}{ccc}
M \times N & \xrightarrow{\phi} & P \\
\mu \downarrow & & \downarrow \hat{\phi} \\
M \otimes_R N & \xrightarrow{\exists !} & P
\end{array}
\]

\( M \otimes_R N \) tensor product.

For \( m \in M, n \in N \), set \( m \otimes n = \mu(m, n) \).
I.e. universal bilinear map is
\[
\mu : M \times N \to M \otimes_R N \\
(m, n) \mapsto m \otimes n.
\]

Note: \( \hat{\phi}(m \otimes n) = \phi(m, n) \).

Fact: \( M \otimes_R N \) is generated by \( \{m \otimes n\} \).
Constructing $R$-linear maps

Example

Let $f : M' \to M$ and $g : N' \to N$ be $R$-linear maps. Then $\exists !$ $R$-linear map

$$f \circ g : M' \otimes_R N' \to M \otimes_R N$$

such that

$$(f \circ g)(m' \otimes n') = f(m') \otimes g(n').$$

In fact, $\mu \circ (f \times g) : M' \times N' \to M \otimes_R N$ is bilinear:

$$M' \times N' \xrightarrow{f \times g} M \times N \xrightarrow{\mu} M \otimes_R N$$

$$\xrightarrow{\mu'} \quad M' \otimes_R N' \xrightarrow{\exists ! f \circ g} M \otimes_R N$$


$M' \otimes_R N'$ generated by $\{m' \otimes n'\}$

$\Rightarrow f \circ g$ is determined by

$$(f \circ g)(m' \otimes n') = f(m') \otimes g(n').$$
Example

A $R$-algebra, i.e. a ring with ring homomorphism $f: R \to A$ such that $f(R) \subseteq Z(A)$.

3! $R$-linear map $m : A \otimes_R A \to A$ such that $m(a_1 \otimes a_2) = a_1 a_2$.

\[
\begin{array}{ccc}
\times & \xrightarrow{\text{mult}} & A \\
\mu & \downarrow & \downarrow m \\
A \otimes_R A & & A
\end{array}
\]

mult is $R$-bilinear.

Example

usually!

$M$ $R$-module. There is no $R$-linear map

\[ f : M \otimes_R M \to M \]

such that $f(m_1 \otimes m_2) = m_1 + m_2$.

Reason:

\[ M \times M \to M \]

\[(m_1, m_2) \mapsto m_1 + m_2 \]

is not bilinear.
Maps to \( M \otimes_R N \)?

**Example**

Given fixed \( u \in N \), there is an \( R \)-linear map:

\[
\tau_u : M \longrightarrow M \otimes_R N \\
\tau_u m \mapsto m \otimes u.
\]

**Example** Let \( M, N, P \) be \( R \)-modules.

Then

\[
\mu : M \times N \times P \longrightarrow (M \otimes N) \otimes P \\
(m, n, p) \mapsto (m \otimes n) \otimes p
\]

is a universal trilinear map.

If \( \phi : M \times N \times P \longrightarrow V \) is any trilinear map, then \( \exists! R \)-linear map

\[
\hat{\phi} : (M \otimes N) \otimes P \longrightarrow V
\]

such that

\[ \phi = \hat{\phi} \circ \mu, \quad \text{or}, \quad \hat{\phi}(m \otimes n \otimes p) = \phi(m, n, p). \]

\[
\begin{array}{ccc}
M \times N \times P & \xrightarrow{\phi} & V \\
\downarrow \mu & & \uparrow \exists! \hat{\phi} \\
(M \otimes N) \otimes P & & 
\end{array}
\]
Proof. For \( p \in P \), define
\[
\phi_p : M \times N \rightarrow V \quad \text{by} \quad \phi_p(m, n) = \phi(m, n, p).
\]
Since \( \phi_p \) is bilinear, \( \exists \) \( R \)-linear map
\[
\hat{\phi}_p : M \otimes_R N \rightarrow V, \quad \hat{\phi}_p(m \otimes n) = \phi(m, n, p).
\]
The map \( \phi' : (M \otimes_R N) \times P \rightarrow V \)
\[
(t, p) \mapsto \hat{\phi}_p(t)
\]
is bilinear.

\( \exists \) \( R \)-linear map
\[
\hat{\phi} : (M \otimes_R N) \otimes_R P \rightarrow V
\]
\[
t \otimes p \mapsto \hat{\phi}_p(t).
\]

Now:
\[
\hat{\phi}((m \otimes n) \otimes p) = \hat{\phi}_p(m \otimes n) = \phi(m, n, p).
\]