Working with tensor products

R commutative ring.

M, N R-modules.

<u>Thm</u> There exists a universal R-bilinear map $\mu : M \times N \longrightarrow M \otimes_R N$.

If $\phi : M \times N \longrightarrow P$ is any R-bilinear map, then $\exists!$ R-linear map $\hat{\phi} : M \otimes_R N \longrightarrow P$ such that $\phi = \hat{\phi} \circ \mu$.

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[dr, "\mu"'] & P \\ & M \otimes_R N \arrow[u, dashed, "\hat{\phi}", "\exists!"'] \end{tikzcd}$$

$M \otimes_R N$ <u>tensor product</u>.

For $m \in M$, $n \in N$, set $m \otimes n = \mu(m, n)$.

I.e. universal bilinear map is

$$\mu : M \times N \longrightarrow M \otimes_R N$$
$$(m, n) \longmapsto m \otimes n.$$

<u>Note</u>: $\hat{\phi}(m \otimes n) = \phi(m, n)$.

<u>Fact</u>: $M \otimes_R N$ is generated by $\{m \otimes n\}$.

Constructing R-linear maps

Example

Let $f: M' \longrightarrow M$ and $g: N' \longrightarrow N$ be R-linear maps. Then $\exists!$ R-linear map

$$f \otimes g : M' \otimes_R N' \longrightarrow M \otimes_R N$$

such that

$$(f \otimes g)(m' \otimes n') = f(m') \otimes g(n').$$

In fact, $\mu \circ (f \times g) : M' \times N' \longrightarrow M \otimes_R N$ is bilinear:

$$\begin{tikzcd}
M' \times N' \arrow[r, "f \times g"] \arrow[d, "\mu'"'] & M \times N \arrow[r, "\mu"] & M \otimes_R N \\
M' \otimes_R N' \arrow[urr, dashed, "\exists!\, f \otimes g"'] & &
\end{tikzcd}$$

$M' \otimes N'$ generated by $\{m' \otimes n'\}$

$\Rightarrow$ $f \otimes g$ is determined by

$$(f \otimes g)(m' \otimes n') = f(m') \otimes g(n').$$

Example

A R-algebra, i.e. A ring with ring homomorphism $f: R \longrightarrow A$ such that $f(R) \subseteq Z(A)$.

$\exists!$ R-linear map $m: A \otimes_R A \longrightarrow A$ such that $m(a_1 \otimes a_2) = a_1 a_2$.

$$\begin{tikzcd} A \times A \arrow[r, "\text{mult}"] \arrow[d, "\mu"'] & A \\ A \otimes_R A \arrow[ur, dashed, "m"'] & \end{tikzcd}$$

mult is R-bilinear.

Example

M R-module. There is usually! no R-linear map

$$f: M \otimes_R M \longrightarrow M$$

such that $f(m_1 \otimes m_2) = m_1 + m_2$.

Reason: $$M \times M \longrightarrow M$$
$$(m_1, m_2) \longmapsto m_1 + m_2$$
is not bilinear.

Maps to $M \otimes_R N$?

<u>Example</u>

Given fixed $n \in N$, there is an R-linear map:

$$t_n : M \longrightarrow M \otimes_R N$$
$$m \longmapsto m \otimes n .$$

<u>Example</u> Let M, N, P be R-modules. Then

$$\mu : M \times N \times P \longrightarrow (M \otimes N) \otimes P$$
$$(m, n, p) \longmapsto (m \otimes n) \otimes p$$

is a universal trilinear map:

If $\phi : M \times N \times P \longrightarrow V$ is any trilinear map, then $\exists!$ R-linear map $\hat{\phi} : (M \otimes N) \otimes P \longrightarrow V$ such that $\phi = \hat{\phi} \circ \mu$, or, $\hat{\phi}((m \otimes n) \otimes p) = \phi(m, n, p)$.

$$\begin{tikzcd}
M \times N \times P \arrow[rr, "\phi"] \arrow[dr, "\mu"'] & & V \\
& (M \otimes N) \otimes P \arrow[ur, "\exists! \hat{\phi}"'] &
\end{tikzcd}$$

<u>Proof</u> For $p \in P$, define

$\phi_p : M \times N \longrightarrow V$ by $\phi_p(m, n) = \phi(m, n, p)$.

Since ϕ_p is bilinear, $\exists!$ R-linear map

$\hat{\phi}_p : M \otimes_R N \longrightarrow V$, $\hat{\phi}_p(m \otimes n) = \phi(m, n, p)$.

The map
$$\begin{aligned} \phi' : (M \otimes_R N) \times P &\longrightarrow V \\ (t, p) &\longmapsto \hat{\phi}_p(t) \end{aligned}$$
is bilinear.

$\exists!$ R-linear map
$$\begin{aligned} \hat{\phi} : (M \otimes_R N) \otimes_R P &\longrightarrow V \\ t \otimes p &\longmapsto \hat{\phi}_p(t). \end{aligned}$$

Now:
$$\hat{\phi}((m \otimes n) \otimes p) = \hat{\phi}_p(m \otimes n) = \phi(m, n, p).$$