Algebra 551 Takehome Midterm, Fall 2022

Instructions: Do all of the following six problems. Give complete proofs of your claims. You may use the book and other tools.

• When relevant, use only one universal property at the time, and explain how you apply it.

• Make sure to introduce all symbols before you use them. For example, "Let G be a group and let $g \in G$ be an element" introduces the symbols "G" and "g". The sentence "We have $x^2 > 4$ whenever $x > 2$" is also fine, since it is clear what x is.

• Write as much as you need to argue that your solutions are correct and complete. There is no need to write more than that.

• Your solutions will be evaluated for correctness, completeness, and clarity, so please write your solutions carefully, clearly, and legibly.

Problem 1.

Let $\{G_i\}_{i \in I}$ be a family of abelian groups. Let P be the product of this family, with projections $\pi_j : P \to P_j$ for each $j \in I$. Let S be the coproduct of the family, with inclusions $\iota_k : G_k \to S$ for each $k \in I$. For $j, k \in I$, let $\phi_{jk} : G_k \to G_j$ denote the identity map when $j = k$, and the zero map when $j \neq k$. Use the defining universal properties of P and S to prove the following. (Can you do it without referring to elements in the groups?)

(a) For each fixed $k \in I$, there exists a group homomorphism $\alpha_k : G_k \to P$, such that $\pi_j \alpha_k = \phi_{jk}$ holds for each $j \in I$.

(b) There exists a group homomorphism $\psi : S \to P$, such that $\pi_j \psi \iota_k = \phi_{jk}$ holds for all $j, k \in I$.

Problem 2.

Let $\mathbb{F}$ be a field, X a set, $\mathbb{F}X$ the free $\mathbb{F}$-vector space on X, $S^\bullet(\mathbb{F}X)$ the symmetric algebra of $\mathbb{F}X$, and $\mathbb{F}[X]$ the free commutative $\mathbb{F}$-algebra on X. Use the universal properties of free objects and symmetric algebras to show that $S^\bullet(\mathbb{F}X)$ is isomorphic to $\mathbb{F}[X]$ as $\mathbb{F}$-algebras.

Problem 3.

Let R be a commutative ring and let M, N, and P be R-modules. Use the defining universal property of tensor products to prove the following. (Beware that an R-module may not have a basis, so don't use any basis elements.)

(a) There exists a unique R-module homomorphism $\alpha : M \otimes_R (N \oplus P) \to M \otimes_R N$, such that $\alpha(m \otimes (n, p)) = m \otimes n$ holds for all $m \in M$ and $n \in N$.

(b) There exists a unique isomorphism of R-modules

$$\phi : M \otimes_R (N \oplus P) \to (M \otimes_R N) \oplus (M \otimes_R P),$$

such that $\phi(m \otimes (n, p)) = (m \otimes n, m \otimes p)$ holds for all $m \in M$, $n \in N$, and $p \in P$.

Problem 4.

Let G be a group and $H \leq G$ a subgroup. Prove that if H is a characteristic subgroup of G, then H is a normal subgroup of G. Give an example of a normal subgroup that is not a characteristic subgroup.

Problem 5.

Let A_n be the group of even permutations of n elements. Show that for $n \geq 5$, all 3-cycles in A_n are conjugate as elements of A_n.

Problem 6.

Find the number of colorings of the edges of a regular hexagon using n colors, up to rotations and reflections. In other words, two colorings are considered the same if one can be obtained from the other by a rotation or reflection of the hexagon.

Hint: There are 92 colorings when using 3 colors.